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On the Constrained LQ Problem and tne Temperature Control of Vinylon Solution

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Abstract

This paper deals with a constrained LQ problem and gives the sufficient and necessary condition for its solubility. The form of the optimal synthesis function is presented. Also, the sufficient condition for the optimal closed-loop system to be globally asymptotically stable is given. The results have been satisfactorily applied to control the temperature of vinylon solution in a vinylon fibre mill, achieving the given temperature of vinylon solution rapidly without overshoot and oscillation.

§ 1. Introduction

Let us consider a single-input LQ problem with the control variable being constrained:

$$\dot{x} = Ax + bu, \tag{1.1}$$

$$J[u] = \frac{1}{2}\int_0^{+\infty} [x^T Q x + r u^2] dt, \tag{1.2}$$

$$|u| \leq M, \tag{1.3}$$

where $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^{n\times 1}$, $u \in \mathbf{R}^1$, $M > 0$, $r > 0$, $Q = C^T C$ is a nonnegative definite symmetric matrix.

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It is known that when there is no constraint on the scalar control u in (1.1) and (1.2), under certain conditinos, for example, (A,b) is completely controllable and (A, C) is completely observable, the optimal synthesis function $u^*(x)$ of (1.1) and (1.2) has the following form:

$$u^*(x) = -r^{-1}b^\tau P^* x, \tag{1.4}$$

where P^* is the unique positive definite symmetric solution matrix of the Riccati matrix equation:

$$PA + A^\tau P + Q - r^{-1}Pbb^\tau P = O_{n\times n} \tag{1.5}$$

Moreover, the optimal closed-loop system

$$\dot{x} = (A - r^{-1}bb^\tau P^*)x \tag{1.6}$$

is globally asymptotically stable [1].

In some engineering problems, such as in the temperature control of vinylon solution, the amplitude of the control variable is bounded. Generally speaking, the restriction on the control variable u may be characterized by (1.3). So it is interesting to discuss the following problems: whether an optimal control exists when there is a constraint on the control variable in (1.1) and (1.2)? What about the stability of the corresponding optimal closed-loop system? If these problems are solved, a lot of constrained optimal regulation problems occuring in practice can be solved at least theoetically.

In paper [2] a single-input linear system with relay control was discussed. The necessary aud sufficient condition for the optimal closed-loop system to be locally stable was obtained. In paper [3] a constrained LQ problem was discussed. By constructing a solution for Bellman equation a sufficient condition for the problem to be solvabe is presented, but the stability of closed-loop system was not discussed. In paper [4] the author considered a nonlinear optimal control law for a linear system in a finite-time interval $[t_0, t_f]$. It was shown that the optimal arcs may end up with a singular trajectory.

This paper will discuss the problem mentioned above, that is, the constrained LQ problem. The existence condition and the analytical form of the solution of the constrained LQ problem are obtained. The sufficient condition for the optimal closed-loop system to be globally symptotically stable is presented. Moreover, the estimation formula for the transition time of a solution of the optimal closed-loop system is also given. The results have been satisfactorily applied to control the temperature in the heating process of vinylon solution in a vinylon fibre mill. As a result, the given temperature has been technologically achieved rapidly without overshoot and oscillation. The control law obtained is intuitive enough to be understood by engineers and it is convenient for use in practice.

§ 2. Solution of the Problem and Stability

Throughout the paper we assume that (A, b) is completely controllable.

Let $\mathbf{u}_{x_0}=\{u(t)\,|\,u(t)$ is piecewise continuous on $[0,+\infty)$, and $|u(t)|\le M$; the solution x(t) of (1.1) corresponding to $u(t)$ with the initial value $x(0)=x_0$ satisfies $\lim\limits_{t\to\infty}x(t)=0_n\}$.

It is easy to prove that $\mathbf{u}_{x_0}$ is a nonempty set under the assumption of controllability for those x_0 belonging to a neighborhood of $x=0_n$.

Let P_n be a positive definite symmetric solution matrix of the Riccati algebraic matrix equation:

$$PA+A^{\tau}P+Q-r^{-1}Pbb^{\tau}P=0_{n\times n}. \tag{2.1}$$

we now take some $u(t)\in\mathbf{u}_{x_0}$. The solution of (1.1) corresponding to $u(t)$ is denoted by $x(t)$.

It is easy to see that:

$$\frac{d}{dt}x^{\tau}(t)P_nx(t)=-[x^{\tau}(t)Qx(t)+ru^2(t)]+r[u(t)+r^{-1}b^{\tau}P_nx(t)]^2.$$

Integrating above equation from 0 to $+\infty$ and noticing $\lim\limits_{t\to\infty}x(t)=0_n$, we have:

$$J[u(t)]=x_0^{\tau}P_nx_0+\int_0^{\infty}r[u(t)+r^{-1}b^{\tau}P_nx(t)]^2dt. \tag{2.2}$$

It follows from (2.2) that under constraint condition (1.3) the necessary and sufficient condition for $J[u(t)]$ to be minimum is that $u^*(t)$ possesses the following form:

$$u^*(t)=\begin{cases}M, & r^{-1}b^{\tau}P_nx^*(t)\le -M,\\ -r^{-1}b^{\tau}P_nx^*(t), & -M\le r^{-1}b^{\tau}P_nx^*(t)\le M,\\ -M, & r^{-1}b^{\tau}P_nx^*(t)\ge M.\end{cases} \tag{2.3}$$

From here we can prove that the optimal synthesis function $u^*(x)$ of (1.1), (1.2) and (1.3) has the form:

$$u^*(x)=\begin{cases} M, & r^{-1}b^TP_nx\le -M, \\ -r^{-1}b^TP_nx, & -M\le r^{-1}b^TP_nx\le M, \\ -M, & r^{-1}b^TP_nx\le M. \end{cases} \quad (2.4)$$

Substituting $u^*(x)$ in (2.4) into (1.1) we obtain the optimal closed-loop system of (1.1), (1.2) and (1.3) as follows:

$$\dot{x}=Ax+bu^*(x), \quad (2.5)$$

which is nonlinear. Let $G^{(n)}$, $G_1^{(n)}$ and $G_2^{(n)}$ be regions defined as:

$G^{(n)}$: $|r^{-1}b^TP_nx|\le M$,

$G_1^{(n)}$: $r^{-1}b^TP_nx\le -M$,

$G_2^{(n)}$: $r^{-1}b^TP_nx\ge M$.

Since in the interior of $G^{(n)}$ the optimal closed-loop system is:

$$\dot{x}=(A-r^{-1}bb^TP_n)x \quad (2.6)$$

and $(A-r^{-1}bb^TP_n)$ is a stable matrix, so the optimal closed-loop system (2.5) is locally asymptotically stable.

It can be shown that for the following two-dimensional constrained LQ problem:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$J[u]=\int_0^\infty [x_1^2+x_2^2+u^2]\,dt$$

$$|u|\le 1$$

the optimal control law exists but the optimal closed-loop system is not globally stable. The phase diagram of the optimal closed-loop system is shown in Fig 1.

Let $f(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0$ be the characteristic polynomial of the $n\times n$ matrix A in (1.1). Then we have:

Theorem If the coefficients of $f(\lambda)$ satisfy:

$$a_0=0,\ a_1>0,\ a_2>0,$$

$$a_h^2-2a_{h-1}a_{h+1}+2a_{h-2}a_{h+2}+\cdots(-1)^{h-1}2a_1a_{2h-1}>0 \quad (2.7)$$

$h=2, 3, \cdots n-1$, $a_n=1$, $a_k=0$, $\forall k>n$. Then we can choose a diagonal

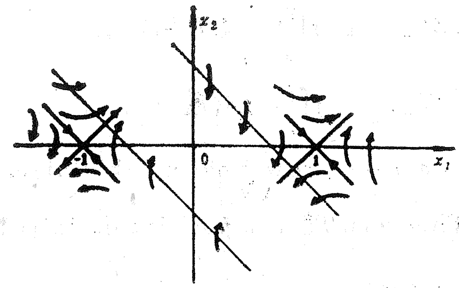

Fig. 1

positive definite symetric matrix Q in (1.2) to make the optimal closed-loop system (2.5) be globally asymptotically stable.

The proof of the Theorem can be found in appendix 1.

From the Theorem it is not difficult to obtain the estimation formula of the optimal transition time and the optimal performance index value. That is, the estimation formula of transition time T_ε reaching the set $\|x\|=\varepsilon$ of the optimal trajectory $x=x^*(t)$ starting from $x_0\in R^n$ is given by

$$T_\varepsilon \le \begin{cases} T_+ - \lambda_+(P_n^*)\ln\left(\dfrac{\lambda_-(P_n^*)}{x_+^\tau P_n^* x_+}\varepsilon^2\right), & x_0\in G_1^{(n)} \\ -\lambda_+(P_n^*)\ln\left(\dfrac{\lambda_-(P_n^*)}{x_0^\tau P_n^* x_0}\varepsilon^2\right), & x_0\in G^{(n)} \\ T_- - \lambda_+(P_n^*)\ln\left(\dfrac{\lambda_-(P_n^*)}{x_-^\tau P_n^* x_-}\varepsilon^2\right), & x_0\in G_2^{(n)} \end{cases}$$

and the optimal performance index value J^* is in the form:

$$J^* = \begin{cases} x_0^\tau P_n^* x_0 + \int_0^{T_+}[M + r^{-1}b^\tau P_n^* x^*(t)]^2 \cdot dt & x_0\in G_1^{(n)} \\ x_0^\tau P_n^* x_0 & x_0\in G^{(n)} \\ x_0^\tau P_n^* x_0 + \int_0^{T_-}[-M + r^{-1}b^\tau P_n^* x^*(t)]^2 \cdot dt, & x_0\in G_2^{(n)} \end{cases}$$

where $T_+ = [-M - r^{-1}b^\tau P_n^* x_0]/M\,(r^{-1}b_1^2 q_{nn})^{1/2}$ $\quad x_0\in G_1^{(n)}$

$$T_- = [-M + r^{-1}b^\tau P_n^* x_0]/M\,(r^{-1}b_1^2 q_{nn})^{1/2} \qquad x_0\in G_2^{(n)}$$

$$x_+ = e^{AT_+}x_0 + M\int_0^{T_+} e^{A(T_+-t)}\, b\cdot dt \qquad x_0 \in G_1^{(n)}$$

$$x_- = e^{AT_-}x_0 - M\int_0^{T_-} e^{A(T_--t)}\, b\cdot dt \qquad x_0 \in G_2^{(n)}$$

q_{nn} is the element on n-th row and n-th column of $Q = \mathrm{diag}[q_{11}\ q_{22}\ \cdots q_{nn}]$ in the Theorem. P_n^* is a positive definite symmetric solution matrix of (2.1) satisfying $b^T P_n^* A = 0_n$, $\lambda_+(P_n^*)$, $\lambda_-(P_n^*)$ is the maximal, minimal eigenvalue of P_n^* respectively.

§3. Simplified Mathematical Model for Controlling a Temperature of Vinylon Solution

Suppose that there is some vinylon solution being heated in a vessel. Let q_i, q_0 be the heat quantity entering and dissipating from the vessel during a unit time respectively. Thus the haat stored in the vessel is $q_i - q_0$. Let T_a be the temperature of vinylon solution and m be the corresponding displacement of the control valve. The target temperature and corresponding displacement is denoted by T_{af}, m_f respectively, $0 < T_a < T_{af}$. Obviously, $q_i = q_i(T_a, m)$, $q_{if} = q_i(T_{af}, m_f)$. Utilizing $Q_1 = M_c \cdot \Delta T_a$, the Equilibrium Law: $dQ_1/dt = q_i - q_0$ and taking the 1-st approximation of $\Delta q_i \triangleq q_i - q_0$ we have:

$$T\frac{d\Delta T_a}{dt} + \Delta T_a \doteq K_0 \Delta q_0 + K_m \Delta m, \tag{3.1}$$

where $T = -M_c \Big/ \left.\frac{\partial q_i}{\partial T_a}\right|_{\substack{T_a = T_{af} \\ m = m_f}} > 0$, $K_m = -\left(\frac{\partial q_i}{\partial m} \Big/ \frac{\partial q_i}{\partial T_a}\right)\Big|_{\substack{T_a = T_{af} \\ m = m_f}}$

$K_0 = -1 \Big/ \left.\frac{\partial q_i}{\partial T_a}\right|_{\substack{T_a = T_f \\ m = m_{fa}}}$, M_c is the total heat capacity, $\Delta T_a = T_a - T_{af}$, $\Delta m = m - m_f$, Δq_0 is considered as a constant disturbance, that is, $d\Delta q_0/dt = 0$, $|\Delta m| \leq \alpha_a$ (some positive constant), $|\Delta q_0| \ll \alpha$.

Introducing the notations: $x_1 = \Delta T_a$, $x_2 = \Delta q_0$, $u = \Delta m$, $a_1 = 1/T > 0$, $a_3 = K_0/T$, $b_1 = K_m/T$, $y_1 = z_1 = x_1$, we obtain a simplified mathematical model of a temperature control process of heated vinylon solution:

$$\dot{x}_1 = -a_1 x_1 + a_3 x_2 + b_1 u,$$

$$\dot{x}_2 = 0. \tag{3.2}$$

In the next paragraph we will find out the control law $u = \Delta m$ in(3.2) which makes the temperature of vinylon solution attain the target value rapidly without overshoot and oscillation.

§ 4. Solution of the Temperature Control Problem

We are going to find the control law of $u = \Delta m$ in the following way: Design a "servo compensator" to cancel out the disturbance and design a "optimal regulator" to make the system be internally stable. According to "Internal Model Principle" the servo compensator cancelling out disturbance x_2 is

$$\dot{x}_{c2} = x_1 \tag{4.1}$$

and when $a_3 = 0$, we can choose u such that the corresponding closed-loop system becomes globally stable [6]. Now, let us consider the following constrained LQ problem:

$$\begin{cases} \dot{x}_{c2} = x_1 \\ \dot{x}_1 = -a_1 x_1 + b_1 u \end{cases} \tag{4.2}$$

$$J[u] = \int_0^\infty [x^T Q_2 x + r u^2] \cdot dt \tag{4.3}$$

$$|u| \le \alpha, \tag{4.4}$$

where $x^T = [x_{c2},\ x_1]$, $Q_2 = \mathrm{diag}[a_1 q_{22},\ q_{22}]$, $r > 0$, $q_{22} > 0$.

Based on the results obtained in §2 we have the nonlinear optimal closed-loop system

$$\begin{cases} \dot{x}_{c2} = x_1 \\ \dot{x}_1 = -a_1 x_1 + b_1 u^*(x_{c2},\ x_1) \end{cases} \tag{4.5}$$

which is globally asymptotically stable, where $u^*(x_{c2},\ x_1)$ is in the following form;

$$u^*(x_{c2},\ x_1) = \begin{cases} \alpha, & r^{-1} b_1 (p^*_{12} x_{c2} + p^*_{22} x_1) < -\alpha \\ -r^{-1} b_1 (p^*_{12} x_{c2} + p^*_{22} x_1), & 1 r^{-1} b_1 (p^*_{12} x_{c2} + p^*_{22} x_1) 1 < \alpha \\ -\alpha, & r^{-1} b_1 (p^*_{12} x_{c2} + p^*_{22} x_1) > \alpha \end{cases}$$

$p^*_{12} = a_1 (r b_1^{-2} q_{22})^{1/2}$, $p^*_{22} = (r b_1^{-2} q_{22})^{1/2}$, the phas ediagram of (4.5) is as follows:

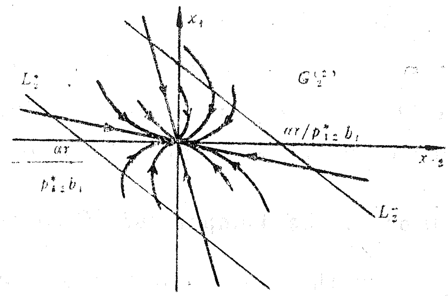

Fig. 2

From the phase diagram of (4.5) it is clear that when the point $(x_{c20}, x_{10}) \in G_2^{(2)}$ (or $\in G_1^{(2)}$), trajectory of (4.5) starting from this point will reach L_2^- (or L_1^+) quickly without overshoot and oscillation. Moreover, the optimal closed-loop system with a constant disturbance x_2

$$\begin{cases} \dot{x}_1 = -a_1 x_1 + a_3 x_2 + b_1 u^*(x_{c2}, x_1) \\ \dot{x}_{c2} = x_1 \\ \dot{x}_2 = 0 \end{cases}$$

possesses the following property: $\lim x_1(t) = 0$. The following graph is the time-temperature curve in the real process of heating vinylon solution in the vinylon fibre mill.

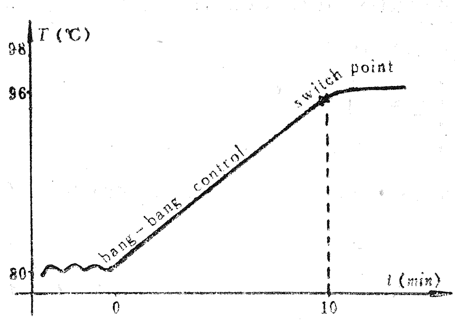

Fig. 3

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Appendix 1. Proof of the Theorem

We prove that under condition (2.7) it is possible to choose a positive definite symmetric matrix Q in (1.2) so that the following simultaneous equations

$$PA + A^{\tau}P + Q - r^{-1}Pbb^{\tau}P = O_{n\times n} \qquad (A)_1$$

$$b^{\tau}PA = O_n^{\tau} \qquad (A)_2$$

have a unique positive definite symmetric matrix solution.

Without loss of generality we suppose that (A,b) is in controllable canonical form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & & -a_3 & \cdots & -a_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_1 \end{pmatrix}, \quad b_1 \neq 0$$

Let $Q = \mathrm{diag}[q_{11}\, q_{11} \cdots q_{nn}]$ and $q_{ii} > 0 (i = 1, 2, \cdots, n)$. We choose q_{ii} such that a positive definite symmetric matrix $P = (p_{ij})$ of $(A)_1$ satisfies $(A)_2$. It follows from $(A)_1$ and $(A)_2$ that

$$-r^{-1}b^{\tau}Pbb^{\tau}Pb+b^{\tau}Qb=0,\quad \text{or}\ p_{nn}=(rb_1^{-2}q_{nn})^{1/2}$$
$$p_{1n}=a_1p_{nn},\ p_{2n}=a_2p_{nn},\ \cdots,\ p_{n-1n}=a_{n-1}p_{nn} \qquad (A)_3$$

By simple computation it is not difficult to obtain:

$$p_{1n}=(rb_1^{-2}q_{11})^{1/2},\ q_{11}=a_1^2q_{nn},\ p_{11}=(a_1+r^{-1}b_1^2p_{12})p_{1n},\ q_{22}=(a_2^2-2a_1a_3)rb_1^{-2}q_{nn},\ q_{hh}=(a_h^2-2a_{h-1}a_{h+1}+\cdots+(-1)^{h-1}2a_1a_{2h-1})rb_1^{-2}q_{nn} \qquad (A)_4^h$$

for $h=2,\ 3,\ \cdots,\ n$ and $a_n=1,\ a_h=0(h>n,\ h<0)$.

Based on condition (2.7) of the Theorem, the q_{hh} defined by $(A)_4^h$ is positive provided $q_{nn}>0$, therefore, $Q=\text{diag}\,[q_{11}\ q_{22}\cdots q_{nn}]$ is a positive definite symmetric matrix. Having those q_{hh} $(h=1,\ 2,\ \cdots,\ n)$ we can find out from $(A)_1$ p_{ij}, $i,\ j=1,\ 2,\ \cdots,\ n$. By the condition of complete controllability of $(A,\ b)$ and the fact of $p_{11}>0$, it can be proved that the solution $P_n^*=(p_{ij})$ of $(A)_1$ found out in this way is unique positive definite symmetric solution of $(A)_1$ and $(A)_2$.

Since $b^{\tau}P_n^*A=O_n^{\tau}$, it is easy to see:

$$\frac{d}{dt}(r^{-1}b^{\tau}P_n^*x)=\begin{cases} r^{-1}Mb^{\tau}P_n^*b=r^{-1}b_1^2Mp_{nn}^*, & x\in G_1^{(n)} \\ -r^{-1}Mb^{\tau}P_n^*b=-r^{-1}b_1^2Mp_{nn}^*, & x\in G_2^{(n)} \end{cases} \qquad (A)_5$$

Relationship $(A)_5$ shows the trajectory of the optimal closed-loop system starting from a point belonging to $(G_1^{(n)})$ $(G_2^{(n)})$ will reach the boundary of $G^{(n)}$ after some finite time and will go into the interior of $G^{(n)}$. In the interior of $G^{(n)}$ the optimal closed-loop system is in the form:

$$\dot{x}=(A-r^{-1}bb^{\tau}P_n^*)x \qquad (A)_6$$

Since $A-r^{-1}bb^{\tau}P_n^*$ is stable matrix, the all trajectories of $(A)_6$ starting from an interior point of $G^{(n)}$ will tend to $x=0_n$. Thus the optimal closed-loop system is globally asymptotically stable.

受约束的LQ问题与维尼纶緐液温度控制

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摘 要

文中考察受约束的线性二次最优控制问题，给出该问题可解的充分必要条件以及最优综合函数的分析形式.文中还给出最优闭环为全局渐近稳定的充分条件.文中给出的例子说明，最优闭环系统不一定是全局渐近稳定的. 本文的结果为某工厂的维尼纶丝液温度控制的原理方案提供了合理的、满意的解释. 并为实际温控系统的快速性能、抗干扰性能、无超调及无振荡性质等等提供了理论依据.

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