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Controllability, Reachability, and Strong Connectivity of Discrete-Time Systems with Control Constraints

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Abstract

In this paper we study the controllability, reachability and strong connectivity of discrete-time linear systems with control constraints. Necessary and sufficient conditions are given to test these properties. Their interrelation is revealed.

Introduction

The study of discrete-time systems of the form

$$X_{k+1} = AX_k + BU_k, \quad k = 0, 1 \cdots.$$

where $X_k \in \mathbf{R}^n$, A is a real constant $n\times n$ matrix, and B is an $n\times m$ real constant matrix, has received considerable attention during the last twenty years. Kalman [1] introduced the concept of complete controllability, and derived an elegant algebraic test for this property when (for the above system) $U_k \in \mathbf{R}^m$, $k = 0, 1 \cdots$.

Further he [2] advanced the concepts of complete reachability and strong connectivity and discovered their equivalency when $\det A \neq 0$, $U_k \in \mathbf{R}^m$, $k=0, 1\cdots$. Later some people considered these problems in the case of control constraints. The best results are in Ref [3].

In this paper we examine the reachability and strong connectivity of the system (A, B) with control constraints, and indicate their relationship and difference.

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Preliminaries

The norms of input can be introduced [3,4]

$$\|U_0U_1\cdots U_{k-1}\|_p = \left\| \begin{pmatrix} U_0 \\ U_1 \\ \vdots \\ U_{k-1} \end{pmatrix} \right\|_p$$

for a control sequence $U_0\ U_1\ \cdots\ U_{k-1}$ of length k, where $p\in\{1,2,\infty\}$. They represent control fuel, control energy and controller opening, respectively. The sets of admissible control can be defined

$$\Omega_p(k,l) \triangleq \{U_0\cdots U_{k-1} : U_i\in \mathbf{R}^m \text{ and } \|U_0\cdots U_{k-1}\|_p \le l\},\ k=0,1,\cdots$$

where l is the resources maximum.

Definition 2.1 The system (A,B) with control resources constraint is said to be completely controllable if for every state X there exists an admissible control sequence $U_0\cdots U_{k-1}\in\Omega_p(k,l)$ of length $k=k(X)$ such that the state can be forced from X to zero in k interation times.

For simplicity, we denote it by

$$(A,B)\in \text{CCS}\begin{cases}\text{CCF} & \text{when } p=1,\\ \text{CCE} & p=2,\\ \text{CCO} & p=\infty.\end{cases}$$

Definition 2.2 The system (A,B) with control resources constraint is said to be completely reachable if for every state X there exists an admissible control sequence $U_0\cdots U_{k-1}\in\Omega_p(k,l)$ of length $k=k(X)$ such that the state can be forced from origin to X in k iteration times.

For simplicity, we denote it by

$$(A,B)\in \text{CRS}\begin{cases}\text{CRF} & \text{when } p=1,\\ \text{CRE} & p=2,\\ \text{CRO} & p=\infty.\end{cases}$$

Definition 2.3 The system (A,B) with control resources constraint is said to be strongly connected if for any both states $\widetilde{X}$, $\widehat{X}$ there exists an admissible control sequence $U_0\cdots U_{k-1}\in\Omega_p(k,l)$ of length $k=k(\widetilde{X},\widehat{X})$ such that the state can be forced frcm $\widetilde{X}$ to $\widehat{X}$ in k

iteration times.

For simplicity, we denote it by

$$(A,B) \in \text{SCS}\begin{cases}\text{SCF} & \text{when } p=1,\\ \text{SCE} & p=2,\\ \text{SCO} & p=\infty.\end{cases}$$

The notation (A,B) $\in$ CC means that (A,B) is completely controllable (Kalman meaning). Kalman [1] found that (A,B)$\in$CC iff rank $W_n=n$ where $W_n=[A^{n-1}B,\cdots AB,B]$. Recent results [3] are given as the following lemma.

Lemma 2.3

1) (A,B)$\in$CCF iff rank $W_n=n$ and $\max\limits_i |\lambda_i(A)|<1$;

2) (A,B)$\in$CCE iff rank $W_n=n$ and $\max\limits_i |\lambda_i(A)|\le 1$;

3) (A,B)$\in$CCO iff rank $W_n=n$ and $\max\limits_i |\lambda_i(A)|\le 1$,

where $\lambda_i(A)$ represent the eigenvalues of A.

Remark The conclusions of Lemma 2.3 do not depend on the magnitude of control resources, i. e., having nothing to do with l.

Reachability

Given a initial state X_0 and a control sequence $U_0\cdots U_{k-1}$ of length k, then the state can be forced to the state X_k in k iteration times where

$$X_k=A^kX_0+A^{k-1}BU_0+\cdots+ABU_{k-2}+BU_{k-1}$$

From this the controllable set of (A,B) with control constraint in k iteration times is given as

$$\mathbf{C}^k_p(A,B)\triangleq\{X_0:X_0=-[A^{-1}B,A^{-2}B,\cdots A^{-k}B]\begin{pmatrix}U_0\\ \vdots\\ U_{k-1}\end{pmatrix},U_0\cdots U_{k-1}\in\Omega_p(k,l)\},$$

the reachable set of (A,B) with control constraint in k iteration times is given as

$$\mathbf{R}^k_p(A,B)\triangleq\{X:X=[A^{k-1}B,\cdots,AB,B]\begin{pmatrix}U_0\\ \vdots\\ U_{k-1}\end{pmatrix},\ U_0\cdots U_{k-1}\in\Omega_p(k,l)\}.$$

the system $(A^{-1},-A^{-1}B)$ is called to be an inverse of the system (A,B) when $\det A\ne 0$. Obviously, (A,B) and $(A^{-1},-A^{-1}B)$ are inverse each other.

Lemma 3.1 $\mathbf{C}^k_p(A,B)=\mathbf{R}^k_p(A^{-1},-A^{-1}B)$;

$\mathbf{R}^k_p(A,B)=\mathbf{C}^k_p(A^{-1},-A^{-1}B)$.

Proof Note the following relations

$$-[A^{-1}B,A^{-2}B,\cdots,A^{-k}B]\begin{pmatrix}U_0\\ \vdots\\ U_{k-1}\end{pmatrix}$$

$$=[(A^{-1})^{k-1}(-A^{-1}B),\cdots,A^{-1}(-A^{-1}B),\ (-A^{-1}B)]\begin{bmatrix}U_{k-1}\\ \vdots\\ U_0\end{bmatrix},$$

$$[A^{k-1}B,\cdots,AB,B]\begin{pmatrix}U_0\\ \vdots\\ U_{k-1}\end{pmatrix}$$

$$=-[(A^{-1})^{-1}(-A^{-1}B),(A^{-1})^{-2}(-A^{-1}B),\cdots,\ (A^{-1})^{-k}(A^{-1}B)]\begin{bmatrix}U_{k-1}\\ \vdots\\ U_0\end{bmatrix},$$

$$\left\|\begin{pmatrix}U_0\\ \vdots\\ U_{k-1}\end{pmatrix}\right\|_p=\left\|\begin{pmatrix}U_{k-1}\\ \vdots\\ U_0\end{pmatrix}\right\|_p,\quad p\in\{1,2,\infty\},$$

We can assert that this lemma is true.

Now we state our main result for this section – the necessary and sufficient condition for complete reachability of (A,B) with control constraint.

Theorem 3.2

1) $(A,B)\in CRF$ iff rank $W_n=n$ and $\min_i|\lambda_i(A)|>1$;

2) $(A,B)\in CRE$ iff rank $W_n=n$ and $\min_i|\lambda_i(A)|\geq 1$;

3) $(A,B)\in CRO$ iff rank $W_n=n$ and $\min_i|\lambda_i(A)|\geq 1$.

Proof $(A,B)\in CRS$ iff $\bigcup_{k=0}^{\infty}\mathbf{R}^k_{p(A,B)}=\mathbf{R}^n$, by Lemma 3.1 that is

$(A,B)\in CRS$ iff $\bigcup_{k=0}^{\infty}\mathbf{C}^k_{p(A^{-1},-A^{-1}B)}=\mathbf{R}^n$ i. e., $(A^{-1},-A^{-1}B)\in CCS$.

By Lemma 2.3,

recall that

$$\lambda_i(A^{-1}) = 1/\lambda_i(A)$$

and rank $[(A^{-1})^{n-1}(-A^{-1}B),\cdots A^{-1}(-A^{-1}B),(-A^{-1}B)]$

$$= \text{rank } \{A^{-n}[B,\ AB,\ \cdots A^{n-1}B]\}$$

$$= \text{rank } [A^{n-1}B,\ \cdots AB,\ B],$$

we have immediately finished the proof.

Remark

(i) The conclusions of Themorem 3.2 do not depend on the magnitude of control resources, i.e., having nothing to do with l.

(ii) It is easy to verify that the system (A,B) $\in$ CRS when A is singular. (omitted here)

strong connectivity

First we give the following lemma, then draw Theorem 4.2.

Lemma 4.1

(A,B) $\in$ SCS iff (A,B) $\in$ CCS and (A,B) $\in$ CRS.

Proof The "ONLY IF" part is very obvious. We will only show the "IF" Part.

Given any both states $\widetilde{X}$, $\hat{X}$, by the assumption (A,B) $\in$ CCS there exists a iteration time k_1 and an input sequence $\widetilde{U}_0 \cdots \widetilde{U}_{k_1-1} \in \Omega_p(k_1,\alpha_p l)$ such that it drive the event* $(o,\widetilde{X})$ to (k_1, o_x), by the assumption (A,B) $\in$ CRS, there exists a iteration time k_2 and an input sequence $\hat{U}_0 \cdots \hat{U}_{k_1-1} \in \Omega_p(k_2,\alpha_p l)$ such that it drives the event (k_1, o_x) to $(k_1+k_2, \hat{X})$ where

$$\alpha_p : \begin{cases} \alpha_1 = 1/2 & p = 1; \\ \alpha_2 = 1/\sqrt{2} & p = 2; \\ \alpha_\infty = 1 & p = \infty. \end{cases}$$

Link up the two input sequences, we obtain a new input sequence of length k_1+k_2

$$U_0 \cdots U_{k_1-1} U_{k_1} \cdots U_{k_1+k_2-1} = \widetilde{U}_0 \cdots \widetilde{U}_{k_1-1} \hat{U}_0 \cdots \hat{U}_{k_2-1}$$

It drives the event $(o,\widetilde{X})$ to $(k_1+k_2, \hat{X})$ in (k_1+k_2) iteration

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* Event (t,X) is used as a state at time t, see Ref [1,4].

times, at the same time we can show $U_0 \cdots U_{k_1-1} U_{k_1} \cdots U_{k_1+k_2-1} \in$ $\Omega_p(k_1+k_2, l)$. In fact, $\|U_0 \cdots U_{k_1-1} U_{k_1} \cdots U_{k_1+k_2-1}\|_1 = \|\tilde{U}_0 \cdots U_{k_1-1}\|_1 + \|\hat{U}_0 \cdots \hat{U}_{k_2-1}\|_1 \leq 2\alpha_1 l = l$;

$$\|U_0 \cdots U_{k_1-1} U_{k_1} \cdots U_{k_1+k_2-1}\|_2 = \{\|\tilde{U}_0 \cdots \tilde{U}_{k-1}\|_2^2 + \|\hat{U}_0 \cdots \hat{U}_{k_2-1}\|_2^2\}^{1/2}$$

$$\leq (2\alpha_2^2 l^2)^{1/2} = l;$$

$$\|U_0 \cdots U_{k_1-1} U_{k_1} \cdots U_{k_1+k_2-1}\|_\infty = \max\{\|\tilde{U}_0 \cdots \tilde{U}_{k_1-1}\|_\infty, \|\hat{U}_0 \cdots \hat{U}_{k_2-1}\|_\infty\}$$

$$\leq \alpha_\infty l = l,$$

From lemma 4.1 above we deduce our main result for this section

Theorem 4.2

1) The case (A,B) ∈ SCF has never been possible;

2) (A,B) ∈ SCE iff rank $W_n = n$ and $|\lambda_i(A)| = 1$ for all $\lambda_i(A)$;

3) (A,B) ∈ SCO iff rank $W_n = n$ and $|\lambda_i(A)| = 1$ for all $\lambda_i(A)$;

where $\lambda_i(A)$ represent the eigenvalues of A.

Remark The conclusions of this theorem do not depend on the magnitude of control resources, i. e., having nothing to do with l.

Conclusion

The controllability, reachability, and strong connectivity of discrete-time systems with control constraints are three profound concepts. Besides the controllability matrix, they are connected with system pole location, respectively. Necessary and sufficient conditions are given to test these properties. They may be significant in practice.

References

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带有控制约束的离散时间系统的可控性可达性强可连性

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摘 要

本文研究带有控制约束的离散时间线性系统的可控性、可达性及强可连性。给出判别这些性质的充分必要条件，并指出它们之间的关系。