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A Necessary and Sufficient Condition for Feedback Strictly Positive Real Output*

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Abstract: This paper concerns with the problem of feedback strictly positive real system with multi-input. The sufficient and necessary condition for a linear system having feedback strictly positive real output is obtained in this paper. The condition itself is succinct in expression and is easy to check.

Key words: feedback strictly positive real system; Hurwitz polynomial; feedback strictly positive real output

1 Introduction

Consider a linear control system $\Sigma(C,A,B), A\in \mathrm{R}^{n\times n}$, $B\in \mathrm{R}^{n\times m}$, $C\in \mathrm{R}^{m\times n}$, $m,n\,(m\leqslant n)$ positive integer.

Definition 1.1 If there exist $n\times n$ positive definite symmetric matrices P, Q (denoted by $P>\Theta_n$, $Q>\Theta_n$, Θ_n $n\times n$ zero matrix) and $K\in \mathrm{R}^{m\times n}$ such that

$$\begin{cases} P(A+BK)+(A+BK)^{\mathrm{T}}P=-Q, \\ PB=C^{\mathrm{T}}. \end{cases} \tag{1.1}$$

Then $\Sigma(C,A,B)$ is said to be a feedback strictly positive real system, or $\Sigma(C,A,B)$ is said to have feedback strictly positive real output.

The concept of feedback positive real system was first proposed by Kokotovic and Sussmann, when they studied global stabilization of nonlinear system in [1]. In fact, there are many applications of feedback strictly positive real property in other control fields, for example, in designing robust controllers[2] and in assigning optimal poles[3] etc. For the case of scalar input, Stalford, H. L. and Chao, C. H. have proved that when the pair (A,b) is in the controllable canonical form, system $\Sigma(c,A,b)$ has feedback strictly positive real output iff c is a stable $(n-1)$-vector[2]. For the case of multiple inputs, they obtained some necessary conditions in investigating the design problem of robust controllers[4]. In this paper, we obtain the sufficient and necessary condition for a linear system to have feedback strictly positive real output in both cases: with the assumption of (A,B) being a controllable pair and without this assumption. That is, system $\Sigma(C,A,B)$ has feedback strictly positive real output iff $CB>\Theta_m$ and $\det(sI_n-A)\det(C$

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• $(sI_n - A)^{-1}B$ is an $(n-m)$-Hurwitz polynomial.

2 Main Result

In this paragraph, we consider the case of (A,B) being a completely controllable pair. Thus it can be seen that system $\Sigma(C,A,B)$ is in the following form [5].

$$\begin{cases} A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ C_{21} & C_{22} & \cdots & C_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{bmatrix}, \\ B = \text{block diag}\{b_1, b_2, \cdots, b_m\}, \end{cases}$$

where

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{ii}^1 & -a_{ii}^2 & -a_{ii}^3 & \cdots & -a_{ii}^{n_i} \end{bmatrix} \in \mathrm{R}^{n_i \times n_i}, \quad i = 1,2,\cdots,m,$$

$$A_{ij} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -a_{ij}^1 & -a_{ij}^2 & -a_{ij}^3 & \cdots & -a_{ij}^{n_i} \end{bmatrix} \in \mathrm{R}^{n_i \times n_j}, \quad i \neq j, \quad i,j = 1,2,\cdots,m,$$

$$b_j = [0,\ 0,\ 0,\ \cdots,\ 1]^{\mathrm{T}} \in \mathrm{R}^{n_j}, \quad j = 1,2,\cdots,m,$$

$$c_{ij} = [c_{ij}^1, c_{ij}^2, \cdots, c_{ij}^{n_j}] \in \mathrm{R}^{1 \times n_j}, \quad i,j = 1,2,\cdots,m,$$

$$\sum_{i=1}^{m} n_i = n.$$

For convenience we introduce some notations:

$$a_{ij} \triangleq [a_{ij}^1, a_{ij}^2, \cdots, a_{ij}^{n_i}], \quad i,j = 1,2,\cdots,m,$$

$$x \triangleq [x_1, x_2, \cdots, x_l] \in \mathrm{R}^{1 \times l}, \quad l \text{ positive integer},$$

$$x(s) \triangleq [1, s, \cdots, s^{l-1}][x_1, x_2, \cdots, x_l]^{\mathrm{T}} = x_1 + x_2 s + \cdots + x_l s^{l-1}.$$

Let
$$A(s) = \begin{bmatrix} p_{11}(s) & a_{12}(s) & \cdots & a_{1m}(s) \\ a_{21}(s) & p_{22}(s) & \cdots & a_{2m}(s) \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}(s) & a_{m2}(s) & \cdots & p_{mm}(s) \end{bmatrix},$$

where $p_{ii}(s) = \det(sI_{n_i} - A_{ii})$, $i = 1,2,\cdots,m$, and

$$C(s) = \begin{bmatrix} c_{11}(s) & c_{12}(s) & \cdots & c_{1m}(s) \\ c_{21}(s) & c_{22}(s) & \cdots & c_{2m}(s) \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1}(s) & c_{m2}(s) & \cdots & c_{mm}(s) \end{bmatrix}.$$

Lemma 2.1 The following relationship holds:

$$H(s) \triangleq C(sI_n - A)^{-1}B = C(s)A^{-1}(s),$$

$$\det C(s) = \det(C(sI_n - A)^{-1}B)\det(sI_n - A). \qquad \text{Q.E.D.}$$

Theorem 2.1 For the system $\Sigma(C,A,B)$ having the form (2.1), it has feedback strictly positive real output iff the following conditions are satisfied

i) $CB > \Theta_n$,

ii) $\Delta(s) \triangleq \det C(s)$ is an $(n-m)$-Hurwitz polynomial.

Proof The proof of necessity can be found in [4] (see Theorem 2.2 of [4]). We will give the proof of sufficiency in the following.

Let

$$K = -(CB)^{-1}C(A + \mu I_n),$$

where $\mu > 0$. Then by straghtforward computation and Lemma 2.1, we have

$$H(s) \triangleq C(sI_n - A - BK)^{-1}B = \frac{1}{s+\mu}C(s)C^{-1}(s)CB.$$

Since $\Delta(s) = \det C(s)$ is an (n−m)-Hurwitz polynomial, we obtain

i) All elements of $H(s)$ are analytic in $\mathrm{Re}(s) \geqslant 0$;

ii) $H(j\omega) + H^T(-j\omega) = \frac{2\mu}{\omega^2 + \mu}CB > \Theta_m, \quad \omega \in (-\infty, +\infty)$;

iii) $\lim_{\omega \to \infty} \omega^2[H(j\omega) + H^T(-j\omega)] = \lim_{\omega \to \infty} \frac{2\omega^2\mu}{\omega^2 + \mu^2}CB = 2\mu CB > \Theta_m$.

According to Theorem 3.1 of [6]. $\Sigma(C,A,B)$ has feedback strictly positive real output.

3 Generalization of Main Result in Section 2

In the following, we assume that $\mathrm{rank}B = m$.

Lemma 3.1 Let (A,B) be a controllable pair. Then system $\Sigma(C,A,B)$ has feedback strictly positive real output iff the following conditions hold:

i) $CB > \Theta_m$;

ii) $\Delta(s) = \det(C(sI_n - A)^{-1}B)\det(sI_n - A)$ is an $(n-m)$-Hurwitz polynomial.

Theorem 3.1 $\Sigma(C,A,B)$ has feedback strictly positive real output iff the following conditions hold:

i) $CB > \Theta_m$;

ii) $\Delta(s) = \det(C(sI_n - A)^{-1}B)\det(sI - A)$ is an $(n-m)$-Hurwitz polynomial.

Proof Firstly, it can be proved that the property of feedback strictly positive real output for $\Sigma(C,A,B)$ is not changed under coordinate transformation. Therefore we can assume that $\Sigma(C,A,B)$ possesses the following form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \Theta_{n_2 \times n_1} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \Theta_{n_2 \times m} \end{bmatrix}, \quad C = [C_1 \quad C_2].$$

where (A_{11}, B_1) is a controllable pair and A_{22} is stable, $A_{11} \in R^{n_1 \times n_1}$, $A_{12} \in R^{n_1 \times n_2}$, $A_{22} \in R^{n_2 \times n_2}$, $B_1 \in R^{n_1 \times m}$, $C_1 \in R^{m \times n_1}$, $C_2 \in R^{m \times n_2}$, $n_1 + n_2 = n$.

By direct calculation, we have

i) $CB = C_1 B_1$, (4.1)

ii) $\Delta(n)=\det(C_1(sI_{n_1}-A_{11})^{-1}B_1)\det(sI_{n_1}-A_{11})\det(sI_{n_2}-A_{22})$. (4. 2)

From (4. 1), (4. 2), we see that if $\Sigma(C,A,B)$ has feedback strictly positive real output, then $\Sigma(C_1,A_{11},B_1)$ has feedback strictly positive real output too, and (A_{11},B_1) is a controllable pair. In terms of Theorem 3. 1 in 3 and Lemma 4. 1, $CB=C_1B_1>\Theta_m$, $\Delta_1(s)=\det(C_1(sI_{n_1}-A_{11})^{-1}B_1)\det(sI_{n_1}-A_{11})$ is an (n_2-m)-Hurwitz polynomial. Since $\det(sI_{n_2}-A_{22})$ is an n_2-Hurwitz polynomial with the leading coefficient being one. Therefore, $\Delta(s)=\Delta_1(s)\det(sI_{n_2}-A_{22})$ is $(n-m)$-Hurwitz polynomial. Thus, the necessity has been proved.

Next, we prove the sufficiency. Since $\Delta(s)$ is a Hurwitz polynomial, so

$$\Delta_1(s)=\det(C(sI_{n_1}-A_{11})^{-1}B_1)\det(sI_{n_1}-A_{11})$$

and $\det(sI_{n_2}-A_{22})$ are Hurwitz polynomials. In addition $C_1B_1=CB>\Theta_m$. With the help of Theorem 3. 1 in 3 and Lemma 4. 1, $\Sigma(C_1,A_{11},B_1)$ has feedback strictly positive real output, that is, there exist matrices $P_{11}>\Theta_{n_1}$, $Q_{11}>\Theta_{n_1}$ and $K_1\in R^{m\times n_1}$ such that

$$\begin{cases} P_{11}(A_{11}+B_1K_1)+(A_{11}+B_1K_1)^TP_{11}=-Q_{11} \\ P_{11}B_1=C_1^T \end{cases}$$

Since $\operatorname{rank}B_1=\operatorname{rank}B=m$, there exists a matrix $P_{21}\in R^{n_2\times n_1}$ such that $P_{21}B_1=C_2^T$. Let's take

$$P_{12}\triangleq P_{21}^T,$$
$$Q_{21}\triangleq-[P_{21}(A_{11}+B_1K_1)+A_{21}^TP_{11}+A_{22}^TP_{12}],$$
$$Q_{12}\triangleq Q_{21}^T.$$

Then, when a positive number α is taken to be large enough, we have

$$\alpha I_{n_2}+A_{12}^TP_{21}^T+P_{12}A_{12}>\Theta_{n_2},$$
$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & \alpha I_{n_2} \end{bmatrix}>\Theta_n$$

and the Lyapunov equation

$$P_{22}A_{22}+A_{22}^TP_{22}=-(\alpha I_{n_2}+A_{12}^TP_{21}+P_{12}A_{12}),$$

possesses unique solution $P_{22}>\Theta_{n_2}$ such that

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}>\Theta_n.$$

Denote

$$P\triangleq\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},\quad Q\triangleq\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & \alpha I_{n_2} \end{bmatrix}$$

and

$$K\triangleq[K_1,\ \Theta_{m\times n_2}].$$

Then, it is easy to verify that matrices $P>\Theta_n$, $Q>\Theta_n$, K defined in above satisfy the following relationship:

$$P(A+BK)+(A+BK)^TP=-Q,$$
$$PB=C^T$$

Thus, the sufficiency has been proved. Q. E. D.

4 Conclusion

As stated above we have obtained the sufficient and necessary condition for a linear system to have feedback strictly positive real output in both cases: with the assumption of (A,B) being a completely controllable pair and without this assumption. This condition is expressed directly by matrices A, B and C depending only on system itself. Therefore, it is convenient to operate. This result has been used to study the problem of feedback stabilization for a class of nonlinear systems.

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反馈严格正实输出的必要与充分条件

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摘要：本文研究线性多输入多输出系统反馈成为严格正实系统的问题，得到了反馈严格正实系统的必要和充分条件.这个条件在表达上十分简洁.由于这个条件的检验仅需考虑系统矩阵、控制矩阵及输出矩阵间的某种确定系统，从而便于实际使用.

关键词：反馈严格正实系统；霍尔维茨多项式；反馈严格正实输出

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