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一种非线性可观型系统和非线性观测器的设计

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摘 要

Li和TAO[2]提出了化非线性系统为一种可观型的条件，用单变量积分解决了Bestle和Zeitz在非线性观测器设计中的计算问题，但是状态变量的求取比较麻烦，本文给出一个新的条件和计算方法，使非线性系统观测器的设计大为简化。

关键词：非线性系统；可观正则型；观测器。

一、引 言

1983年Bestle和Zeitz[1]引入了一种非性线可观型系统

$$\dot{x}^* = \underbrace{\begin{pmatrix} 0 & & \mathbf{0} \\ 1 & \ddots & \\ \mathbf{0} & 1 & 0 \end{pmatrix}}_{E_n} x^* - \underbrace{\begin{pmatrix} \alpha_0(t, x_n^*) \\ \vdots \\ \alpha_{n-1}(t, x_n^*) \end{pmatrix}}_{\alpha(t, x_n^*)}, \quad y = (0 \cdots 0\ 1)x^*$$

讨论了非线性系统

$$\dot{x} = f(t,x), \quad y = h(t,x)$$

化可观型的条件，并用极点配置的方法设计了如下形式的非线性观测器

$$\dot{\tilde{x}}^* = E_n \tilde{x}^* - \frac{\partial \alpha}{\partial x_n^*}\bigg|_{(t,\hat{x})} \tilde{x}_n^* - K^*(t, \hat{x}^*)\tilde{x}_n^* + O(x_n^{*2})$$

$\tilde{x}(t_0) = \hat{x}_0^* - x_0^*$，$K^*(t, \hat{x}^*)$是增益向量

由于状态变换的计算问题没有解决，Bestle和Zeitz的方法只具有理论意义。1986年Li和TAO[2]提出了一个化可观型系统的条件，用单变量积分解决了状态变换的计算问题，改进了Bestle和Zeitz的观测器设计。但是这种方法的证明和计算都比较麻烦。

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这里将给出一个新的条件和两种化可观型的方法，减少了求逆矩阵的困难，使观测器的设计大为简化。

二、化非线性可观型系统的条件

为推导简明起见，仅考虑非线性定常系统

$$\dot{x}=f(x),\quad y=h(x),\quad x(t_0)=x_0 \tag{1}$$

$x\in R^n$，$y\in R^1$，$f(x)$和$h(x)$具有n阶连续偏导。

假定存在一个状态变换$x=\omega(x^*)$，$\frac{\partial\omega}{\partial x^*}$满秩，能把系统（1）化成可观型系统

$$\dot{x}^*=\begin{pmatrix}0 & & & \mathbf{0}\\ 1 & \ddots & & \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & 0\end{pmatrix}x^*-\begin{pmatrix}\alpha_0(x_n^*)\\ \vdots\\ \alpha_{n-1}(x_n^*)\end{pmatrix}=f^*,\quad y=(0\cdots0\ 1)x^* \tag{2}$$

把$x=\omega(x^*)$代入系统（1）得

$$f(x)=\dot{x}=\frac{\partial\omega}{\partial x^*}\dot{x}^*=\frac{\partial\omega}{\partial x^*}f^*$$

两边对x_k^*求偏导

$$\frac{\partial f}{\partial x}\left(\frac{\partial\omega}{\partial x_k^*}\right)^T=\left(\frac{\partial}{\partial x_k^*}\frac{\partial\omega}{\partial x^*}\right)f^*+\frac{\partial\omega}{\partial x^*}\left(\frac{\partial f^*}{\partial x_k^*}\right)^T$$

交换求偏导次序

$$\frac{\partial f}{\partial x}\left(\frac{\partial\omega}{\partial x_k^*}\right)^T=\left(\frac{\partial}{\partial x^*}\frac{\partial\omega}{\partial x_k^*}\right)^T f^*+\frac{\partial\omega}{\partial x^*}\left(\frac{\partial f_*}{\partial x_k^*}\right)^T=\left(\frac{\partial}{\partial x}\frac{\partial\omega}{\partial x_k^*}\right)^T f+\frac{\partial\omega}{\partial x_{k+1}^*}$$

$$\frac{\partial\omega}{\partial x_{k+1}^*}=\frac{\partial f}{\partial x}\left(\frac{\partial\omega}{\partial x_k^*}\right)^T-\left(\frac{\partial}{\partial x}\frac{\partial\omega}{\partial x_k^*}\right)^T f,\quad 1\leqslant k\leqslant n-1$$

定义

$$\frac{\partial f}{\partial x}\left(\frac{\partial\omega}{\partial x_k^*}\right)^T-\left(\frac{\partial}{\partial x}\frac{\partial\omega}{\partial x_k^*}\right)^T f\triangleq N\left[\frac{\partial\omega}{\partial x_k^*}\right]$$

则得

$$\frac{\partial\omega}{\partial x_k^*}=N\left[\frac{\partial\omega}{\partial x_{k-1}^*}\right]=\cdots=N^{k-1}\left[\frac{\partial\omega}{\partial x_1^*}\right],\quad 1\leqslant k\leqslant n \tag{3}$$

再定义

$$C\left[\frac{\partial h}{\partial x}\right] \triangleq \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}f\right),\quad C^{k+1}\left[\frac{\partial h}{\partial x}\right] \triangleq \frac{\partial}{\partial x}\left(C^{k}\left[\frac{\partial h}{\partial x}\right]f\right),\quad 1\leqslant k\leqslant n-1$$

$$\frac{\partial h}{\partial x}\left(\frac{\partial \omega}{\partial x^*}\right)^T=\frac{\partial y}{\partial x}\left(\frac{\partial \omega}{\partial x^*}\right)^T=\frac{\partial y}{\partial x^*}=(0\cdots 0\ \ 1) \tag{4}$$

$$\frac{\partial h}{\partial x}f=\frac{\partial y}{\partial x}\dot{x}=\dot{y}=\dot{x}_n^*=x_{n-1}^*-\alpha_{n-1}(x_n^*) \tag{5}$$

将（5）两边对x^*求导得

$$C\left[\frac{\partial h}{\partial x}\right]\left(\frac{\partial \omega}{\partial x^*}\right)^T=\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}f\right)\left(\frac{\partial \omega}{\partial x^*}\right)^T=\left(0\cdots 0\ \ 1\ \ -\frac{\partial \alpha_{n-1}(x_n^*)}{\partial x_n^*}\right) \tag{6}$$

将（5）沿系统（1）的状态轨线求导得

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}f\right)\right]f=\dot{x}_{n-1}^*-D\alpha_{n-1}=x_{n-2}^*-\alpha_{n-2}(x_n^*)-D\alpha_{n-1}$$

即

$$C\left[\frac{\partial h}{\partial x}\right]f=x_{n-2}^*-D\alpha_{n-1}-\alpha_{n-2}(x_n^*) \tag{7}$$

其中 $D\alpha_{n-1}=\dfrac{\partial \alpha_{n-1}(x_n^*)}{\partial x^*}f^*=\dfrac{\partial \alpha_{n-1}(x_n^*)}{\partial x_n^*}(x_{n-1}^*-\alpha_{n-1}(x_n^*))$。

将（7）两边对x^*求导得

$$C^2\left[\frac{\partial h}{\partial x}\right]\left(\frac{\partial \omega}{\partial x^*}\right)^T=\left(0\cdots 0\ \ 1\ \ *\ \ -\frac{\partial D\alpha_{n-1}}{\partial x_n^*}-\frac{\partial \alpha_{n-2}}{\partial x_n^*}\right) \tag{8}$$

将（7）沿系统（1）状态轨线求导得

$$C^2\left[\frac{\partial h}{\partial x}\right]f=\dot{x}_{n-2}^*-D^2\alpha_{n-1}-D\alpha_{n-2}=x_{n-3}^*-\alpha_{n-2}(x_n^*)-D^2\alpha_{n-1}-D\alpha_{n-2} \tag{9}$$

照上面方法做下去可得

$$C^k\left[\frac{\partial h}{\partial x}\right]\left(\frac{\partial \omega}{\partial x^*}\right)^T=\left(\underbrace{0\cdots 0\ \ 1}_{n-k}\ \ *\cdots*\ \ -\sum_{i+j=k}\frac{\partial D^j\alpha_{n-i}}{\partial x_n^*}-\frac{\partial \alpha_{n-k}}{\partial x_n^*}\right) \tag{10}$$

$$R=1,\cdots,n-1,\quad i,j=1,\cdots,n-1$$

$$C^{n-1}\left[\frac{\partial h}{\partial x}\right]\left(\frac{\partial \omega}{\partial x^*}\right)^T=\left(1\ \ *\cdots*\ \ -\sum_{i+j=n-1}\frac{\partial D^j\alpha_{n-i}}{\partial x_n^*}-\frac{\partial \alpha_1}{\partial x_n^*}\right) \tag{11}$$

$$C^n\left[\frac{\partial h}{\partial x}\right]\left(\frac{\partial \omega}{\partial x^*}\right)^T=\left(\ast\cdots\ast\quad -\sum_{i+j=n-1}\frac{\partial D^j\alpha_{n-i}}{\partial x_n^*}-\frac{\partial \alpha_0}{\partial x_n^*}\right) \tag{12}$$

由(4)、(6)、(8)、(10)、(11)得

$$\begin{pmatrix}\dfrac{\partial h}{\partial x}\\ C\left[\dfrac{\partial h}{\partial x}\right]\\ \vdots\\ C^{n-1}\left[\dfrac{\partial h}{\partial x}\right]\end{pmatrix}\left(\frac{\partial \omega}{\partial x^*}\right)^T=\begin{pmatrix} & & & 1\\ & \mathbf{0} & 1 & -\dfrac{\partial \alpha_{n-1}(x_n^*)}{\partial x_n^*}\\ & \iddots & & \vdots\\ 1 & \ast\cdots\cdots\ast & & -\sum\limits_{i+j=n-1}\dfrac{\partial D^j\alpha_{n-i}}{\partial x_n^*}-\dfrac{\partial \alpha_1}{\partial x_n^*}\end{pmatrix} \tag{13}$$

由(13)我们可以证明下面结果:

定理 1　设$f(x)$和$h(x)$具有n阶连续偏导,则非线性系统(1)能经状态变换 $x=\omega(x^*)$化成可观型系统(2)的充要条件是:

i)　$H(x)=\left(\dfrac{\partial h^T}{\partial x}\quad C\left[\dfrac{\partial h}{\partial x}\right]^T\cdots C^{n-1}\left[\dfrac{\partial h}{\partial x}\right]^T\right)^T$满秩

ii)　$Q^{-1}(x)=(q(x)\quad N[q(x)]\cdots N^{n-1}[q(x)])^{-1}$满足可积性,即存在$\omega^{-1}(x)\in R^n$,使$\dfrac{\partial\omega^{-1}(x)}{\partial x}=Q^{-1}(x)$,其中 $q(x)$是$H^{-1}(x)$的最后一列。

iii)　由下面方法递推出的$\alpha_{n-1}(x),\cdots,\alpha_0(x)$仅与$h(x)$函数相关

$$\alpha_{n-1}(x)=-\int C\left[\frac{\partial h}{\partial x}\right]N^{n-1}[q(x)]\mathrm{d}h(x)$$

$$\alpha_{n-k}(x)=-\int C^k\left[\frac{\partial h}{\partial x}\right]N^{n-1}[q(x)]\mathrm{d}h(x)-\sum_{i+j=k}D^j\alpha_{n-i},\quad 2\leqslant k\leqslant n-1$$

$$\alpha_0(x)=C^{n-1}\left[\frac{\partial h}{\partial x}\right]f-\sum_{i+j=n}D^j\alpha_{n-i}$$

证　必要性:由(13)条件i)显然,只需证ii)和iii)。

由(13)可解得

$$\frac{\partial \omega}{\partial x_1^*}=H^{-1}(x)(0\cdots0\quad 1)=q(x)$$

由(3)式

$$\frac{\partial \omega}{\partial x^*}=(q(x)\quad N[q(x)]\cdots N^{n-1}[q(x)])=Q(x)$$

利用雅可比阵性质和反函数求导法则得

$$Q^{-1}(x)=\frac{\partial \omega^{-1}(x)}{\partial x}=\left(\frac{\partial \omega}{\partial x^*}\right)^{-1}=(q(x)\quad N[q(x)\cdots N^{n-1}[q(x)])^{-1}$$

由假定 $x^*=\omega^{-1}(x)$存在；$\frac{\partial \omega^{-1}(x)}{\partial x}$满足可积性。

$\alpha_{n-i}(x)$，$i=1,\cdots,n$的递推式子可由（10）、（12）两式得出，因为α_{n-i}仅是x_n^*的函数，iii)成立。

充分性：由ii）存在$\omega^{-1}(x)\in R^n$使得$\frac{\partial \omega^{-1}(x)}{\partial x}=Q^{-1}(x)$定义状态变换

$$x^*=\omega^{-1}(x),\quad x=\omega(x^*)$$

不难验证，经上面定义的状态变换确能把系统（1）化成可观型系统（2）。

三、化可观型系统的方法

利用前面定理可导出下面两种化可观型系统的方法

方法 1

（1）计算$H(x)$

若$H(x)$不满秩，该系统不能化成可观型系统；若$H(x)$满秩，转（2）

（2）计算$Q(x)$

（3）计算$\alpha_{n-1}(x)$，$\alpha_{n-2}(x)$，$\cdots$，$\alpha_0(x)$

若计算中出现某个$\alpha_{n-i}(x)$违背定理中的条件iii)，停止计算，该系统不能化成可观型系统；若$\alpha_{n-i}(x)$皆满足条件iii)，转（4）

（4）利用关系式$\dot{x}_k^*=x_{k-1}^*-\alpha_{k-1}(x)$和$x_n^*=h(x)$ 依次求出$x_{n-1}^*(x)$，$x_{n-2}^*(x)$，$\cdots$，$x_1^*(x)$

（5）计算 $\frac{\partial x^*(x)}{\partial x}$

若 $\frac{\partial x^*(x)}{\partial x}\cdot Q(x)=I$（单位阵），则

$$x^*=\omega^{-1}(x)=x^*(x)=(x_1^*(x)\cdots x_n^*(x))$$

就是化系统（1）为可观型系统（2）的状态变换，若$\frac{\partial x^*(x)}{\partial x}\cdot Q(x)\neq I$，该系统不能

成可观型系统。

（6）利用求出的$\alpha(x)$写出可观型系统

例 1　$\dot{x}=f(x)=(x_2 \quad x_1x_2^2)^T$，$y=h(x)=x_1$

$$H(x)=\begin{pmatrix} \dfrac{\partial h}{\partial x} \\ C\left[\dfrac{\partial h}{\partial x}\right] \end{pmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q(x)=(q(x) \quad N[q(x)])=\begin{bmatrix} 0 & 1 \\ 1 & 2x_1x_2 \end{bmatrix}$$

$$\alpha_1(x)=-\int C\left[\frac{\partial h}{\partial x}\right]N[q(x)]\mathrm{d}h(x)=-\int(0 \quad 1)\begin{bmatrix} 1 \\ 2x_1x_2 \end{bmatrix}\mathrm{d}x_1=-x_1^2x_2$$

$$=-h^2(x)x_2$$

不满足定理条件iii)，该系统不能化成可观型系统（2）。

例 2

$$\dot{x}=f(x)=\begin{pmatrix} \sin\ln x_3-(1+2x_2)(x_1+x_2+x_2^2+\ln x_3) \\ x_1+x_2+x_2^2+\ln x_3 \\ x_2x_3+x_3\ln^3x_3 \end{pmatrix}$$

$$y=h(x)=\ln x_3$$

$$H(x)=\begin{pmatrix} \dfrac{\partial h}{\partial x} \\ C\left[\dfrac{\partial h}{\partial x}\right] \\ C^2\left[\dfrac{\partial h}{\partial x}\right] \end{pmatrix}=\begin{pmatrix} 0 & 0 & \dfrac{1}{x_3} \\ 0 & 1 & \dfrac{3}{x_3}\ln^2x_3 \\ 1 & 1+2x_2+3\ln^2x_3 & \dfrac{1}{x_3}(1+6x_2\ln x_3+15\ln^4x_3) \end{pmatrix}$$

$$q(x)=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ Q(x)=(q(x) \quad N[q(x)] \quad N^2[q(x)])=\begin{pmatrix} 1 & -(1+2x_2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_3 \end{pmatrix}$$

$$\alpha_2(x)=-\int C\left[\frac{\partial h}{\partial x}\right]N^2[q(x)]\mathrm{d}h(x)=-\int\left(0 \quad 1 \quad \frac{3}{x_3}\ln^2x_3\right)(0 \quad 0 \quad x_3)^Td\ln x_3$$

$$=-3\int\ln^2x_3d\ln x_3=-\ln^3x_3$$

$$\alpha_1(x) = -\int C^2\left[\frac{\partial h}{\partial x}\right]N^2[q(x)]dh(x) - D\alpha_2(x)$$

$$= -\int(1 \quad 1+2x_2+3\ln^2 x_3 \quad \frac{1}{x_3}(1+6x_2 l_n x_3+15\ln^4 x_3))(0 \quad 0 \quad x_3)^T d\ln x_3$$

$$-\frac{\partial \alpha_2(x)}{\partial x}f(x) = -\int(1+6x_2\ln x_3+15\ln^4 x_3)d\ln x_3+3x_2\ln^2 x_3+3\ln^5 x_3$$

$$= -\ln x_3$$

$$\alpha_0(x) = -C^2\left[\frac{\partial h}{\partial x}\right]f(x) - D^2\alpha_2 - D\alpha_1$$

$$= -(1 \quad 1+2x_2+3\ln^2 x_3 \quad \frac{1}{x_3}(1+6x_2\ln x_3+15\ln^4 x))f(x)$$

$$+(0 \quad 3\ln^2 x_3 \quad \frac{1}{x_3}(6x_2\ln x_3+15\ln^4 x_3))f+(0 \quad 0 \quad \frac{1}{x_3})f$$

$= -\sin\ \ln x_3$ $\alpha_2(x)$，$\alpha_1(x)$，$\alpha_0(x)$皆满足定理条件iii)。

计算$x^* = x^*(x)$

$$x_3^* = h(x) = \ln x_3,\quad x_2^* = \dot{x}_3^* + \alpha_2(x) = \frac{\partial x_3^*}{\partial x}f + \alpha_2(x) = x_2$$

$$x_1^* = \dot{x}_2^* + \alpha_1(x) = \frac{\partial x_2^*}{\partial x}f - \ln x_3 = x_1 + x_2 + x_2^2$$

$$\frac{\partial x^*(x)}{\partial x}Q(x) = \begin{pmatrix} 1 & 1+2x_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{x_3}\end{pmatrix}\begin{pmatrix} 1 & -(1+2x_2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_3\end{pmatrix} = I$$

所以经状态变换$x^* = \omega^{-1}(x) = (x_1+x_2+x_2^2 \quad x_2 \quad \ln x_3)^T$系统(1)化为可观型系统

$$\dot{x}^* = E_n x^* - \alpha(x) = E_n x^* - (-\sin\ \ln x_3 \quad -\ln x_3 \quad -\ln^3 x_3)^T$$

方法 2

(1)、(2)两步同方法1.

(3)若$Q^{-1}(x)$满足可积性，定义$x^* = \omega^{-1}(x)$，否则该系统不能化成可观型系统。

(4)把$x^* = \omega^{-1}(x)$代入(1)验证。

仍以前面的两个例题证明这种算法。

例 1 $Q^{-1}(x) = \begin{bmatrix} -2x_1x_2 & 1 \\ 1 & 0 \end{bmatrix}$ 显然不可积，不能化成可观型系统。

例 2　$Q^{-1}(x)=\begin{pmatrix}1 & 1+2x_2 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{x_3}\end{pmatrix}$, $\int Q^{-1}(x)dx=\begin{pmatrix}x_1+x_2+x_2^2\\ x_2\\ \ln x_3\end{pmatrix}$

定义　$x^*=\omega^{-1}(x)=(x_1+x_2+x_2^2 \quad x_2 \quad \ln x_3)^T$

$$\dot{x}^*=Q^{-1}(x)f=\begin{pmatrix}\sin\ln x_3\\ x_1+x_2+x_2^2+\ln x_3\\ x_2+\ln^3 x_3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}x^*+\begin{pmatrix}\sin x_3^*\\ x_3^*\\ x_3^{*3}\end{pmatrix}$$

四、非线性系统观测器的设计

定理 2　如果系统（1）满足定理1的三个条件，其中条件ii)可用算法 1 的（4）和（5）代替，系统（1）可建立如下形式的观测器

$$\dot{\hat{x}}=f(\hat{x})+K(\hat{x})[y-h(\hat{x})],\quad \hat{x}(t_0)=x_0$$

证　因为系统（1）满足化可观型的条件，可经状态变换$x=\omega(x^*)$或$x^*=\omega^{-1}(x)$化成可观型系统（2），对可观型系统（2）建立观测器如下：

$$\dot{\hat{x}}^*=E_n\hat{x}^*-\alpha(y)+K^*(y-\hat{x}_n^*)=f^*(\hat{x}^*)+K^*(y-\hat{x}_n^*)$$

$$\hat{x}^*(t_0)=x_0^*$$

其中 $K^*=(K_1^*\cdots K_n^*)$是增益向量

于是可得误差方程

$$\dot{\tilde{x}}^*=\dot{x}^*-\dot{\hat{x}}^*=E_n\tilde{x}^*-K^*\tilde{x}_n^*$$

显然增益向量K^*可由极点配置方便选定。由$x^*=\omega^{-1}(x)$相应原系统(1)的观测器为

$$\dot{\hat{x}}=f(\hat{x})+K(\hat{x})[y-h(\hat{x})],\quad \hat{x}(t_0)=x_0$$

其中
$$K(\hat{x})=\frac{\partial\omega}{\partial x^*}K^*$$

由上面过程可以看出，对系统（1）能用极点配置的方法设计观测器的关键是化可观型系统，本文给出的两种算法解决了这个问题。

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Non-linear Observer Canonical Form and the Design of Observer for Non-linear Systems

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Abstract

Li and TAO have given a necessary and suffcient condition under which the non-linear observer canonical form exists. By integrating single variable functions, they solved the problem of design of observer Bestle and Zeitz unsolved. But their conditions and computations of state transformation is too complex. In this paper a new existence conditions and two algorithms are given. According to this the design of non-linear observer is well simplified.

Key words——Nonlinear systems; Observable canonical form; Observer.