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# Optimal Birth Control of A Nonlinear Population Diffusion with External Constraint*

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**Abstract**: The optimal birth (boundary) control of age-dependent population dynamics with unilateral constraint diffusion are considered. Existence of optimal control and the necessary conditions for optimality are presented.

**Key words**: population dynamical systems; maximun principle; optimal control

## 1 Introduction

In 1973, M. E. Gurtin[1] introduced the first model of age-dependent population with diffusion in a bounded set of $R^3$. Since then, many similar models were proposed and the existence and uniqueness of their solutions were considered by using various methods (see [2～6]). On the other hand there has been much interest in the optimal control of age-dependent population models (without diffusion) ([7～10]). For a variational inequality model, Anita S. [11] recently considered an "optimal harvesting" problem with only distributed control.

In this paper, in the spirit of [8,9,11], we consider the optimal birth control of population diffusion with unilateral constraint and in which both the birth rate and death rate are dependent of the spatial variable. By using approximate smooth methods[12], we obtain the existence as well as the necessary conditions for optimality.

Assume that a population is free to move in $\Omega$, a bounded and open set of $R^3$. with $\partial\Omega$ of $C^1$-class. The dynamics of population is described by a function $l(a,x,t)$, which represents the density of the individuals of age $a$ at time $t$ and at position $x$. We study the general variational inequalities model.

$$\begin{cases} l \leqslant \psi, \\ \dfrac{\partial l}{\partial t} + \dfrac{\partial l}{\partial a} + \mu l - \Delta_x l \leqslant 0, \qquad 0 < t < T, \quad 0 < a < A, \quad x \in \Omega, \\ \left(\dfrac{\partial l}{\partial t} + \dfrac{\partial l}{\partial a} + \mu l - \Delta_x l\right)(l - \psi) = 0, \end{cases}$$

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$$\begin{cases} \dfrac{\partial l}{\partial \eta} = 0, & 0 < t < T, \quad 0 < a < A, \quad x \in \partial\Omega, \\ l(0,a,x) = l_0(a,x), & 0 < a < A, \quad x \in \Omega, \\ l(0,a,x) = \displaystyle\int_0^A \beta(t,a,x) l(t,a,x) \mathrm{d}a, & t > 0, \quad x \in \Omega. \end{cases} \tag{1}$$

Here, $\mu(t,a,x)$ is the rate of mortality; $\beta$ represents the rate of fertility; $\eta$ is the exterior normal of $\Omega$, (1) shows that $l(t,a,x)$ permanently remains less than or equal to $\psi(t,a,x)$, a given function.

Let $Q=(0,T)\times(0,A)$, $\partial_t$ and $\partial_a$ the partial differentiation in $D'(Q;(H^1(\Omega))^*)$. In order to obtain the existence and uniqueness of solution of (1), we need the following hypotheses[3]:

$(\mu)_1$: $\mu \in C^0([0,T]\times[0,A]\times\bar{\Omega})$, $\mu(t,a,x)\geqslant 0$, in $[0,T]\times[0,A]\times\Omega$,

$(\mu)_2$: $\begin{cases} 0<t<A, \quad x\in\Omega, \quad \lim\limits_{a\to A}\displaystyle\int_0^A \mu(\tau,a-t+\tau,x)\mathrm{d}\tau=\infty, \\ 0<t<T, \quad x\in\Omega, \quad \lim\limits_{a\to A}\displaystyle\int_0^A \mu(t-a+a,a,x)\mathrm{d}a=\infty; \end{cases}$

$(\mu)_3$: $\nabla\mu\in[L^\infty(Q\times\Omega)]^N$,

$(\beta)_1$: $\beta\in L^\infty(Q\times\Omega)$,

$(\beta)_2$: $\beta(t,a,x)\geqslant 0$, a. e. in $(Q\times\Omega)$,

$(\beta)_3$: $\sup\limits_{(t,x)\in(0,T)\times\Omega}\displaystyle\int_0^A[\beta^2(t,a,x)+|\nabla\beta|^2(t,a,x)]\mathrm{d}a\leqslant C_1<+\infty$,

$(l)_0$: $l_0\in L^2((0,A);H^1(\Omega))$, $l_0(a,x)\geqslant 0$, a. e. in $(0,A)\times\Omega$.

The "obstacle" $\psi$ is a regular function, more precisely, $\psi$ satisfies:

$$(\psi)_1 \quad \begin{cases} \psi \in L^2(Q;H^2(\Omega)), & \\ \dfrac{\partial\psi}{\partial\eta}\geqslant 0, & \text{on } Q\times\partial\Omega, \\ (\partial_t+\partial_a)\psi\in L^2(Q\times\Omega), & \end{cases}$$

$$(\psi)_2 \quad \begin{cases} \psi(t,a,x)\geqslant 0, & \text{a. e. in } Q\times\Omega, \\ \psi(0,a,x)\leqslant l_0(a,x), & \text{a. e. in } (0,A)\times\Omega, \\ \psi(t,0,x)\leqslant \displaystyle\int_0^A\beta(t,a,x)\psi(t,a,x)\mathrm{d}a, & \text{a. e. in } (0,T)\times\Omega. \end{cases}$$

Let

$$\varphi(r) = \begin{cases} 0, & \text{for } r<0, \\ R^+, & \text{for } r=0. \end{cases}$$

(1) can be written as:

$$\begin{cases} \dfrac{\partial l}{\partial t}+\dfrac{\partial l}{\partial a}+\mu l-\Delta_x l+\varphi(l-\psi)\ni 0, & (t,a,x)\in Q\times\Omega, \\ \dfrac{\partial l}{\partial\eta}\geqslant 0, & \text{a. e. in } (0,A)\times\partial\Omega, \\ l(0,a,x)=l_0(a,x), & (a,x)\in(0,A)\times\Omega, \\ l(t,0,x)=\displaystyle\int_0^A\beta(t,a,x)l(t,a,x)\mathrm{d}a, & (t,a)\in(0,T)\times\Omega. \end{cases} \tag{2}$$

**Lemma 1. 1**[3] Under the above hypotheses, the equation (1) (or (2)) admits a unique solution $l \in L^2(Q; H^1(\Omega))$ and $\frac{\partial l}{\partial t}+\frac{\partial l}{\partial a}+\mu l \in L^2(Q \times \Omega)$. Other properties about the solution were presented in [3].

## 2 Optimal Birth Control Problems

In this section, we consider several optimal problems by taking the rate of fertility $\beta$ as a boundary control.

### 2.1 Maximizing the Total Birth Quantities

First, we study

($P_1$) Maximize

$$\int_0^T\int_\Omega l(t,0,x)\mathrm{d}x\mathrm{d}t = \int_0^T\int_0^A\int_\Omega \beta(t,a,x)l(t,a,x)\mathrm{d}x\mathrm{d}a\mathrm{d}t,$$

subject to (1) (or (2)), w. r. t. $\beta$.

Problem ($P_1$) means that under the constraint "$l \leqslant \psi$" one maximizes the total birth quantities.

Denoted by $U_{ad}$ the admissible set

$$U_{ad} = \left\{ \beta \middle| \begin{array}{l} \beta \in W^{1,\infty}(Q \times \Omega), \quad 0 \leqslant \beta \leqslant \beta_1, \text{ a. e. in } Q \times \Omega \\ |\beta|_{W^{1,\infty}} \leqslant \beta_2, \quad \psi(t,0,x) \geqslant \int_0^A \beta(t,a,x)\psi(t,a,x)\mathrm{d}a, \end{array} \right\}$$

where $\beta_1$, $\beta_2$ are constants, $\beta_1 < \beta_2$.

In order to obtain the existence of ($P_1$), we need additional assumptions:

$(\Omega_1)$: $\Omega$ is a bounded domain of $R^3$ such that $(0,T)\times(0,A)\times\Omega$ has the cone property.

$(\psi)_3$: there exists a small constant $\delta$ such that $0<\delta<\beta_1$ and

$$\psi(t,0,x) \geqslant \delta\int_0^A \psi \mathrm{d}a, \quad \text{a. e. } (t,x) \in (0,T) \times \Omega.$$

It follows from $(\Omega)_1$ and $(\psi)_3$ and the Sobolev's imbedding theorem that $U_{ad}$ is a non-empty convex compact set of $L^\infty(Q\times\Omega)$.

$U_{ad}$ can be expressed as the intersection of three sets in $L^\infty(Q\times\Omega)$, i. e.

$$U_{ad} = K_1 \cap K_2 \cap K_3, \tag{3}$$

where

$$K_1 = \{\beta | \beta \in L^\infty(Q \times \Omega)\ 0 \leqslant \beta \leqslant \beta_1, \text{ a. e. in } Q \times \Omega\},$$
$$K_2 = \{\beta | \beta \in W^{1,\infty}(Q \times \Omega),\ |\beta|_{W^{1,\infty}} \leqslant \beta_2\},$$
$$K_3 = \{\beta | \beta \in L^\infty(Q \times \Omega),\ \psi(t,0,x) \geqslant \int_0^A \beta(t,a,x)\psi(t,a,x)\mathrm{d}a\}.$$

It is clear that $K_i (i=1,2,3)$ is a closed set of $L^\infty(Q\times\Omega)$.

**Lemma 2.1** Denote by $T_K(u)$ the tangent cone to $K$ at $u$ in $K$ and by $N_K(u)$ the normal cone to $K$ at $u$, we have

$$T_{U_{ad}}(\beta) = T_{K_1}(\beta) \cap T_{K_2}(\beta) \cap T_{K_3}(\beta); \text{ for any } \beta \in K$$

and

$$N_{U_{ad}}(\beta) \supset N_{K_1}(\beta) \cup N_{K_2}(\beta) \cup N_{K_3}(\beta).$$

**Proof** We have by $(\psi)_3$, for any $v_i \in \frac{\delta}{2}B(i=1,2,3)$

$$(K_1 - v_1) \cap (K_2 - v_2) \cap (K_3 - v_3) \neq \varnothing,$$

where $B$ is a open unit ball of $L^\infty(Q\times\Omega)$. So the result follows[13].

**Theorem 2.1** Problem ($P_1$) has at least one optimal pair.

**Proof** Let $d=\sup\{\int_0^T\int_\Omega l(t,0,x)\mathrm{d}x\mathrm{d}t,\ \beta\in U_{ad},\ (\beta,l)$ satisfies (1)$\}$ and $\{(\beta_n,l_n)\}_n$ be a sequence of pairs such that

$$d-\frac{1}{n}<\int_0^T\int_0^A\int_\Omega \beta_n l_n \mathrm{d}x\mathrm{d}a\mathrm{d}t,$$

where $l_n$ is the solutions of (2) w. r. t. $\beta_n\in K$. Since $K$ is a compact set of $L^\infty(Q\times\Omega)$, there exists a subsequence of $\{\beta_n\}$, (also denoted by $\{\beta_n\}$), such that

$$\beta_n \to \beta^* \text{ in } L^\infty(Q\times\Omega), \text{ as } n\to\infty. \tag{4}$$

It follows from (2) that

$$\int_0^T\int_0^A <\frac{\partial(l_n-l_m)}{\partial t}+\frac{\partial(l_n-l_m)}{\partial a}+\mu(l_n-l_m),l_n-l_m>\mathrm{d}a\mathrm{d}t$$
$$+\int_0^T\int_0^A\int_\Omega |\nabla_x(l_n-l_m)|^2\mathrm{d}x\mathrm{d}a\mathrm{d}t=0,$$

here $\langle\cdot,\cdot\rangle$ denotes the inner product in $L_2(\Omega)$.

Hence

$$\frac{1}{2}\int_0^A\int_\Omega |l_n(t,a,x)-l_m(t,a,x)|^2\mathrm{d}x\mathrm{d}a$$
$$\leqslant\frac{1}{2}\int_0^t\int_\Omega |l_n(s,0,x)-l_m(s,0,x)|^2\mathrm{d}x\mathrm{d}s$$
$$=\frac{1}{2}\int_0^t\int_\Omega\Big[\int_0^A\beta_n l_n\mathrm{d}a-\int_0^A\beta_m l_m\mathrm{d}a\Big]^2\mathrm{d}x\mathrm{d}s$$
$$\leqslant C\Big[\int_0^t\int_0^A\int_\Omega |l_n-l_m|^2\mathrm{d}x\mathrm{d}a\mathrm{d}s+\int_0^t\int_0^A\int_\Omega|\beta_n-\beta_m|^2\mathrm{d}x\mathrm{d}a\mathrm{d}s\Big].$$

Using Gronwall's inequality, we obtain

$$l_n\to l^* \text{ in } C([0,T];X), \text{ as } n\to\infty,$$

where $X=L_2((0,A)\times\Omega)$. Now we prove that $l^*$ is a variational solution of (1). Indeed, multiplying (2) by $l_n$ and integrating by parts, we get

$$\frac{1}{2}\|l_n(T)\|^2+\frac{1}{2}\int_0^T\int_\Omega l_n^2(t,A,x)\mathrm{d}x\mathrm{d}t+\int_0^T\int_0^A\int_\Omega(\mu l_n^2+|\nabla_x l_n|^2)\mathrm{d}x\mathrm{d}a\mathrm{d}t$$
$$\leqslant\frac{1}{2}\int_0^T\int_\Omega\Big[\int_0^A\beta_n l_n\mathrm{d}a\Big]^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\|l_0\|^2, \tag{5}$$

where $\|\cdot\|$ is the norm of $X$.

From (5) we deduce that $\{(\partial_t+\partial_a)l_n+\mu l_n\}$ is bounded in $L^2((0,T)\times(0,A);(H^1(\Omega))^*)$, so we can extract a subsquence $\{l_k\}$ such that for $k\to\infty$,

$$l_k\to l^* \text{ in } L^2((0,T)\times(0,A);H^1(\Omega)),$$
$$(\partial_t+\partial_a)l_k+\mu l_k\to h \text{ in } L^2((0,T)\times(0,A);(H^1(\Omega))^*).$$

Since $\{(\partial_t+\partial_a)l_k+\mu l_k\}$ converges to $(\partial_t+\partial_a)l^*+\mu l^*$, so $h=(\partial_t+\partial_a)l^*+\mu l^*$. We may conclude that $l^*$ is a variational solution of (1), and

$$\int_0^T\int_0^A\int_\Omega \beta^* l^* \mathrm{d}x\mathrm{d}a\mathrm{d}t = d = \max_{\beta\in U_{ad}}\int_0^T\int_0^A\int_\Omega \beta l \mathrm{d}x\mathrm{d}a\mathrm{d}t.$$

That is, $(\beta^*, l^*)$ is an optimal pair of problem $(P_1)$.

To obtain necessary conditions of problem $(P_1)$, we first approximate $(P_1)$ by the following problems:

$(P_1^\varepsilon)$ Maximize

$$\int_0^T\int_0^A\int_\Omega \beta l_\varepsilon \mathrm{d}x\mathrm{d}a\mathrm{d}t - \frac{1}{2}\|\beta - \beta^*\|^2_{L^2((0,T)\times(0,A)\times\Omega)},$$

subject to

$$\begin{cases}\dfrac{\partial l_\varepsilon}{\partial t} + \dfrac{\partial l_\varepsilon}{\partial a} + \mu l_\varepsilon - \Delta_x l_\varepsilon + \Phi_\varepsilon(l_\varepsilon - \psi) = 0 \text{ in } (O,T)\times(0,A)\times\Omega, \\ \dfrac{\partial l_\varepsilon}{\partial \eta} = 0 \text{ in } (0,T)\times(0,A)\times\partial\Omega, \\ l_\varepsilon(t,0,x) = \displaystyle\int_0^A \beta(t,a,x) l_\varepsilon(t,a,x)\mathrm{d}a \text{ in } (0,T)\times\Omega, \\ l_\varepsilon(0,a,x) = l_0(a,x) \text{ in } (0,A)\times\Omega,\end{cases} \tag{6}$$

$$\varphi_\varepsilon(r) = \begin{cases} 0, & \text{for } r < 0, \\ \dfrac{1}{2\varepsilon^2} r^2, & \text{for } r\in[0,\varepsilon), \\ \dfrac{1}{\varepsilon} - \dfrac{1}{2}, & \text{for } r \geq \varepsilon,\end{cases} \tag{7}$$

where $\beta^*$ is an optimal solution of problem $(P_1)$.

It is clear that $(P_1^\varepsilon)$ has at least one optimal pair, denoted by $(\beta_\varepsilon^*, l_\varepsilon^*)$, for any $\varepsilon > 0$. Therefore, for any $\beta \in T_{U_{ad}}(\beta_\varepsilon^*)$, we get

$$\begin{aligned}&\int_0^T\int_0^A\int_\Omega \beta_\varepsilon^* l_\varepsilon^* \mathrm{d}x\mathrm{d}a\mathrm{d}t - \frac{1}{2}\|\beta_\varepsilon^* - \beta^*\|^2_{L^2((0,T)\times(0,A)\times\Omega)} \\ \geq &\int_0^T\int_0^A\int_\Omega (\beta_\varepsilon^* + \delta\beta) l_\varepsilon^\delta \mathrm{d}x\mathrm{d}a\mathrm{d}t - \frac{1}{2}\|\beta_\varepsilon^* + \delta\beta - \beta^*\|^2_{L^2((0,T)\times(0,A)\times\Omega)}\end{aligned} \tag{8}$$

for any $\delta$, $0<\delta<\delta_0$, $\delta_0$ is a constant,
where $l_\varepsilon^\delta$ is the solution of (6) w. r. t. $\beta_\varepsilon^* + \delta\beta$.

Denote by $z^\varepsilon = \mathrm{d}_\beta l_\varepsilon^{\beta_\varepsilon^*}(\beta)$ the Gateaux differential of $l_\varepsilon$ w. r. t. $\beta$ at $\beta_\varepsilon^*$, we have

$$\begin{cases}\dfrac{\partial z^\varepsilon}{\partial t} + \dfrac{\partial z^\varepsilon}{\partial a} + \mu z^\varepsilon - \Delta_x z^\varepsilon + \dot{\varphi}_\varepsilon(l_\varepsilon^* - \psi) z^\varepsilon = 0, & \text{a. e. in } Q\times\Omega, \\ \dfrac{\partial z^\varepsilon}{\partial \eta} = 0, & \text{a. e. in } Q\times\partial\Omega, \\ z^\varepsilon(t,0,x) = \displaystyle\int_0^A \beta_\varepsilon^* z^\varepsilon \mathrm{d}a + \int_0^A \beta l_\varepsilon^* \mathrm{d}a, & \text{a. e. in } (0,T)\times\Omega, \\ z^\varepsilon(0,a,x) = 0, & \text{a. e. in } (0,A)\times\Omega.\end{cases} \tag{9}$$

From (8)

$$\int_0^T\int_0^A\int_\Omega (\beta_\varepsilon^* z^\varepsilon + \beta l_\varepsilon^*)\mathrm{d}x\mathrm{d}a\mathrm{d}t - \int_0^T \langle \beta, \beta_\varepsilon^* - \beta^* \rangle \leq 0. \tag{10}$$

Introduce the dual system of (9).

$$\begin{cases} \dfrac{\partial p_\varepsilon}{\partial t} + \dfrac{\partial p_\varepsilon}{\partial a} - \mu p_\varepsilon + \Delta_x p_\varepsilon - \dot{\varphi}_\varepsilon(l_\varepsilon^* - \psi) p_\varepsilon & \\ = \beta_\varepsilon^* (1 - p_\varepsilon(t,0,x)), & \text{a. e. in } Q \times \Omega, \\ \dfrac{\partial p_\varepsilon}{\partial \eta} = 0, & \text{a. e. in } Q \times \partial\Omega, \\ p_\varepsilon(T,a,x) = 0, & \text{a. e. in } (0,A) \times \Omega. \end{cases} \tag{11}$$

**Lemma 2.2** For any $\varepsilon > 0$, there exists a unique solution $p \in L^2(Q, H^1(\Omega))$ for (11).

**Proof** For any $\lambda > 0$, let $\tilde{p}_\varepsilon = e^{\lambda t} p_\varepsilon$, then $\tilde{p}_\varepsilon$ satisfies:

$$\begin{cases} \dfrac{\partial \tilde{p}_\varepsilon}{\partial t} + \dfrac{\partial \tilde{p}_\varepsilon}{\partial a} - (\lambda + \mu)\tilde{p}_\varepsilon + \Delta_x \tilde{p}_\varepsilon - \dot{\varphi}_\varepsilon(l_\varepsilon^* - \psi)\tilde{p}_\varepsilon & \\ = \beta_\varepsilon^* e^{\lambda t} - \beta_\varepsilon^* \tilde{p}_\varepsilon(t,0,x), & \text{in } Q \times \Omega, \\ \dfrac{\partial \tilde{p}_\varepsilon}{\partial \eta} = 0, & \text{a. e. in } Q \times \partial\Omega, \\ \tilde{p}_\varepsilon(T,a,x) = 0, & \text{a. e. in } (0,A) \times \Omega. \end{cases} \tag{12}$$

We only need to prove that there exists a unique solution $\tilde{p}_\varepsilon$ that satisfies (12).

For any $d \in \mathscr{H} = \{d \mid d \in L^2(Q, H^1(\Omega)), \dfrac{\partial d}{\partial t} + \dfrac{\partial d}{\partial a} \in L^2(Q, (H^1(\Omega))^*)\}$, define the map $H(d): \mathscr{H} \to L^2(Q \times \Omega)$ by

$$H(d)(t,a,x) = \mu(t,a,x)d(t,a,x) + \dot{\varphi}_\varepsilon(l_\varepsilon^* - \psi)d(t,a,x) - \beta_\varepsilon^*(t,a,x)d(t,0,x).$$

We have by the "trace theorem"[4] $d(\cdot, 0, \cdot) \in L^2((0,T) \times \Omega)$ so that $H(d) \in L^2(Q \times \Omega)$, and hence $H \in \mathscr{L}(\mathscr{H}, L^2(Q \times \Omega))$.

Consider the auxiliary problem, for any $h \in L^2(Q \times \Omega)$,

$$\begin{cases} \dfrac{\partial \tilde{p}}{\partial t} + \dfrac{\partial \tilde{p}}{\partial a} - \lambda \tilde{p} + \Delta_x \tilde{p} = \beta_\varepsilon^* e^{\lambda t} + h, & \text{in } Q \times \Omega, \\ \dfrac{\partial \tilde{p}}{\partial \eta} = 0, & \text{a. e. in } Q \times \partial\Omega, \\ \tilde{p}(T,a,x) = 0, & \text{a. e. in } (0,A) \times \Omega. \end{cases} \tag{13}$$

The existence of solution of (13) can be shown by the Galerkin method[11].

For any $h \in L^2(Q \times \Omega)$, $S(h)$ denotes the solultion of (13), that is, $S$ is the solution map of (13) from $L^2(Q \times \Omega)$ to $\mathscr{H}$.

Since $\lambda$ is arbitrary, we can deduce that the composit operator $SH$ from $\mathscr{H}$ to itself is strictly constracting. So, there exists a unique fixed point $\tilde{p}_\varepsilon \in \mathscr{H}$ such that $\tilde{p}_\varepsilon = SH(\tilde{p}_\varepsilon)$, i. e. $\tilde{p}_\varepsilon$ is the solution of (11). This complete the proof.

By (9) and (11), and noticing $p_\varepsilon(t,A,x) = 0$[13], we have:

$$\int_0^T \int_0^A \int_\Omega \beta_\varepsilon^* z^\varepsilon \mathrm{d}x\mathrm{d}a\mathrm{d}t$$

$$= \int_0^T \int_0^A \int_\Omega \left[ \frac{\partial p_\varepsilon}{\partial t} + \frac{\partial p_\varepsilon}{\partial a} - \mu p_\varepsilon + \Delta_x p_\varepsilon - \dot{\varphi}_\varepsilon(l_\varepsilon^* - \psi) p_\varepsilon + \beta_\varepsilon^* p_\varepsilon(t,0,x) \right] z^\varepsilon \mathrm{d}x\mathrm{d}a\mathrm{d}t$$

$$= - \int_0^T \int_\Omega p_\varepsilon(t,0,x) z^\varepsilon(t,0,x) \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^A \int_\Omega \beta_\varepsilon^*(t,a,x) p_\varepsilon(t,0,x) z^\varepsilon(t,a,x) \mathrm{d}x\mathrm{d}a\mathrm{d}t$$

$$-\int_0^T\int_0^A\int_\Omega p_\varepsilon\left[\frac{\partial z^\varepsilon}{\partial t}+\frac{\partial z^\varepsilon}{\partial a}+\mu z^\varepsilon-\Delta_x z^\varepsilon+\dot{\varphi}_\varepsilon(l_\varepsilon^*-\psi)z\right]\mathrm{d}x\mathrm{d}a\mathrm{d}t$$
$$=-\int_0^T\int_\Omega p_\varepsilon(t,0,x)\Big[\int_0^A\beta_\varepsilon^* z^\varepsilon \mathrm{d}a+\int_0^A\beta l_\varepsilon^*\mathrm{d}a\Big]\mathrm{d}x\mathrm{d}t+\int_0^T\int_0^A\int_\Omega\beta_\varepsilon^* p_\varepsilon(t,0,x)z^\varepsilon(t,a,x)\mathrm{d}x\mathrm{d}a\mathrm{d}t$$
$$=-\int_0^T\int_0^A\int_\Omega p_\varepsilon(t,0,x)\beta l_\varepsilon^*\mathrm{d}x\mathrm{d}a\mathrm{d}t.$$

It follows from (10) that

$$\int_0^T\int_0^A\int_\Omega[1-p_\varepsilon(t,0,x)]\beta l_\varepsilon^*\mathrm{d}x\mathrm{d}a\mathrm{d}t-\int_0^T\langle\beta,\beta_\varepsilon^*-\beta^*\rangle\mathrm{d}t\leqslant 0 \tag{14}$$

for any $\beta\in T_{U_{ad}}(\beta_\varepsilon^*)$.

**Theorem 2.2** Let $\varepsilon\searrow 0$, then we have

$$\beta_\varepsilon^*\to\beta^* \text{ in } L^\infty(Q\times\Omega)\quad l_\varepsilon^*\to l^* \text{ in } L^2(Q\times\Omega),$$

where $(\beta^*,l^*)$ is the optimal pair of problem $(\mathrm{P}_1)$.

**Proof** Since $\{\beta_\varepsilon^*\}\subset U_{ad}$ and $U_{ad}$ is a compact subset of $L^\infty(Q\times\Omega)$, so there exists a subsequence, also denoted by $\{\beta_\varepsilon^*\}$, such that

$$\beta_\varepsilon^*\to\beta_1^* \text{ in } L^\infty(Q\times\Omega). \tag{15}$$

We can easily obtain that (similiar to [3])

$$l_\varepsilon^*\to l_1^* \text{ in } L^2(Q\times\Omega) \tag{16}$$

where $l_1^*$ is the solution of (2) corresponding to $\beta_1^*$.

Now let us prove $l_1^*=l^*$, $\beta_1^*=\beta^*$. Since $(\beta_\varepsilon^*, l_\varepsilon^*)$ is the optimal pair of $(P_1^\varepsilon)$, so we have

$$\int_0^T\int_0^A\int_\Omega\beta_1^* l_1^*\mathrm{d}x\mathrm{d}a\mathrm{d}t-\frac{1}{2}\|\beta_\varepsilon^*-\beta^*\|^2_{L^2(Q\times\Omega)}\geqslant\int_0^T\int_0^A\int_\Omega\beta^* l_\varepsilon^{\beta^*}\mathrm{d}x\mathrm{d}a\mathrm{d}t, \tag{17}$$

where $l_\varepsilon^{\beta^*}$ is the solution of (6) corresponding to $\beta^*$.

Let $\varepsilon\searrow 0$ in (17), we have, by (15) and (16),

$$\int_0^T\int_0^A\int_\Omega\beta_1^* l_1^*\mathrm{d}x\mathrm{d}a\mathrm{d}t-\frac{1}{2}\|\beta_1^*-\beta^*\|^2_{L^2(Q\times\Omega)}\geqslant\int_0^T\int_0^A\int_\Omega\beta^* l^*\mathrm{d}x\mathrm{d}a\mathrm{d}t.$$

Noticing that $(\beta^*,l^*)$ is the optimal pair of $(\mathrm{P}_1)$, we have $\beta_1^*=\beta^*$, $l_1^*=l^*$. Q. E. D.

Arguing as in [12], we obtain the existence of $p\in L^2(Q,H^1(\Omega))$ such that,

$$\begin{cases}\left(\dfrac{\partial p}{\partial t}+\dfrac{\partial p}{\partial a}-\Delta_x p+\mu p+\beta^* p(t,0,x)\right)^a=\beta^*, & \text{a. e. in } [l^*<\psi],\\ p\left(\mu l^*-\Delta_x l^*+\dfrac{\partial l^*}{\partial t}+\dfrac{\partial l^*}{\partial a}\right)=0, & \text{a. e. in } [l^*=\psi],\\ \dfrac{\partial p}{\partial\eta}=0, & \text{a. e. in } Q\times\partial\Omega,\\ p(T,a,x)=0, & \text{a. e. in } (0,A)\times\Omega.\end{cases} \tag{18}$$

Let $\varepsilon\searrow 0$ in (14), we have

$$\int_0^T\int_0^A\int_\Omega[1-p(t,0,x)]\beta l^*\mathrm{d}x\mathrm{d}a\mathrm{d}t\leqslant 0 \tag{19}$$

for any $\beta\in T_{U_{ad}}(\beta^*)$.

That is
$$(1-p(\cdot,0,\cdot))l^*\in N_{U_{ad}}(\beta^*), \tag{20}$$

or equivalently

$$\int_0^T\int_0^A\int_\Omega \beta^* l^*(1-p(t,0,x))\mathrm{d}x\mathrm{d}a\mathrm{d}t = \max_{\beta\in U_{ad}}\int_0^T\int_0^A\int_\Omega \beta l^*(1-p(t,0,x))\mathrm{d}x\mathrm{d}a\mathrm{d}t. \tag{21}$$

Let $\widetilde{\beta}\in K_2\cap K_3$ such that

$$\int_0^T\int_0^A\int_\Omega \widetilde{\beta} l^*(1-p(t,0,x))\mathrm{d}x\mathrm{d}a\mathrm{d}t = \max_{\beta\in K_2\cap K_3}\int_0^T\int_0^A\int_\Omega \beta l^*(1-p(t,0,x))\mathrm{d}x\mathrm{d}a\mathrm{d}t. \tag{22}$$

We have the following maximum principle:

**Theorem 2.3** If $(\beta^*, l^*)$ is an optimal pair for problem $(P_1)$, then there exists a solution $p$ of (18), $p\in L^2(Q,H^1(\Omega))$, and $\beta^*$ is an "bang-bang control",

$$\beta^*(t,a,x) = \begin{cases} 0, & \text{for } 1 < p(t,0,x), \\ \inf\{\widetilde{\beta}(t,a,x),\beta_1\}, & \text{for } 1 > p(t,0,x). \end{cases}$$

## 2.2 Optimal Population Distribution at Time $T$

This problem is to best apprroximate a given age distribution by birth control.

(P2) Minimize

$$\frac{1}{2}\int_0^A\int_\Omega (l(T,a,x)-\bar{l}(a,x))^2\mathrm{d}x\mathrm{d}a,$$

where $\beta\in U_{ad}$, $l$ satisfies (2). i. e.

$$\begin{cases} \dfrac{\partial l}{\partial t}+\dfrac{\partial l}{\partial a}+\mu l-\Delta_x l+\varphi(l-\psi)\ni 0, & (t,a,x)\in Q\times\Omega, \\ \dfrac{\partial l}{\partial \eta}=0, & \text{a. e. in } (0,A)\times\partial\Omega, \\ l(0,a,x)=l_0(a,x), & (a,x)\in(0,A)\times\Omega, \\ l(t,0,x)=\displaystyle\int_0^A \beta(t,a,x)l(t,a,x)\mathrm{d}a, & (t,a)\in(0,T)\times\Omega, \end{cases} \tag{23}$$

where the given distribution $\bar{l}\in L^2((0,A)\times\Omega)$.

It is clear that problem $(P_2)$ has at least one optimal pair. Let $(\beta^*, l^*)$ be one optimal pair, and introduce the smooth problem. for $\varepsilon>0$,

$(P_2^\varepsilon)$ Minimize

$$\frac{1}{2}\int_0^T\int_0^A\int_\Omega|\beta-\beta^*|^2\mathrm{d}x\mathrm{d}a\mathrm{d}t+\frac{1}{2}\int_0^A\int_\Omega(l_\varepsilon(T,a,x)-\bar{l}(a,x))^2\mathrm{d}x\mathrm{d}a,$$

where $\beta\in U_{ad}$, $l_\varepsilon$ satisfies (6).

Let $(\beta_\varepsilon^*, l_\varepsilon^*)$ be one optimal pair for problem $(P_2^\varepsilon)$. Therefore, for any $\beta\in T_{U_{ad}}(\beta_\varepsilon^*)$, we have

$$\frac{1}{2}\|\beta_\varepsilon^*-\beta\|^2_{L^2(Q\times\Omega)}+\frac{1}{2}\int_0^A\int_\Omega(l_\varepsilon^*-\bar{l}(a,x))^2\mathrm{d}x\mathrm{d}a$$

$$\leqslant\frac{1}{2}\|(\beta_\varepsilon^*+\delta\beta)-\beta^*\|^2_{L^2(Q\times\Omega)}+\frac{1}{2}\int_0^A\int_\Omega(l_\varepsilon^\delta(T,a,x)-\bar{l}(a,x))^2\mathrm{d}x\mathrm{d}a, \tag{24}$$

where $l_\varepsilon^\delta$ is the solution of (6) corresponding to $\beta_\varepsilon^*+\delta\beta$, $\delta>0$.

Denote by $z^*=d_\beta l_\varepsilon^{\beta_\varepsilon^*}(\beta)$ the Gateaux differential of $l_\varepsilon$ w. r. t. $\beta$ at $\beta_\varepsilon^*$, we then have (9).

From (24) and (9), it is clear that

$$\int_0^A\int_\Omega z^\varepsilon(T,a,x)(l_\varepsilon^* - \bar{l}(a,x))\mathrm{d}x\mathrm{d}a + \int_0^T\langle\beta,\beta_\varepsilon^* - \beta^*\rangle\mathrm{d}t \geqslant 0 \tag{25}$$

for any $\beta\in T_{U_{ad}}(\beta_\varepsilon^*)$.

Introduce the dual system of (9).

$$\begin{cases}\dfrac{\partial p_\varepsilon}{\partial t} + \dfrac{\partial p_\varepsilon}{\partial a} - \mu p_\varepsilon + \Delta_x p_\varepsilon - \dot{\varphi}_\varepsilon(l^* - \psi)p_\varepsilon + \beta_\varepsilon^* p_\varepsilon(t,0,x) = 0, & \text{in } Q\times\Omega,\\ \dfrac{\partial p_\varepsilon}{\partial \eta} = 0, & \text{a. e. in } Q\times\partial\Omega,\\ p_\varepsilon(T,a,x) + l_\varepsilon^*(T,a,x) = \bar{l}(a,x), & \text{a. e. in } (0,A)\times\Omega.\end{cases} \tag{26}$$

Similiar to lemma 2. 2, we can deduce that there is a unique solution $p_\varepsilon\in L^2(Q,H^1(\Omega))$ for the equation (26).

From (25), (26) and (9), similar to section 2. 1 we can deduce that

$$\int_0^A\int_\Omega z^\varepsilon(T,a,x)(l_\varepsilon^*(T,a,x) - \bar{l}(a,x))\mathrm{d}x\mathrm{d}a = -\int_0^T\int_0^A\int_\Omega p_\varepsilon(t,0,x)\beta l_\varepsilon^*\mathrm{d}x\mathrm{d}a\mathrm{d}t.$$

From (25), we have

$$-\int_0^T\int_0^A\int_\Omega p_\varepsilon(t,0,x)\beta l_\varepsilon^*\mathrm{d}x\mathrm{d}a\mathrm{d}t + \int_0^T\langle\beta,\beta_\varepsilon^* - \beta\rangle\mathrm{d}t \geqslant 0 \tag{27}$$

for any $\beta\in T_U(\beta_\varepsilon^*)$.

Similiar to Theorem 2. 2, if $\varepsilon\searrow 0$, then we have

$$\beta_\varepsilon^* \to \beta^* \text{ in } L^\infty(Q\times\Omega), \tag{28}$$

$$l_\varepsilon^* \to l^* \text{ in } L^2(Q\times\Omega). \tag{29}$$

Arguing as [12], we obtain

$$p_\varepsilon \to p \text{ in } L^2(Q,\ H^1(\Omega)), \tag{30}$$

$$\frac{\partial p_\varepsilon}{\partial t} + \frac{\partial p_\varepsilon}{\partial a} - \mu p_\varepsilon \to \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu p \text{ in } L^2((0,T);\ H^1(\Omega)^*) \tag{31}$$

as $\varepsilon\to 0$, thus we have

$$p_\varepsilon(\cdot,0,\cdot) \to p(\cdot,0,\cdot) \text{ in } L^2((0,T)\times\Omega), \tag{32}$$

and $p\in L^2(Q;H^1(\Omega))$ satisfies:

$$\begin{cases}\left(\dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - \mu p + \Delta_x p + \beta^* p(t,0,x)\right)^a = 0, & \text{a. e. in } [l^* < \psi],\\ p\left(\mu l^* - \Delta_x l^* + \dfrac{\partial l^*}{\partial t} + \dfrac{\partial l^*}{\partial a}\right) = 0, & \text{a. e. in } [l^* = \psi],\\ \dfrac{\partial p}{\partial \eta} = 0, & \text{a. e. in } Q\times\partial\Omega,\\ p(T,a,x) + l^*(T,a,x) = \bar{l}(a,x), & \text{a. e. in } (0,A)\times\Omega.\end{cases} \tag{33}$$

Let $\varepsilon\searrow 0$ in (27), we have

$$\int_0^T\int_0^A\int_\Omega p(t,0,x)\beta l^*\mathrm{d}x\mathrm{d}a\mathrm{d}t \leqslant 0$$

for any $\beta\in T_{U_{ad}}(\beta^*)$.

That is
$$p(\cdot,0,\cdot)l^* \in N_{U_{ad}}(\beta^*), \tag{34}$$
or equivalently
$$\int_0^T\int_0^A\int_\Omega \beta^* l^* p(t,0,x)\mathrm{d}x\mathrm{d}a\mathrm{d}t = \max_{\beta\in U_{ad}}\int_0^T\int_0^A\int_\Omega \beta l^* p(t,0,x)\mathrm{d}x\mathrm{d}a\mathrm{d}t. \tag{35}$$
Let $\widetilde{\beta}\in K_2\bigcap K_3$, such that
$$\int_0^T\int_0^A\int_\Omega \widetilde{\beta} l^* p(t,0,x)\mathrm{d}x\mathrm{d}a\mathrm{d}t = \max_{\beta\in K_2\bigcap K_3}\int_0^T\int_0^A\int_\Omega \beta l^* p(t,0,x)\mathrm{d}x\mathrm{d}a\mathrm{d}t. \tag{36}$$
We have

**Theorem 2.4** if $(\beta^*,l^*)$ is an optimal pair for problem $(P_2)$, then there exists a solution $p$ of (33), $p\in L^2(Q,H^1(\Omega))$ and the optimal control $\beta^*$ is a "bang-bang control",
$$\beta^*(t,a,x) = \begin{cases} 0, & \text{for } p(t,0,x) < 0, \\ \inf\{\widetilde{\beta}(t,a,x),\beta_1\}, & \text{for } p(t,0,x) > 0. \end{cases} \tag{37}$$

**Remark 2.1** We may also consider the general optimal birth control problem:

$(P_3)$ Minimize
$$\int_0^T g(l(t))\mathrm{d}t + \varphi_0(l(T)),$$
where $\beta\in U_{ad}$, $l$ satisfies (2); $g$, $\varphi_0: X=L^2((0,A)\times\Omega)\to R$ are convex proper and l. s. c. function

Similiar to section 2.1~2.2, we have

**Theorem 2.5** If $(\beta^*,l^*)$ is an optimal pair for $(P_3)$, then there exists $a$ $p$ such that
$$\begin{cases} \left(\dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - \mu p + \Delta_x p + \beta^* p(t,0,x)\right)^a \in \partial g(l^*), & \text{a. e. in } [l^* < \psi], \\ p\left(\mu l^* - \Delta_x l^* + \dfrac{\partial l^*}{\partial t} + \dfrac{\partial l^*}{\partial a}\right) = 0, & \text{a. e. in } [l^* = \psi], \\ \dfrac{\partial p}{\partial \eta} = 0, & \text{a. e. in } Q\times\partial\Omega, \\ p(T,a,x) + \partial\varphi_0(l^*(T,a,x)) \ni 0, & \text{a. e. in } (0,A)\times\Omega, \end{cases} \tag{38}$$
and the optimal control $\beta^*$ is a "bang-bang" control
$$\beta^*(t,a,x) = \begin{cases} 0, & \text{for } p(t,0,x) < 0, \\ \inf\{\widetilde{\beta}(t,a,x),\beta_1\}, & \text{for } p(t,0,x) > 0, \end{cases} \tag{39}$$
wehre $\widetilde{\beta}$ satisfies (36)

The proof of theorem 2.5 is ommited.

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# 带有外界约束的人口扩散系统的最优出生率控制

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**摘要**：本文考虑单边约束的依赖于年龄的人口扩散系统，讨论其最优出生率(边界)控制，得到其最优控制的存在性的必要条件.

**关键词**：人口动态系统；最大值原理；最优控制

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