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On Unconditional Stability of Large-Scale Discrete Systems With Delays

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Abstract

In this paper the problem of unconditional stability for a class of linear time-invariant large-scale discrete systems with delays is investigated. By using the properties of m matrices, a sufficient condition of unconditional stability for low-order discrete systems with delays is first derived. Then, a comparison principle is presented for discrete systems with delays. Finally, a sufficient condition of unconditional stability for large-scale systems is established by using the aggregation technique based on vector Lyapunov functions. A numerical example is also given to illustrate the applicability of the stability criterion obtained in this paper.

1. Introduction

The stability problem for a class of large-scale systems described by differential-difference equations was fruitfully studied by some researchers [1]-[3]. with the increasing complexity and sophistication of engineering systems and with the rapid development of computer techniques in our country, it will be an inevitable trend that large-scale engineering systems are controlled by using micro-computers in a decentralized manner. Hence, it is of practical significance to study the stability properties of large-scale discrete systems with delays. In this paper a sufficient condition under which the large-scale discrete system is stably independent of delays is derived by using the aggregation technique based on vector Lyapunov functions. This criterion results from testing whether a matrix with dimensions equal to the number of subsystems is an m matrix or not.

2. A Sufficient Condition of Unconditional Stability for

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Discrete-Delay Systems

In the stability analysis of dynamic systems, a class of matrices called **m** matrices play a very important role. Before establishing the main result of this section, we must have a discussion on some related properties of **m** matrices.

Definition 1 A matrix $\mathbf{A}\in R^{n\times n}$ is called an **m** matrix if (a) all off-diagonal elements of **A** are nonpositive; (b) every eigenvalue of **A** has a positive real part.

Lemma 1 If **A** is an **m** matrix, then there exists a positive diagonal matrix **D** such that **AD** is positive dominant diagonal.

The proof of this lemma and the definition of positive dominant diagonal matrix can be found in [5], and are omitted here.

Definition 2 For an **m** matrix $\mathbf{A}\in R^{n\times n}$, a class of matrices related to **A** is defined by $\underline{\mathbf{B}}(\mathbf{A})=\{\mathbf{B}\in C^{n\times n}\,|\,|b_{ii}|\geq a_{ii};\ |b_{ij}|\leq -a_{ij},\ \text{for } i\neq j\}$.

Lemma 2 If $\mathbf{A}\in R^{n\times n}$ is an **m** matrix, then every matrix $\mathbf{B}\in\underline{\mathbf{B}}(\mathbf{A})$ is nonsingular.

Proof Since **A** is an **m** matrix, according to lemma 1 there must be a positive diagonal matrix **D** such that **AD** is positive dominant diagonal, i. e., $d_i a_{ii} > \sum_{\substack{j=1\\ j\neq i}}^{n} d_j|a_{ij}|$. For every matrix $\mathbf{B}\in\underline{\mathbf{B}}(\mathbf{A})$, it is not difficult to show that **BD** is dominant diagonal, since

$$d_i|b_{ii}|\geq d_i a_{ii} > \sum_{\substack{j=1\\ j\neq i}}^{n} d_j|a_{ij}| \geq \sum_{\substack{j=1\\ j\neq i}}^{n} d_j|b_{ij}|.$$

We know that if a matrix is dominant diagonal, then it is nonsingular. Hence, every matrix $\mathbf{B}\in\underline{\mathbf{B}}(\mathbf{A})$ is nonsingular, because of the nonsingularity of the matrix **D**. This completes the proof of this lemma.

Now we are in a position to give the main result of this section.

Consider the system described by N scalar difference-delay equations

$$x_i(k+1)=\sum_{j=1}^{N} a_{ij}x_j(k)+\sum_{j=1}^{N} b_{ij}x_j(k-m_{ij}),\quad k>0,\quad i=1 \text{ to } N \qquad (1)$$

where m_{ij} is any nonnegative integer.

Definition 3 The system (1) is said to be unconditionally asymptotically stable if it is asymptotically stable for any nonnegative integer m_{ij}.

Theorem 1 Define a matrix $C=\{c_{ij}\}$,

$$c_{ij}=\delta_{ij}-|a_{ij}|-|b_{ij}|, \text{where } \delta_{ij} \text{ is Kronecker delta symbol.} \tag{2}$$

If $\mathbf{C}$ is an $\mathbf{m}$ matrix, then the system (1) is unconditionally asymptotically stable.

Proof Taking the z transformation of system (1), we obtain the characteristic matrix $\mathbf{D}(z)=\{d_{ij}(z)\}$ of system (1), where $d_{ij}(z)=z\delta_{ij}-a_{ij}-b_{ij}/z^{m_{ij}}$. When $|z|\geq 1$, we have

$$|d_{ii}(z)|=|z-a_{ii}-b_{ii}/z^{m_{ii}}|\geq|z|-|a_{ii}|-|b_{ii}|/|z|^{m_{ii}}\geq 1-|a_{ii}|-|b_{ii}|=c_{ii}$$

$$|d_{ij}(z)|=|-a_{ij}-b_{ij}/z^{m_{ij}}|\leq|a_{ij}|+|b_{ij}|/|z|^{m_{ij}}\leq|a_{ij}|+|b_{ij}|=-c_{ij},$$

This implies that $\mathbf{D}(z)\in \underline{\mathbf{B}}(\mathbf{C})$. According to lemma 2, $\mathbf{D}(z)$ is nonsingular when $|z|\geq 1$. This implies the fact that all the poles of system (1) lie inside the unit circle of the complex plane. That is, system (1) is unconditionally asymptotically stable. Q. E. D.

3. A Comparison Theorem

Consider the following vector difference-delay inequality

$$V(k+1)\leq \mathbf{A}_0V(k)+\mathbf{A}_1V(k-\tau_1)+\cdots+\mathbf{A}_sV(k-\tau_s) \tag{3}$$

where τ_i is a nonnegative integer and satisfies $\tau_i<\tau_{i+1}$, s is some finite positive integer, $V\in R^N$, $\mathbf{A}_i\in R^{N\times N}$.

Let the comparison system of (3) be

$$W(k+1)=\mathbf{A}_0W(k)+\mathbf{A}_1W(k-\tau_1)+\cdots+\mathbf{A}_sW(k-\tau_s). \tag{4}$$

For difference-delay equation (4), W(k) is called the instantaneous state of (4), the complete state of (4) is a finite point set with τ_s+1 elements in R^N defined by $W_k(\theta)=W(k-\theta)$, $\theta\in T=\{0,1,2,\cdots,\tau_s\}$. Hence, the initial condition of (4) must be a point set $W_0(\theta)$, $\theta\in T$. The above discussion is also true for the inequality (3).

Let the initial conditions of (3) and (4) be $V_0(\theta)$ and $W_0(\theta)$, $\theta\in T$ respectively. Then we can give the comparison theorem as follows:

Theorem 2 If $V_0(\theta)=W_0(\theta)$, $\theta\in T$, and for $i=0,1,2,\cdots,s$, $\mathbf{A}_i$ is a nonnegative matrix, then for all $k>0$, the solutions of (3) and (4) satisfy the inequality $V_k(\theta)\leq W_k(\theta)$, $\theta\in T$.

Proof We prove it by using induction. Noting that $\mathbf{A}_i$ is a nonnegative matrix for all $i=0,1,\cdots,s$, and that $V_0(\theta)=W_0(\theta)$, $\theta\in T$, we can prove the inequality $V(k)\leq W(k)$ for $1\leq k\leq\tau_s+1$ through τ_s+1

steps of iteration:

$$V(1)\le A_0V(0)+A_1V(-\tau_1)+\cdots+A_sV(-\tau_s)$$
$$=A_0W(0)+A_1W(-\tau_1)+\cdots+A_sW(-\tau_s)=W(1)$$
$$V(2)\le A_0V(1)+A_1V(1-\tau_1)+\cdots+A_sV(1-\tau_s)$$
$$\le A_0W(1)+A_1W(1-\tau_1)+\cdots+A_sW(1-\tau_s)=W(2)$$
$$\cdots\cdots$$

Suppose that $V(k)\le W(k)$ is true for $n-\tau_s-1\le k\le n-1$, where n is an arbitrary natural number, then through τ_s+1 steps of iteration as above we can prove that the inequality $V(k)\le W(k)$ is true for $n\le k\le \tau_s+n$. Thus the proof of this theorem is completed.

4. A Sufficient Condition of Unconditional Stability for Large-Scale Systems

Consider the following largs-scale system composed of N subsystems

$$X_i(k+1)=A_{ii}X_i(k)+\sum_{\substack{j=1\\ j\ne i}}^{N}A_{ij}X_j(k)+\sum_{j=1}^{N}B_{ij}X_j(k-m_{ij}),\quad i=1 \text{ to } N \tag{5}$$

where $X_i\in R^{n_i}$, $X=[X_1^T,X_2^T,\cdots,X_N^T]^T\in R^n$ are the instantaneous state of the i th subsystem and the whole system respectively and $\sum_{i=1}^{N}n_i=n$, m_{ij} is any nonnegative integer, A_{ij} and B_{ij} are real constant matrices with compatible dimensions.

Consider the i th isolated subsystem of the system (5)

$$X_i(k+1)=A_{ii}X_i(k),\quad i=1 \text{ to } N \tag{6}$$

Suppose that for all $i=1$ to N, the n_i eigenvalues of A_{ii} lie inside the unit circle of the complex plane, i. e., all N isolated subsystems are asymptotically stable. Then, for any given symmetric positive definite matrix G_i there always exists unique symmetric positive definite matrix H_i such that

$$A_{ii}^TH_iA_{ii}-H_i=-G_i \tag{7}$$

Let $V_i(k)=X_i^T(k)H_iX_i(k)$ be the Lyapunov function for the i th isolated subsystem. From (6) and (7), it is easy to show the following two inequalities

$$\lambda_m(\mathbf{H}_i)\|X_i(k)\|^2 \le V_i(k) \le \lambda_M(\mathbf{H}_i)\|X_i(k)\|^2 \tag{8}$$

$$\Delta V_i(k)|_{(6)} = V_i(k+1)|_{(6)} - V_i(k) \le -\eta_i V_i(k) \tag{9}$$

where $\eta_i = \lambda_m(\mathbf{G}_i)/\lambda_M(\mathbf{H}_i)\cdot\lambda_m(.)$ and $\lambda_M(.)$ denote the minimum and the maximum eigenvalues of the matrices within the brackets respectively. $\|\cdot\|$ denotes the Euclidian norm. For the matrices the induced norm compatible with the Euclidian norm is used. Since every isolated subsystem is asymptotically stable, the matrix $\mathbf{G}_i$ can be chosen appropriately such that the inequality $0 < \eta < 1$ is always satisfied.

Let

$$\left.\begin{aligned} &\|\mathbf{A}_{ij}\| = \lambda_M^{1/2}(\mathbf{A}_{ij}^T\mathbf{A}_{ij}) = \beta_{ij}, \qquad \|\mathbf{B}_{ij}\| = \lambda_M^{1/2}(\mathbf{B}_{ij}^T\mathbf{B}_{ij}) = \gamma_{ij} \\ &\lambda_M(\mathbf{H}_i)/\lambda_m(\mathbf{H}_i) = \alpha_{ij}, \quad \sum_{\substack{j=1\\ j\ne i}}^{N}\beta_{ij} = \varphi_i, \quad \sum_{j=1}^{N}\gamma_{ij} = \phi_i \end{aligned}\right\} \tag{10}$$

Define a matrix $\mathbf{C} \in R^{N\times N}$ as follows:

$$\left.\begin{aligned} &C_{ii} = \eta_i - \alpha_{ii}\beta_{ii}\gamma_{ii} - \alpha_{ii}(\varphi_i + \phi_i)(\beta_{ii} + \gamma_{ii}) \\ &C_{ij} = -\alpha_{ij}(\beta_{ii} + \varphi_i + \phi_i)(\beta_{ij} + \gamma_{ij}) \quad \text{for } i \ne j \end{aligned}\right\} \tag{11}$$

Now we are ready to establish the main result of this paper.

Theorem 3 The large-scale discrete-deley system (5) is unconditionally asymptotically stable if the following two conditions are satisfied:

(a) all N isolated subsystems in (6) are asymptotically stable;

(b) the matrix $\mathbf{C}$ defined in (11) is an $\mathbf{m}$ matrix.

Proof Take $V_i(k) = X_i(k)^T\mathbf{H}_i X_i(k)$ as the Lyapunov function of the ith subsystem. Then we have

$$V_i(k+1) = \Big[\mathbf{A}_{ii}X_i(k) + \sum_{\substack{j=1\\ j\ne i}}^{N}\mathbf{A}_{ij}X_j(k) + \sum_{j=1}^{N}\mathbf{B}_{ij}X_j(k-m_{ij})\Big]^T\mathbf{H}_i\Big[\mathbf{A}_{ii}X_i(k)$$

$$+ \sum_{\substack{j=1\\ j\ne i}}^{N}\mathbf{A}_{ij}X_j(k)\ \sum_{j=1}^{N}\mathbf{B}_{ij}X_j(k-m_{ij})\Big] - X_i^T(k)\mathbf{H}_iX_i(k) + V_i(k)$$

$$\leq (1-\eta_i)V_i(k) + 2\|A_{ii}\|\|H_i\|\|X_i(k)\| \sum_{\substack{j=1 \\ j\neq i}}^{N} \|A_{ij}\|\|X_j(k)\|$$

$$+2\|A_{ii}\|\|H_i\|\|X_i(k)\| \sum_{j=1}^{N} \|B_{ij}\|\|X_j(k-m_{ij})\| + \sum_{\substack{i=1 \\ j\neq i}}^{N} \sum_{\substack{s=1 \\ s\neq i}}^{N} \|A_{ij}\|\|A_{is}\| \cdot$$

$$\cdot \|H_i\|\|X_j(k)\|\|X_s(k)\| + 2\sum_{j=1}^{N} \sum_{\substack{s=1 \\ s\neq i}}^{N} \|A_{is}\|\|H_i\|\|B_{ij}\|\|X_s(k)\|\|X_j(k-m_{ij})\|$$

$$+ \sum_{s=1}^{N} \sum_{j=1}^{N} \|B_{ij}\|\|H_i\|\|B_{is}\|\|X_j(k-m_{ij})\|\|X_s(k-m_{is})\|. \qquad (12)$$

By noting (8) and (10) and by using the fundamental inequality $a^2+b^2 \geq 2ab$, through simple algebraic operatons we can obtain the following inequality from (12):

$$V_i(k+1) \leq \left[1-\eta_i+\alpha_{ii}\beta_{ii}(\varphi_i+\phi_i)\right]V_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \alpha_{ij}\beta_{ij}(\beta_{ii}+\phi_i+\phi_i)V_j(k)$$

$$+ \sum_{j=1}^{N} \alpha_{ij}\gamma_{ij}(\beta_{ii}+\varphi_i+\phi_i)V_j(k-m_{ij}), \quad i=1 \text{ to } N. \qquad (13)$$

Let (14) be the comparison system of (13).

$$W_i(k+1) = \left[1-\eta_i+\alpha_{ii}\beta_{ii}(\varphi_i+\phi_i)\right]W_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \alpha_{ij}\beta_{ij}(\beta_{ii}+\varphi_i+\phi_i)W_j(k)$$

$$+ \sum_{j=1}^{N} \alpha_{ij}\gamma_{ij}(\beta_{ii}+\varphi_i+\phi_i)W_j(k-m_{ij}), \quad i=1 \text{ to } N. \qquad (14)$$

It is obvious that (13) and (14) can be rewritten in the form of (3) and (4), and that the possible maximum number of terms in (3) and (4) must be $N\times N+1$ when m_{ij}'s are all different for $i,j=1$ to N. Since $\eta_i<1$, all the coefficients in (13) and (14) are nonnegative, i.e., after transformation all the coefficient matrices A_0, $A_1, \cdots, A_{N\times N}$ are nonnegative. We know from theorm 2 that if the system (14) is unconditionally asymptotically stable, so is the large-scale system (5). Since the matrix **C** defined in (11) is an m matrix by assumption, the theorem 1 shows that the system (14) is unconditionally asymptotically stable. Q. E. D.

To test if a real square matrix with nonpositive off-diagonal elements is an m matrix, we present the following lemma:

Lemma 3 Let $\mathbf{A}\in R^{n\times n}$ be a matrix with nonpositive off-diagonal elements, then **A** is an m matrix if and only if either of the following two conditons is satisfied:

(a) all its leading principal minors are positive;

(b) $\mathbf{A}^{-1}$ exists and $\mathbf{A}^{-1}$ is a nonnegative matrix.

The proof of this lemma ean be found in [6]-[7] and is omitted here.

5. Example

Consider the following system:

$$\left.\begin{aligned} X_1(k+1) &= \mathbf{A}_{11}X_1(k)+\mathbf{A}_{12}X_2(k)+\mathbf{B}_{11}X_1(k-m_{11})+\mathbf{B}_{12}X_2(k-m_{12}) \\ X_2(k+1) &= \mathbf{A}_{21}X_1(k)+\mathbf{A}_{22}X_2(k)+B_{21}X_1(k-m_{21})+\mathbf{B}_{22}X_2(k-m_{22}) \end{aligned}\right\} \tag{15}$$

where

$$\mathbf{A}_{11}=\begin{pmatrix} 0.001 & 0 & 0 \\ 0 & -0.02 & 0.01 \\ 0 & -0.01 & -0.02 \end{pmatrix},\quad \mathbf{A}_{12}=\begin{pmatrix} 0 & 0.01 \\ 0 & -0.05 \\ 0.02 & 0 \end{pmatrix},$$

$$\mathbf{A}_{21}=\begin{bmatrix} 0.02 & 0.01 & 0 \\ 0.03 & 0 & 0.05 \end{bmatrix},\quad \mathbf{A}_{22}=\begin{bmatrix} -0.005 & 0.3 \\ -0.03 & -0.005 \end{bmatrix},$$

$$\mathbf{B}_{11}=\begin{pmatrix} 0.04 & 0 & 0.05 \\ 0 & -0.08 & 0 \\ -0.2 & 0 & 0.1 \end{pmatrix},\quad \mathbf{B}_{12}=\begin{pmatrix} 0.03 & 0 \\ 0.01 & -0.1 \\ 0 & 0.04 \end{pmatrix},$$

$$\mathbf{B}_{21}=\begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.03 & 0 \end{bmatrix},\quad \mathbf{B}_{22}=\begin{bmatrix} -0.01 & 0.05 \\ 0 & -0.02 \end{bmatrix}$$

$m_{11}, m_{21}, m_{12}, m_{22}$ are any nonnegative integers.

Now let us test the stability of the system (15) by using theorem 3.

Taking $\mathbf{G}_1=\mathrm{diag}\{0.999999,\ 0.9995,\ 0.9995\}$, $\mathbf{G}_2=\{0.999075,\ 0.999075\}$, and solving the Lyapunov equation we obtain $H_1=I_3$ and

$H_2=I_2$, where I_n denotes the identity matrix with dimension n.

It is clear that the two isolated subsystems of (15) are asymptotically stable. Then we can calculate the following parameters.

$\eta_1=0.9995$, $\eta_2=0.999075$, $\alpha_{11}=\alpha_{12}=\alpha_{21}=\alpha_{22}=1$, $\beta_{11}=0.0224$,
$\beta_{12}=0.051$, $\beta_{21}=0.0594$, $\beta_{22}=0.0305$, $\gamma_{11}=0.225$, $\gamma_{12}=0.109$,
$\gamma_{21}=0.05$, $\gamma_{22}=0.0547$, $\varphi_1=0.051$, $\varphi_2=0.0594$, $\phi_1=0.334$,
$\phi_2=0.1047$

According to (11), we obtain the matrix $\mathbf{C}$:

$$\mathbf{C}=\begin{bmatrix} 0.899211 & -0.065184 \\ -0.02129 & 0.98343 \end{bmatrix}$$

Since the leading principal minors of $\mathbf{C}$ are all positive and the off-diagonal elements of $\mathbf{C}$ are all negative, $\mathbf{C}$ is an $\mathbf{m}$ matrix. Then we can conclude that the system (15) is unconditionally asymptotically stable.

6. Conclusions

In this paper a stability criterion is established for a class of linear time-invariant large-scale discrete systems with delays. The main advantage of our result is that it can be applied to a very general class of systems. If $m_{ij}=m$ for all $i,j=1$ to N, we obtain the stability criterion for large scale systems with commensurate delay; and if in (5) all the coefficient matrices $\mathbf{B}_{ij}=0$ for all $i,j=1$ to N, we obtain the stability criterion for large-scale discrete systems without delays. The result obtained in this paper can be further applied to decentralized stabilization of large-scale discrete system with delays [9].

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离散时滞大系统的无条件稳定性

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摘　要

本文研究一类线性定常离散时滞大系统的无条件稳定性问题。首先利用m阵的性质，给出了一般离散时滞系统无条件渐近稳定的充分条件；接着建立了适应于离散时滞系统的比较原理；最后用向量李雅普诺夫函数法和集结（aggregatioa）方法，推得了线性定常离散时滞大系统无条件稳定的充分条件。这一条件归结为判定一个阶数等于子系统个数的实矩阵是否是一个m阵。一个简单的数值例子说明了本文得出的结果的应用。