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The Design of Robust H_∞ Controller for Linear Systems with Uncertain Parameters*

SHI Guojun, YANG Chengwu and ZOU Yun
(The Eighth Department, Nanjing University of Science & Technology·Nanjing, 210014, PRC)

Abstract: This paper focuses on the problem of robust H_∞ control for linear systems with time-varying parameter uncertainty. The uncertainty under consideration is assumed to be unknown but bounded and can exist in both the state and input matrices. An approach is proposed for designing a state feedback control law which will quadratically stabilize the plant and guarantee a disturbance attenuation constraint as well as the stability margin for all admissible uncertainties. It is shown that a suitable stabilizing feedback matrix can be constructed in terms of a positive solution to a certain parameter-dependent algebraic Riccati equation.

Key words: H_∞ control; uncertain systems; stability margin; state feedback; algebraic Riccati equation; quadratic stabilization

1 Introduction

The standard robust H_∞ control problem is depicted as Fig. 1.

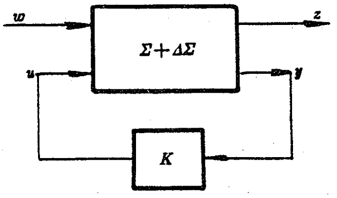

Fig. 1 Dragram for standard H_∞ control

In this diagram, Σ is a nominal time-invariant linear system; $\Delta\Sigma$ represents plant uncertainties, which may be caused by several practical reasons, such as modeling errors, neglected non-linearity, non-ideal implementation, component failure, etc; w is an external disturbance vector; u is the control input; z is the controlled output; and y is the measured output. The standard robust H_∞ control problem involves designing a compensator $K(s)$ such that the resulting closed-loop system is (quadratically) stable with a disturbance attenuation constraint for all admissible uncertainties $\Delta\Sigma$.

In the present paper, attention is focused on the robust H_∞ control problem of linear uncertain systems containing time-varying uncertainties in both the state and input matrices. The time-varying uncertainties are assumed to be unknown but bounded. It is assumed that the full states of the plant are available for feedback. The approach adopted here relies on a different parameter-dependent algebraic Riccati equation (ARE). Based on the ARE, the paper presents a state feedback design which will quadratically stabilize the plant and guarantee a disturbance attenuation constraint for all admissible uncertainties. Furthermore, the desired state feedback will result in the optimal stability margin to the closed-loop nominal system, i. e. the resulting closed-loop

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system has the optimal transient behaviour in some sense.

Notation In the sequel, we shall use the following notion: For symmetric matrices P and Q, $P>0 (P\geqslant 0)$ denotes the fact that P is positive definite (positive semidefinite). Moreover, $P>Q$ $(P\geqslant 0)$ denotes $P-Q>0$ $(P-Q\geqslant 0)$. $\lambda_{max}[\cdot]$ represents the maximum eigenvalue of a matrix. $L_2[0,\infty)$ is the space of L_2 integrable functions on $[0,\infty)$ and $\|\cdot\|_2$ stands for norm on $L_2[0,\infty)$. The notation $\|A\|$ denotes the spectral norm of matrix A.

2 System and Definitions

The class of linear uncertain systems under consideration is described by state equations of the form.

$$\dot{x}(t) = [A + \Delta A(q(t))]x(t) + B_1w(t) + [B_2 + \Delta B_2(r(t))]u(t), \tag{2.1a}$$

$$z(t) = C_1(t) + D_1u(t), \tag{2.1b}$$

$$y(t) = C_2x(t), \tag{2.1c}$$

where $x(t)\in R^n$ is the state, $u(t)\in R^m$ is the control input, $w(t)\in R^p$ is the disturbance input, $y(t)\in R^l$ is measured output, $z(t)\in R^q$ is the controlled output, A, B_1, B_2, C_1, D_1 and C_2 are constant real matrices of appropriate dimensions, $q(t)\in R^v$, $r(t)\in R^s$ are vectors of time-varying uncertain parameters, $\Delta A(\cdot): R^v\to R^{n\times n}$ and $\Delta B_2(\cdot): R^s\to R^{n\times m}$ are continuous matrix functions which represent the state matrix uncertainty and input matrix uncertainty, respectively. It is assumed that the uncertainties

$$q(\cdot): R\to \Omega\subset R^v,$$

$$r(\cdot): R\to \Psi\subset R^s.$$

are Lebesgue measurable, where Ω and Ψ are prescribed compact subsets of appropriate spaces. Furthermore, $\Delta A(\cdot)$ and $\Delta B_2(\cdot)$ are bounded as follows

$$\Delta A(\cdot)\in \mathbf{A}: =\{\Delta A(\cdot): \Delta A^T(\cdot)\Delta A(\cdot)\leqslant \bar{A},$$
$$\bar{A} \text{ is a known positive semidefinite } n\times n \text{ matrix}\};$$

$$\Delta B_2(\cdot)\in \mathbf{B}: =\{\Delta B_2(\cdot): \Delta B_2^T(\cdot)\Delta B_2(\cdot)\leqslant \bar{B},$$
$$\bar{B} \text{ is a known positive definite } m\times m \text{ matrix}\}.$$

Note that in practice the norm bound $\bar{B}$ may be positive semidefinite. In such a case, we can always chose a sufficiently small parameter μ such that $\bar{B}_0: =\bar{B}+\mu I>0$. It leads to a positive definite norm bound $\bar{B}_0$ of and $\Delta B_2(\cdot)$.

For a technical simplification, we shall make the following assumption

A1 $$D_1^TC_1 = 0$$

It should be noted that the assumption A1 causes no loss of generality, see [2].

This paper considers the problem of state feedback robust H_∞ control for system (2.1). We assume that perfect state information is available for feedback and we are concerned with designing a state feedback control law to regulate the system (2.1) with a prescribed disturbance attenuation constraint as well as a given stability margin for all admissible uncertainties $\Delta A(\cdot)\in \mathbf{A}$ and $\Delta B_2(\cdot)\in \mathbf{B}$.

Let the control law for system (2.1) be given by $u(t)=Kx(t)$. Then, the resulting closed-loop system is as follows

$$\dot{x}(t) = A_C(t)x(t) + B_1w(t), \tag{2.2a}$$

$$z(t) = C_Cx(t), \tag{2.2b}$$

$$y(t) = C_2x(t), \tag{2.2c}$$

where

$$A_C(t) := A + B_2K + \Delta A(t) + \Delta B_2(t)K, \tag{2.3a}$$

$$C_C := C_1 + D_1K. \tag{2.3b}$$

In this paper, we shall use the following notion of stabilizability related to system(2.1)

Definition 2.1 ((γ, δ)-quadratic stabilization)

Let constants $\gamma>0$ and $\delta>0$ be prespecified. The system (2.1) is said to be quadratically stabilizable with disturbance attenuation γ and stability margin δ via state feedback if there exists a state feedback control $u(t)=Kx(t)$ such that, for all admissible uncertainties $\Delta A(t)$ and $\Delta B_2(t)$, the following conditions are satisfied:

i) There exist a symmetric matrix $P>0$ and a scalar $\varepsilon>0$ such that the Lyapunov function $V(t)=x^TPx$ satisfies

$$\dot{V}(x) = 2x^TPA_c(t)x < -\varepsilon\|x\|^2, \quad x \in R^n, \ x \neq 0;$$

ii) Subject to the assumption of zero initial condition, the controlled output z satisfies $\|z\|_2=\gamma\|w\|_2$;

iii) The nominal closed-loop system matrix $A_0:=A+B_2K$ has a stability margin δ, i. e. $\mathrm{Re}\lambda(A+B_2K)\leqslant-\delta$.

In the remainder of this section we shall present a fact that will be needed in the proof of the main results.

Fact 2.1 Let X, Y be matrices of appropriate dimensions with Y positive semidefinite symmetric. Then,

$$X^TY + YX \leqslant \alpha Y + \frac{1}{\alpha}X^TYX \quad \text{for any} \quad \alpha > 0.$$

Proof Factorize $Y=(Y^{1/2})(Y^{1/2})$ and define the matrix

$$W := \alpha^{1/2}Y^{1/2} - \alpha^{-1/2}Y^{1/2}X.$$

Then, we have

$$W^TW = \alpha^T + \frac{1}{\alpha}X^TYX - X^TY - YX \geqslant 0.$$

Hence the results follows.

3 Main Results

The approach adopted here in this paper to solve the robust H_∞ control problem involves solving a parameter-dependent algebraic Riccati equation associated with a disturbance attenuation constraint γ and a stability margin constant δ as well as the uncertainties in the state space model. Given the system (2.1) and the desired constants $\gamma>0$ and $\delta>0$, we define the follwoing algebraic Riccati equation corresponding to the problem of (γ,δ)-quadratic stabilizability.

$$(A+\delta I)^{\mathrm{T}}P+P(A+\delta I)+\gamma^{-2}PB_1B_1^{\mathrm{T}}P-PB_2R_1^{-1}(P)B_2^{\mathrm{T}}P+R_2(P)+\varepsilon I=0, \tag{3.1}$$

where

$$R_1(P):=E+\frac{1}{a}\|P\|\overline{B},\quad R_2(P):=C_1^{\mathrm{T}}C_1+\frac{1}{2\delta-a}\|P\|\overline{A}.$$

ε is a sufficiently small constant, $a>0$ is a design parameter.

Now, we present the main result of this paper.

Theorem 3.1 Let the performance constants $\gamma>0$ and $\delta>0$ be prescribed. The system (2.1) is (γ,δ)-quadratically stabilizable for all admissible uncertainties if for a sufficiently small $\varepsilon>0$, there exists a constant $a\in(0,2\delta)$ such that the Riccati equation (3.1) has a positive definite solution P. In this case, a suitable feedback control law is given by

$$u(t)=Kx(t),\quad K=-R_1^{-1}(P)B_2^{\mathrm{T}}P. \tag{3.2}$$

Proof Suppose that there exists a constant $a\in(0.2\delta)$ such that the Riccati equation (3.1) has a solution $P>0$. We will show that, with the state feedback control law (3.2), the system (2.1) is (γ,δ)-quadratic stable. First, let us consider the resultant closed-loop system (2.2) and show that

$$A_C^{\mathrm{T}}(t)P+PA_C(t)+\gamma^{-2}PB_1B_1^{\mathrm{T}}P+C_C^{\mathrm{T}}C_C<0. \tag{3.3}$$

Note that, for any symmetric matrix P

$$P\leqslant\|P\|I.$$

Thus, we have

$$\Delta A^{\mathrm{T}}(t)P\Delta A(t)\leqslant\|P\|\Delta A^{\mathrm{T}}(t)\Delta A(t)\leqslant\|P\|\overline{A},$$
$$\Delta B_2^{\mathrm{T}}(t)P\Delta B_2(t)\leqslant\|P\|\Delta B_2^{\mathrm{T}}(t)\Delta B_2(t)\leqslant\|P\|\overline{B}.$$

Then, it follows from fact 2.1 that

$$\begin{aligned}
&A_C^{\mathrm{T}}(t)P+PA_C(t)\\
&=(A+B_2K)^{\mathrm{T}}P+P(A+B_2K)+\Delta A^{\mathrm{T}}(t)P+P\Delta A(t)+K^{\mathrm{T}}\Delta B_2^{\mathrm{T}}(t)P+P\Delta B_2(t)K\\
&\leqslant(A+B_2K)^{\mathrm{T}}P+P(A+B_2K)+(2\delta-a)P+\frac{1}{2\delta-a}\Delta A^{\mathrm{T}}(t)P\Delta A(t)+aP\\
&\quad+\frac{1}{a}K^{\mathrm{T}}\Delta B_2^{\mathrm{T}}(t)P\Delta B_2(t)K\\
&\leqslant(A+\delta I)^{\mathrm{T}}P+P(A+\delta I)+\frac{1}{2\delta-a}\|P\|\overline{A}+K^{\mathrm{T}}B_2^{\mathrm{T}}P+PB_2K+\frac{1}{a}\|P\|K^{\mathrm{T}}\overline{B}K.
\end{aligned} \tag{3.4}$$

Furthermore, considering assumption A1, it follows

$$C_C^{\mathrm{T}}C_C=(C_1+D_1K)^{\mathrm{T}}(C_1+D_1K)=C_1^{\mathrm{T}}C_1+K^{\mathrm{T}}EK, \tag{3.5}$$

where

$$E=D_1^{\mathrm{T}}D.$$

Combining (3.4) and (3.5), we have

$$\begin{aligned}
&A_C^{\mathrm{T}}(t)P+PA_C(t)+\gamma^{-2}PB_1B_1^{\mathrm{T}}P+C_C^{\mathrm{T}}C_C\\
&\quad\leqslant(A+\delta I)^{\mathrm{T}}P+P(A+\delta I)+\gamma^{-2}PB_1B_1^{\mathrm{T}}P+C_1^{\mathrm{T}}C_1+\frac{1}{2\delta-a}\|P\|\overline{A}
\end{aligned}$$

$$+ K^{T}B_2^{T}P + PB_2K + K^{T}(E + \frac{1}{a}\|P\|\overline{B})K. \tag{3.6}$$

Substituting (3.2) into (3.6), it follows

$$\begin{aligned} &A_C^{T}(t)P + PA_C(t) + \gamma^{-2}PB_1B_1^{T}P + C_C^{T}C_C \\ &\leqslant (A + \delta I)^{T}P + P(A + \delta I) + \gamma^{-2}PB_1B_1^{T}P - PB_2R_1^{-1}(P)B_2^{T}P + R_2(P) \\ &\leqslant -\varepsilon I < 0. \end{aligned}$$

Hence, (3.3) is true.

Now, we show that the resultant closed-loop system is (γ,δ)-quadratically stable.

Consider the Lyapunov function

$$V(x) = -x^{T}Px.$$

Hence, the time derivative of $V(x)$ along the autonomous trajectory of (2.2) is

$$\dot{V}(t) = x^{T}(A_C^{T}(t)P + PA_C(t)]x.$$

Thus, it follows from (3.3) that the closed-loop system (2.2) is quadratically stable.

In order to establish the upper bound $\gamma\|w\|_2$ for $L_2[0,\infty)$-norm of z, we assume $x(0)=0$. Now, let us introduce

$$J = \int_0^\infty (z^{T}z - \gamma^2 w^{T}w)\mathrm{d}t.$$

Then, it follows from (3.3) that, for any $w \in L_2[0,\infty)$

$$\begin{aligned} J &= \int_0^\infty [z^{T}z - \gamma^2 w^{T}w + \frac{\mathrm{d}}{\mathrm{d}t}(V(x))]\mathrm{d}t - x^{T}(\infty)Px(\infty) \\ &= \int_0^\infty \{x^{T}C_C^{T}C_C x - \gamma^2 w^{T}w + [x^{T}A_C^{T}(t) + w^{T}B_1^{T}]Px + x^{T}P[A_C(t)x + B_1 w]\}\mathrm{d}t \\ &= \int_0^\infty x^{T}[A_C^{T}(t)P + PA_C(t) + \gamma^{-2}PB_1B_1^{T}P + C_C^{T}C_C]x\mathrm{d}t \\ &\quad - \int_0^\infty (\gamma w - \gamma^{-1}B_1^{T}Px]^{T}[\gamma w - \gamma^{-1}B_1^{T}Px]\mathrm{d}t < 0, \end{aligned}$$

i.e. $\|z\|_2 < \gamma\|w\|_2$.

From the properties of Riccati equation, it follows that

$$A + \delta I - B_2R_1^{-1}(P)B_2^{T}P$$

is stable. Consider the nominal closed-loop system matrix

$$A_0 := A + B_2K = A - B_2R_1^{-1}(P)B_2^{T}P.$$

Then, we conclude that

$$\mathrm{Re}\lambda(A_0 + \delta I) < 0.$$

i.e.

$$\mathrm{Re}\lambda(A_0) \leqslant -\delta.$$

That is, the nominal closed-loop system has a lower bound of stability margin δ. This completes the proof.

Remark 3.1 From the theorem we conclude that non- zero initial states of the closed-loop system should decay at least as fast as $\exp(-\delta t)$.

References

[1] Khargonekar, P. P., Petersen, I. R. and Zhou, K.. Robust Stabilization of Uncertain Linear Systems: Quadratic Stabilizability and H_∞ Control Theory. IEEE Trans. Automat. Contr., 1990, AC-35(3):356—361

[2] Doyle, J. C., Glover, K., Khargonekar, P. P. and Francis, B. A.. State-Space Solutions to Standard H_2 and H_∞ Control Problems. IEEE Trans. Automat. Contr., 1989, AC-34(8):831—847.

[3] Petersen, I. R.. Disturbance Attenuation and H_∞ Optimization: A Design Method Based on the Algebraic Riccati Equation. IEEE Trans. Automat. Contr., 1987, 32(5):427—429

[4] Wang, S. D., Kuo, T., Lin, Y., Hsu, C. and Juang, Y.. Robust Control Design for Linear Systems with Uncertain Parameters, Int. J. Contr., 1987, 46(5):1557—1567.

[5] Scherzinger, B. M.. The Design of Servomechanism Controllers for Uncertain Plants. Int. J. Contr., 1984, 40(5): 1015—1022

[6] Gilman, A. S. and Rhodes, I. B.. Cone-Bounded Nonlinearities and Mean-Square Bounds-Quadratic Regulation Bounds. IEEE Trans. Automat. Contr., 1976, AC-21(4):472—483

线性参数不确定系统的 H_∞ 鲁棒控制器设计

史国军 杨成梧 邹 云

(南京理工大学八系·南京,210014)

摘要：本文主要讨论了带有未知不确定参数的线性系统的 H_∞ 鲁棒控制问题. 这里假定其中的不确定性是时变有界的,并同时存在于系统的状态矩阵和输入矩阵中. 文中提出了一种同时使受控对象保持二次稳定并满足干扰约束的状态反馈设计方法. 结果表明,该类反馈控制律可以通过求解一个含有参数的代数 Riccati 方程得到.

关键词：H_∞控制；不确定系统；稳定裕度；状态反馈；二次稳定；代数 Riccati 方程

本文作者简介

史国军 1965年生. 分别于1985年,1988年和1992年在华东工学院获工学学士,硕士和博士学位. 主要研究兴趣为 H_∞控制,滞留时间控制,非线性系统理论等. 现在河南平顶山股份有限公司从事科技开发工作.

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