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The Exponential Stability of Nonautonomous Difference Equations for Adaptive Control

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Abstract

In discrete time adaptive control problems the following equation has come up

$$\begin{bmatrix} e(k+1) \\ \phi(k+1) \end{bmatrix} = \begin{bmatrix} A-\dfrac{mbw^T(k)Qw(k)h^T}{1+mpw^T(k)Qw(k)} & \dfrac{bw^T(k)}{1+mpw^T(k)Qw(k)} \\ -\dfrac{Qw(k)h^T}{1+mpw^T(k)Qw(k)} & I-\dfrac{pw(k)Qw^T(k)}{1+mpw^T(k)Qw(k)} \end{bmatrix} \begin{bmatrix} e(k) \\ \phi(k) \end{bmatrix} \quad (1)$$

Where, $p+h^T(Iz-A)^{-1}b$ is a strictly positive real transfer function, $w(k)$ is the function of $e(k)$ and $\phi(k)$. The stability analysis of the system is presented in this paper. It is concluded that if $w(k)$ is sufficienly rich, the system is exponentially stable.

Key words——Discrete time; Adaptive control; Exponentially stable.

1. Introduion

In 1977, Morgan and Narendra[1] presented their remarkable paper on the stability analysis of a nonautonomous differential equation $\dot{x} = [A+B(t)]x$ with a skew symmetric matrix $B(t)$. In 1980, Bitmead and Anderson[2] gave an exponential stability proof for a discrete time-varying free linear system $x(k+1)=F(k)x(k)$. In most discrete adaptive control problems, the error system described by equation (1) is nonlinear in nature therefore, the above results can not be directly applicable for discrete adaptive control system analysis.

This paper presents an exponential stability proof for the system described by equation (1) which actually is the counterpart of

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Morgan and Narendra's results in discrete time case.

In Narendra's discrete time model reference adaptive control (MRAC) scheme the following equations were derived[3]

$$e(k+1)=Ae(k)+bv(k), \tag{2}$$

$$e_1(k)=h^Te(k)+pv(k), \tag{3}$$

$$v(k)=\phi^T(k)w(k)-mw^T(k)Qw(k)e_1(k), \quad m\geq 1/2 \tag{4}$$

$$\phi(k+1)=\phi(k)-Qe_1(k)w(k), \qquad Q=Q^T>0 \tag{5}$$

It is not difficult to bring above equations to equation (1). For convenience of discussion the set of difference equations (2)—(5) is investigated. If the system described by this set of equations is exponentially stable, so is the system described by equation(1).

2. Stability Analysis

From the discrete version of Kalman-Yacubovich lemma, if $p+h^T(Iz-A)^{-1}b$ is strictly positive real then there exists a matrix $P=P^T>0$ and a vector q such that

$$A^TPA-P=-qq^T-\varepsilon L,$$

$$A^TPb=h/2+rq,$$

$$p-b^TPb=r^2.$$

for some $L=L^T>0$ and scalars $\varepsilon>0$, $r>0$.

Choose a Lyapunov function candidate $V(k)$

$$V(k)=2e^T(k)Pe(k)+\phi^T(k)Q^{-1}\phi(k)$$

Then the forward difference of the Lyapunov function candidate along the trajectory of equations (2)—(5) can be written as

$$\Delta V(k)=V(k+1)-V(k)=-2[e^T(k)q-rv(k)]^2-2\varepsilon e^T(k)Le(k)$$
$$+(1-2m)w^T(k)Qw^T(k)\,Qw(k)e_1^2(k) \tag{6}$$

It is seen that $\Delta V(k)$ is negative semidefinite, i. e., the Lyapunov function $V(k)$ is a nonincreasing function of time. Hence the system is stable and $e(k)$ and $\phi(k)$ are bounded if $e(0)$ and $\phi(0)$ are bounded. Furthermore it follows that

$$e(k)\to 0; \qquad \phi(k)\to 0; \qquad e_1(k)\to o, \text{ as } k\to\infty.$$

In adaptive control problems the elements of vector $w(k)$ are the filtered signals of the $e(k)$ and $\phi(k)$, therefore $w(k)$ is also uniformly bounded. The aforementioned stability results could be found in ref. 3. Then we start with introducing a definition of sufficent richness of a vector signal.

Definition 1 A bounded n-vector input sequence $\{u(k)\}$ is said to be sufficiently rich if there exists an integer I_0 and ε_0. such that for any given time instant $k_1>k_0$. for any constant nonzero n-vector d there is a time instant $k\ \varepsilon[k_1,\ k_1+I_0]$ such that

$$|d^T u(k)|\geqslant\varepsilon_0 \tag{7}$$

Next we shall present two lemmas which are necessary for obtaining the exponential stability conclusion for system(2)—(5).

Lemma 1 Consider a system described by equations (2)—(5), let $\delta>0$, $\varepsilon_1>0$ be given. Suppose the transfer function $p+h^T(Iz-A)^{-1}b$ is strictly positive real, $w(k)$ is uniformly bounded and sufficiently rich and $x^T=[e^T,\ \phi^T]$ is a solution of this system with $\|x(k_1)\|\leqslant\varepsilon_1$. Assume there is an integer I such that $\|\phi(k)\|\geqslant\delta$ for all $k\varepsilon[k_1,k_1+I]$, then there exists a $k_2\varepsilon[k_1,\ k_1+I]$ and $\varepsilon>0$ such that $\|e(k_2)\|\geqslant\varepsilon$.

Proof 1) From (3) one has

$$v(k)=\frac{e_1(k)-h^Te(k)}{p} \tag{8}$$

Eliminating $v(k)$ in (2) yields

$$e(k+1)=(A-bh^T/p)e(k)+be_1(k)/p \tag{9}$$

It follows from (9) that

$$\|b/p\|\|e_1(k)\|=\|be_1(k)/p\|\leqslant\|e(k+1)\|+\|A-bh^T/p\|\|e(k)\|$$

or

$$\|e_1(k)\|\leqslant[\|e(k+1)\|+\|A-bh^T/p\|\|e(k)\|]/\|b/p\| \tag{10}$$

Thus, if
$$\|e(k)\|\leqslant\varepsilon,\quad\forall k\varepsilon[k_1,\ k_1+I] \tag{11}$$
one has
$$\|e_1(k)\|\leqslant\beta\varepsilon,\quad\forall k\varepsilon[k_1,\ k_1+I-1] \tag{12}$$

where, β is a positive constant. An inequality comes from (5)

$$\|\phi(k_1+J)-\phi(k_1)\|\leqslant\|Q\|\sum_{i=0}^{J-1}\|e_1(k_1+i)\|\|w(k_1+i)\|$$

$$\leqslant\lambda_{\max}\{Q\}w_mI\beta\varepsilon\triangleq\gamma\varepsilon,\quad\forall J\leqslant I \tag{13}$$

Where γ is a positive constant, w_m is the maximun value of $\|w(k)\|$ which exists because of the boundedness of $w(k)$.

2) An inequality follows from (2) and (4)

$$\|e(k+1)\|\geqslant\|b\phi^T(k)w(k)\|-\|mbw^T(k)Qw(k)e_1(k)-Ae(k)\| \tag{14}$$

Now, we shall outline how to prove lemma 1. First, we make an assumption that $\|e(k)\|\leqslant\varepsilon$ for all $k\varepsilon[k_1,k_1+I]$ and calculate three terms in the right hand side of inequality (14). The result will lead to a contradiction to the assumption. Then the lemma holds.

$$\text{Let} \quad \varepsilon' = \varepsilon_0'/2\|b\|w_m \text{ and } I = I_0 + 1 \tag{15}$$

From (13) and (11) there is a corresponding value θ such that if

$$\|e(k)\| \leqslant \theta, \qquad \forall k \varepsilon [k_1, k_1 + I]$$

then, $\|\phi(k_1 + J) - \phi(k_1)\| \leqslant \varepsilon'$, $\forall J \leqslant I$

Where, ε_0' and I_0 arise from the sufficient richness condition on $w(k)$, i.e., there is a $k\varepsilon[k_1, k_1 + I_0]$ such that

$$\|bd^T w(k)\| = \|b\|\|d^T w(k)\| \geqslant \|b\|\varepsilon_0 \triangleq \varepsilon_0' \tag{16}$$

Now define an unit vector $d = \phi(k_1)/\|\phi(k_1)\|$, one has

$$\|b[d^T\|\phi^T(k_1)\| - \phi^T(k)]w(k)\| = \|b[\phi^T(k_1) - \phi^T(k)]w(k)\|$$

$$\leqslant \|b\|w_m\varepsilon' = \delta\varepsilon_0'/2, \quad \forall k \varepsilon [k_1, k_1 + I]$$

or

$$\|bd^T\|\phi^T(k_1)\|w(k) - b\phi^T(k)w(k)\| \leqslant \delta\varepsilon_0'/2$$

$$\|\phi(k_1)\|\|bd^T w(k)\| - \|b\phi^T(k)w(k)\| \leqslant \delta\varepsilon_0'/2$$

$$\|b\phi^T(k)w(k)\| \geqslant \delta\varepsilon_0' - \delta\varepsilon_0'/2 = \delta\varepsilon_0'/2 \text{ for some } k\varepsilon [k_1, k_1 + I] \tag{17}$$

Choose $\varepsilon = \min\{\delta\varepsilon_0'/8, \ \delta\varepsilon_0'/8m\|b\|\lambda_{\max}\{Q\}w_m^2\beta, \ \theta\}$. It follows from (11) and (12) that

$$\|mbw^T(k)Qw(k) - Ae(k)\| \leqslant \|mbw^T(k)Qw(k)e_1(k)\| + \|Ae(k)\|$$

$$\leqslant m\|b\|\lambda_{\max}\{Q\}w_m^2\beta\varepsilon + \varepsilon \leqslant \delta\varepsilon_0'/4, \quad \forall k\varepsilon [k_1, k_1 + I - 1] \tag{18}$$

It should be pointed out that here we suppose $\|A\| \leqslant 1$. For $\|A\| > 1$, by replacing $\delta\varepsilon_0'/8$ as $\delta\varepsilon_0'/8\|A\|$ in the expression of ε, the same conclusion could be obtained. Substuting (17) and (18) into (14) yields

$$\|e(k+1)\| \geqslant \delta\varepsilon_0'/2 - \delta\varepsilon_0'/4 = \delta\varepsilon_0'/4 > \varepsilon \qquad \text{for some } k \in [k_1, k_1 + I_0]$$

This is contradictive to the assumption made before, therefore, the lemma follows. Actually this lemma says that if $\phi(k)$ is permanently large then $e(k)$ is periodically large.

Lemma 2 Consider a system described by equations (2)—(5). Let $\delta > 0$ and $\varepsilon_1 > 0$ be given. Suppose $p + h^T(Iz - A)^{-1}b$ is a strictly positive real transfer function, $w(k)$ is uniformly bounded and sufficiently rich and $x^T=[e^T, \phi^T]$ is a solution of this system with $\|x(k_1)\| \leq \varepsilon_1$. Then there exists an integer $I=I(\varepsilon_1,\delta)$ and some $k_2 \varepsilon [k_1, k_1+I]$ such than $\|\phi(k)_2\| \leq \delta$.

Proof From equation (6), one has

$$\Delta V(k) = -2[e^T(k)q - rv(k)]^2 - 2\varepsilon e^T(k)Le(k) + (1-2m)w^T(k)Qw(k)e_1^2(k)$$

$$\leq -2\varepsilon\lambda_{min}\{L\}\|e(k)\|^2 \qquad (19)$$

Let $\varepsilon_1 > \varepsilon_2 > 0$, if $\|x(k_1)\| \leq \varepsilon_1$, $\|e(k)\| \geq \varepsilon_2$, then it follows from (19)

$$V(k) = 2e^T(k)Pe(k) + \phi^T(k)Q\phi(k) \leq \beta\|x(k)\|^2 \leq \beta\varepsilon_1^2 \qquad \forall k > k_1$$

for some positive constant β. Thus one has an inequality

$$V(k)/\|\Delta V(k)\| \leq \beta\varepsilon_1^2/2\varepsilon\lambda_{min}\{L\}\varepsilon_2^2 \qquad (20)$$

This inequality implies that there is an uniform limit on the amount of time, a solution $x(k)$, starting inside the ε_1-ball and $\|e(k)\|$ can remain outside the ε_2-ball. This also implies that given $\varepsilon_1 > \varepsilon_2 > 0$, there is a $T>0$ such that if $x(k)$ is a solution of the system with $\|x(k)\| \leq \varepsilon_1$, then, there is a $k_2 \varepsilon [k_1, k_1+T]$ such that $\|e(k_2)\| \leq \varepsilon_2$. The conclusion can then be obtained that $e(k)$ tends to zero. By the above comments and lemma 1, let $\delta>0$, then $\|\phi(k)\| \geq \delta$ implies that there is an $\varepsilon>0$ such that $\|e(k)\|$ is repeatedly both less than $\varepsilon/2$ and greater than ε if we choose $\varepsilon_1=\varepsilon$, $\varepsilon_2=\varepsilon/2$. This eventually leads to a contradiction with the above comments. Since all these results are uniform, we conclude that $\|\phi(k)\| \leq \delta$ repeatedly.

Theorem Consider a system described by equations (2)—(5) If $p+h^T(Iz-A)^{-1}b$ is a strictly positive real transfer function, $w(k)$ is uniformly bounded and sufficiently rich, then the system is exponentially stable in the large.

Proof Form a Lyapunov function candidate

$$V(k) = 2e^T(k)Pe(k) + \phi^T(k)Q^{-1}\phi(k) \qquad (21)$$

with

$$c_1 e^T(k)e(k) \leq 2e^T(k)Pe(k) \leq c_2 e^T(k)e(k) \qquad (22)$$

$$c_3\phi^T(k)\phi(k) \leq \phi^T(k)Q^{-1}\phi(k) \leq c_4\phi^T(k)\phi(k) \qquad (23)$$

Where, c_1, c_2, c_3 and c_4 are positive constants. Now, we shall show that given $\varepsilon_1 > \varepsilon_2 > 0$, there is a θ with $1>\theta>0$ and an integer $M>0$ such that if $\varepsilon_2 \leq V(k) < \varepsilon_1$, $\forall\ k \varepsilon [k_1, k_1+M]$ then there is a $k_3 \varepsilon [k_1, k_1+M]$ such that $V(k_3) \leq \theta V(k_1)$.

From Lemma 2 there is a $k_2 \varepsilon [k_1, k_1+I]$ such that $\|\phi(k_2)\| \leqslant \delta$ for a given $\delta > 0$. It turns out that

$$V(k_2) = 2e^T(k_2)Pe(k_2) + \phi^T(k_2)Q^{-1}(k_2) \leqslant c_2 \|e(k_2)\|^2 + c_4\delta^2 \tag{24}$$

Let $\delta = \sqrt{c_0\varepsilon_2}$. It follows from (24) that

$$\|e(k_2)\|^2 \geqslant (1 - c_4c_0)V(k_2)/c_2 \tag{25}$$

From (6), one has

$$V(k_2) - V(k_2+1) = 2[e^T(k_2)q - rv(k_2)]^2 + 2\varepsilon e^T(k_2)Le(k_2)$$
$$+ 2(m - 0.5)w^T(k_2)Qw(k_2)e_1^2(k_2)$$
$$\geqslant 2\varepsilon\lambda_{min}\{L\}(1 - c_4c_0)v(k_2)/c_2$$

This leads to

$$V(k_2+1) \leqslant [1 - 2\varepsilon\lambda_{min}\{L\}(1 - c_4c_0)/c_2]V(k_2)$$

It is always possible to choose $c_0 > 0$ such that $0 < 1 - 2\varepsilon\lambda_{min}\{L\}(1 - c_4c_0)/c_2 < 1$. Let $k_3 = k_2 + 1$ and $\theta = 1 - 2\varepsilon\lambda_{min}\{L\}(1 - c_4c_0)/c_2$. It turns out that $V(k_3) \leqslant \theta V(k_2) \leqslant \theta V(k_1)$

Inequality (26) implies that quadratic Lyapunov function $V(k)$ is exponentially convergent to zero. However exponential convergence of $V(k)$ is a necessary and sufficient condition for exponential stability of the system when matrices P and Q are positive definite.

3. Conclusion

The exponential stability of Narendra's discrete adaptive control system described by equations (2)—(5) is proved under certain assumptions. The conclusion is useful to analyse the robustness of Narendra's discrete adaptive control systems when those systems are corrupted by external disturbances, measurement noise and modelling errors.

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自适应控制的非自治差分方程的指数稳定性

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摘 要

在某些离散自适应控制问题中，会遇到如下非自治差分方程：

$$\begin{bmatrix} e(k+1) \\ \phi(k+1) \end{bmatrix} = \begin{bmatrix} A - \dfrac{mbw^T(k)Qw(k)h^T}{1+mpw^T(k)Qw(k)} & \dfrac{bw^T(k)}{1+mpw^T(k)Qw(k)} \\ -\dfrac{Qw(k)h}{1+mpw^T(k)Qw(k)} & I - \dfrac{pw(k)Qw^T(k)}{1+mpw^T(k)Qw(k)} \end{bmatrix} \begin{bmatrix} e(k) \\ \phi(k) \end{bmatrix}$$

其中，$p+h^T(Iz-A)^{-1}b$ 是严格正实传递函数，$w(k)$ 是 $e(k)$ 和 $\phi(k)$ 的函数。本文给出了这类系统的稳定性分析。结论是：当$w(k)$充分丰富时，该系统是指数稳定的。

关键词： 离散时域；自适应控制；指数稳定。