第12卷第5期
1995年10月
控制理论与应用
CONTROL THEORY AND APPLICATIONS
Vol. 12, No. 5
Oct., 1995

An Optimal Control Problem for a Class of Deterministic Systems *

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Abstract: In this paper, we obtain the maximum principle in optimal control problems for a class of deterministic forward and backward system applying Ekeland's variational principle. We also prove that the maximum condition not only is necessary but also is sufficient for a linear case.

Key words: forward and backward system; maximum principle; optimal control

1 Statement of the Problem and Our Main Result

In this paper, we consider the following optimal control problem. Minmizing a cost function

$$S(v(\cdot)) = h(x(T)) + \gamma(y(0)) \tag{1}$$

over $\mathscr{U}_{ad}$, subject to

$$\begin{cases} \dot{x} = f(x,v), \\ x(0) = x_0, \quad G_1(x(T)) = 0, \\ \dot{y} = g(x,y,v), \\ y(T) = y_T, \quad G_2(y(0)) = 0. \end{cases} \tag{2}$$

where

$$f:\ \mathrm{R}^n \times \mathrm{R}^k \to \mathrm{R}^n,$$
$$g:\ \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^k \to \mathrm{R}^m,$$
$$G_1:\ \mathrm{R}^n \to \mathrm{R}^{n_1}, \quad n_1 < n,$$
$$G_2:\ \mathrm{R}^m \to \mathrm{R}^{m_1}, \quad m_1 < m,$$
$$h:\ \mathrm{R}^n \to \mathrm{R}^1, \quad \gamma:\ \mathrm{R}^m \to \mathrm{R}^1.$$

and $\mathscr{U}_{ad}$ is the set of admissable controls defined by

$$\mathscr{U}_{ad} = \{v(\cdot) \in L^\infty(0,T) : v(t) \in U, \text{a. e. } t \in [0,T]\}.$$

U is a closed subset of R^k.

There are some works relevant to this problem. Pontryagin[2] discussed an optimal control problem with variable endpoint constraints applying a convex cone method. In our paper, we obtain the maximum principle applying a spike variation and Ekeland's variational

* This work was supported by the Chinese State Education Commission Foundation.
Manuscript received Jan. 26, 1994, revised Nov. 17, 1994.

rinciple, the transversality conditions we obtain are described more precisely.

For the above problem, we give our assumptions

H1) f, g, h, γ, G_1, and G_2 are continuous with respect to x, y, v, t; f, g, h, γ, G_1, and G_2 e continuously differentiable with respect to x, y.

H2) f_x, g_x and g_y are bounded.

We have the following results

Theorem 1 Suppose H1) and H2) hold. Let $(u(\cdot), x(\cdot), y(\cdot))$ be an optimal lution to our problem (1) and (2), $(p(\cdot), q(\cdot))$ be the corresponding solution of the following adjoint equation

$$\begin{cases} -\dot{p} = f_x^*(x,u)p + g_x^*(x,y,u)q, \\ p(T) = -(h_x^*(x(T))h_0 + G_{1x}^*(x(T))h_1), \\ -\dot{q} = g_y^*(x,y,u)q, \\ q(0) = \gamma_y^*(y(0))h_0 + G_{2y}^*(y(0))h_2, \end{cases} \tag{3}$$

ıen, the following maximum condition holds

$$H(x(t), y(t), u(t), p(t), q(t), t) = \max_{v\in U} H(x(t), y(t), v, p(t), q(t), t) \quad \text{a.e.} \quad t \in [0,T]. \tag{4}$$

here, $H(x,y,v,p,q,t) \triangleq \langle p, f(x,v)\rangle + \langle q, g(x,y,v)\rangle$ is the corresponding Hamiltonian ıction, $h_0 \in \mathrm{R}^1, h_1 \in \mathrm{R}^{n_1}$ and $h_2 \in \mathrm{R}^{m_1}$ are constant vectors with $\sum_{i=0}^{2} \| h_i \|^2 = 1$.

This paper is organized as follows. We give the proof of Theorem 1 in Section 2. In ction 3, we study the optimal control problem for another type of forward and backward stem and the corresponding maximum principle is given. We give a sufficient result for a ear system in the last section.

The Proof of Theorem 1

For the optimal control u(•), we define a spike control

$$u^\varepsilon(t) = \begin{cases} v, & \tau \leqslant t \leqslant \tau + \varepsilon, \\ u(t), & \text{otherwise}, \end{cases}$$

ere, $v \in U, \tau \in [0,T)$, $\varepsilon > 0$ is sufficiently small.

Let's consider the following system:

$$\begin{cases} \dot{x} = f(x,v), & x(0) = x_0, \\ \dot{y} = y(x,y,v), & g(T) = y_T. \end{cases} \tag{5}$$

denote the solution of (5) as $(x(t,v), y(t,v))$ and $(x^\varepsilon(\cdot), y^\varepsilon(\cdot)) \triangleq (x(t,u^\varepsilon), y(t,$). For convenience, we use the following notation in this paper:

$$f(u^\varepsilon) = f(x,u^\varepsilon), \quad f(u) = f(x,u),$$

$$g(u^\varepsilon) = g(x,y,u^\varepsilon), \quad g(u) = g(x,y,u), \text{etc.}$$

introduce the variational equation as follows

$$\delta\dot{x} = f_x(u)\delta x + f(u^\varepsilon) - f(u), \quad \delta x(0) = 0,$$

$$\delta \dot{y} = g_x(u)\delta x + g_y(u)\delta y + g(u^\varepsilon) - g(u), \quad \delta y(T) = 0$$

and have the following result. (6

Lemma 1 Suppose H1) and H2) hold. For δx and δy, we have the following estimations:

$$x^\varepsilon(t) = x(t) + \delta x(t) + o(\varepsilon), \quad \forall\, t \in [0,T], \quad (7$$

$$y^\varepsilon(t) = y(t) + \delta y(t) + o(\varepsilon), \quad \forall\, t \in [0,T]. \quad (8$$

Proof we first prove (7). From (5) and (6), we have

$$\begin{aligned} & x^\varepsilon(t) - x(t) - \delta x(t) \\ &= \int_0^t [f(x^\varepsilon,u^\varepsilon) - f(x,u^\varepsilon) - f_x(x,u)\delta x]\mathrm{d}s \\ &= \int_0^t \left[\int_0^1 (f_x(x + \lambda(x^\varepsilon - x),u^\varepsilon))\mathrm{d}\lambda (x^\varepsilon - x) - f_x(x,u)\delta x\right]\mathrm{d}s. \end{aligned}$$

Then, it follows

$$\begin{aligned} & |x^\varepsilon(t) - x(t) - \delta x(t)| \\ &\leqslant \int_0^t |f_x(x,u)|\,|x^\varepsilon - x - \delta x|\mathrm{d}s + \left|\int_0^t A^\varepsilon(x^\varepsilon - x)\mathrm{d}s\right|, \end{aligned}$$

with

$$A^\varepsilon = \int_0^1 (f_x(x + \lambda(x^\varepsilon - x),u^\varepsilon) - f_x(x,u))\mathrm{d}\lambda.$$

Applying Gronwall's inequality to the above relation, it yields that

$$|x^\varepsilon(t) - x(t) - \delta x(t)| \leqslant C\left|\int_0^T A^\varepsilon(x^\varepsilon - x)\mathrm{d}s\right| = o(\varepsilon), \quad t \in [0,T].$$

Then (7) is obtained. We can prove (8) similarly.

Now we give the proof of Theorem 1.

Proof of Theorem 1 We define a metric in $\mathscr{U}_{ad}$. For $v_1(\cdot), v_2(\cdot) \in \mathscr{U}_{ad}$, let

$$d(v_1(\cdot).v_2(\cdot)) \triangleq \mathrm{mes}\{t \in [0,T] : v_1(t) \neq v_2(t)\},$$

where, mes{ · } is the Lebesgue's measure. With this metric, $(\mathscr{U}_{ad}, d(\cdot,\cdot))$ is a complete metric space[1].

For any $v(\cdot) \in \mathscr{U}_{ad}$, we define the following cost function of system (5):

$$F_\varepsilon(v(\cdot)) = \{\|G_1(x(T;v))\|^2 + \|G_2(y(0;v))\|^2 + (h(x(T;v)) + \gamma(y(0;v)) - h(x(T)) - \gamma(y(0)) + \varepsilon)^2\}^{\frac{1}{2}}. \quad (9)$$

It can be proved that $F_\varepsilon : \mathscr{U}_{ad} \to \mathrm{R}^1$ is continuous, and

$$F_\varepsilon(v(\cdot)) \geqslant 0, \quad F_\varepsilon(u(\cdot)) = \varepsilon.$$

Obviously,

$$F_\varepsilon(u(\cdot)) \leqslant \inf_{v(\cdot)\in\mathscr{U}_{ad}} F_\varepsilon(v(\cdot)) + \varepsilon.$$

Then from Ekeland's variational principle, there exists $u_\varepsilon(\cdot) \in \mathscr{U}_{ad}$ such that

$$\begin{cases} \text{i)} \quad F_\varepsilon(u_\varepsilon(\cdot)) \leqslant F_\varepsilon(u(\cdot)) = \varepsilon, \\ \text{ii)} \quad d(u_\varepsilon(\cdot),u(\cdot)) \leqslant \sqrt{\varepsilon}, \\ \text{iii)} \quad F_\varepsilon(w(\cdot)) \geqslant F_\varepsilon(u_\varepsilon(\cdot)) - \sqrt{\varepsilon}\, d(w(\cdot),u_\varepsilon(\cdot)), \quad \forall\, w(\cdot) \in \mathscr{U}_{ad}. \end{cases} \quad (10)$$

We make a variational control of $u_\varepsilon(\cdot)$:

$$u_\varepsilon^\rho(t) = \begin{cases} v, & \tau \leqslant t \leqslant \tau + \rho, \\ u_\varepsilon(t), & \text{otherwise}, \end{cases}$$

where, $v \in U, \tau \in \rho[0,T), \rho > 0$ is sufficiently small. Then $u_\varepsilon^\rho(\cdot) \in \mathscr{U}_{ad}$, and

$$d(u_\varepsilon^\rho(\cdot), u_\varepsilon(\cdot)) \leqslant \rho.$$

It follows from (10) iii) that

$$F_\varepsilon(u_\varepsilon^\rho(\cdot)) - F_\varepsilon(u_\varepsilon(\cdot)) + \sqrt{\varepsilon}\,\rho \geqslant 0. \tag{11}$$

For notational simplification, we denote

$$x_\varepsilon^\rho(t) \triangleq x(t; u_\varepsilon^\rho), \quad x_\varepsilon(t) \triangleq x(t; u_\varepsilon).$$

Let $(\delta x_\varepsilon, \delta y_\varepsilon)$ be the solution of

$$\begin{cases} \delta \dot{x}_\varepsilon = f_x(x_\varepsilon, u_\varepsilon)\delta x_\varepsilon + f(x_\varepsilon, u_\varepsilon^\rho) - f(x_\varepsilon, u_\varepsilon), \\ \delta x_\varepsilon(0) = 0, \\ \delta \dot{y}_\varepsilon = g_x(x_\varepsilon, y_\varepsilon, u_\varepsilon)\delta x_\varepsilon + g_y(x_\varepsilon, y_\varepsilon, u_\varepsilon)\delta y_\varepsilon + g(x_\varepsilon, y_\varepsilon, u_\varepsilon^\rho) - g(x_\varepsilon, y_\varepsilon, u_\varepsilon), \\ \delta y_\varepsilon(T) = 0. \end{cases}$$

From Lemma 1, we have

$$x_\varepsilon^\rho(t) = x_\varepsilon(t) + \delta x_\varepsilon(t) + o(\rho),$$
$$y_\varepsilon^\rho(t) = y_\varepsilon(t) + \delta y_\varepsilon(t) + o(\rho).$$

Thus from (9) and the above relation, it can be derived that

$$\begin{aligned} &F_\varepsilon^2(u_\varepsilon^\rho(\cdot)) - F_\varepsilon^2(u_\varepsilon(\cdot)) \\ =& 2\langle G_{1x}(x_\varepsilon(T))\delta x_\varepsilon(T), G_1(x_\varepsilon(T))\rangle + 2\langle G_{2y}(y_\varepsilon(0))\delta y_\varepsilon(0), G_2(y_\varepsilon(0))\rangle \\ &+ 2\langle h_x(x_\varepsilon(T))\delta x_\varepsilon(T) + \gamma_y(y_\varepsilon(0))\delta y_\varepsilon(0), h(x_\varepsilon(T)) + \gamma(y_\varepsilon(0)) \\ &- h(x(T)) - \gamma(y(0)) + \varepsilon\rangle + o(\rho). \end{aligned} \tag{12}$$

Since

$$u_\varepsilon^\rho(\cdot) \to u_\varepsilon(\cdot), \quad \rho \to 0,$$
$$F_\varepsilon(u_\varepsilon^\rho(\cdot)) \to F_\varepsilon(u_\varepsilon(\cdot)), \quad \rho \to 0,$$

and

$$F_\varepsilon(u_\varepsilon(\cdot)) > 0,$$

from (11) and (12), it follows that

$$\begin{aligned} &\langle G_{1x}^*(x_\varepsilon(T))h_1^\varepsilon + h_x^*(x_\varepsilon(T))h_0^\varepsilon, \delta x_\varepsilon(T)\rangle \\ &+ \langle G_{2y}^*(y_\varepsilon(0))h_2^\varepsilon + \gamma_y^*(y_\varepsilon(0))h_0^\varepsilon, \delta y_\varepsilon(0)\rangle + o(\rho) + \rho\sqrt{\varepsilon} \geqslant 0, \end{aligned} \tag{13}$$

with

$$\begin{cases} h_0^\varepsilon = \dfrac{h(x_\varepsilon(T)) + \gamma(y_\varepsilon(0)) - h(x(T)) - \gamma(y(0)) + \varepsilon}{F_\varepsilon(u_\varepsilon(\cdot))}, \\ h_1^\varepsilon = \dfrac{G_1(x_\varepsilon(T))}{F_\varepsilon(u_\varepsilon(\cdot))}, \\ h_2^\varepsilon = \dfrac{G_2(y_\varepsilon(0))}{F_\varepsilon(u_\varepsilon(\cdot))}. \end{cases}$$

Let $(p_\varepsilon, q_\varepsilon)$ be the solution of

$$\begin{cases} -\dot{p}_\varepsilon = f_x^*(x_\varepsilon, u_\varepsilon)p_\varepsilon + g_x^*(x_\varepsilon, y_\varepsilon, u_\varepsilon)q_\varepsilon, \\ p_\varepsilon(T) = -(G_{1x}^*(x_\varepsilon(T))h_1^\varepsilon + h_x^*(x_\varepsilon(T))h_0^\varepsilon), \\ -\dot{q}_\varepsilon = g_y^*(x_\varepsilon, y_\varepsilon, u_\varepsilon)q_\varepsilon, \\ q_\varepsilon(0) = G_{2y}^*(y_\varepsilon(0))h_2^\varepsilon + \gamma_y^*(y_\varepsilon(0))h_0^\varepsilon. \end{cases}$$

Then from (13), we have

$$\int_0^T [H(x_\varepsilon, y_\varepsilon, u_\varepsilon, p_\varepsilon, q_\varepsilon, t) - H(x_\varepsilon, y_\varepsilon, u_\varepsilon^\rho, p_\varepsilon, q_\varepsilon, t)]dt + o(\rho) + \rho\sqrt{\varepsilon} \geqslant 0, \tag{14}$$

where, $H(x, y, v, p, q, t) \triangleq \langle p, f(x, v)\rangle + \langle q, g(x, y, v)\rangle$.

Multipling by $\frac{1}{\rho}$ on both sides of (14) and letting $\rho \to 0$, it follows that

$$H(x_\varepsilon(t), y_\varepsilon(t), u_\varepsilon(t), p_\varepsilon(t), q_\varepsilon(t), t) - H(x_\varepsilon(t), y_\varepsilon(t), v, p_\varepsilon(t), q_\varepsilon(t), t) + \sqrt{\varepsilon} \geqslant 0, \quad \text{a.e.} \quad t \in [0, T]. \tag{15}$$

Since $\sum_{i=0}^{2} \| h_i^\varepsilon \|^2 = 1$, there exists a convergent subsequence of $\{h_i^\varepsilon\}$ such that

$$h_i^\varepsilon \to h_i, \quad \varepsilon \to 0, \quad i = 0, 1, 2,$$

with $\sum_{i=0}^{2} \| h_i \|^2 = 1$.

From (10) ii), it yields

$$u_\varepsilon(\cdot) \to u(\cdot), \quad \varepsilon \to 0,$$

so we have

$$(x_\varepsilon(t), y_\varepsilon(t)) \to (x(t), y(t)), \quad \varepsilon \to 0, \quad \forall t \in [0, T],$$
$$(p_\varepsilon(t), q_\varepsilon(t) \to (p(t), q(t)), \quad \varepsilon \to 0, \quad \forall t \in [0, T],$$

where $(p(\cdot), q(\cdot))$ is the solution of equation (3).

Let $\varepsilon \to 0$ in (15), then we have

$$H(x(t), y(t), u(t), p(t), q(t), t) - H(x(t), y(t), v, p(t), q(t), t) \geqslant 0, \quad \forall v \in U, \quad \text{a.e.} \quad t \in [0, T].$$

The proof is complete.

3 Optimal Control for Another Forward and Backward System

We consider another forward and backward system

$$\begin{cases} \dot{x} = f(x, y, v), \\ x(0) = x_0, \quad G_1(x(T)) = 0, \\ \dot{y} = g(y, v), \\ y(T) = y_T, \quad G_2(y(0)) = 0. \end{cases} \tag{16}$$

Our optimal control problem is to minimize the cost function (1) over $\mathscr{U}_{ad}$, where $f: \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}^k \to \mathrm{R}^n$, $g: \mathrm{R}^m \times \mathrm{R}^k \to \mathrm{R}^m$.

Under the assumptions H1) and H3), we can prove the following result similarly, where

H3) f_x, f_y and g_y are bounded.

Theorem 2 Suppose H1) and H3) hold. Let $(u(\cdot), x(\cdot), y(\cdot))$ be an optimal solution to our optimal control problem (16) and (1), $(p(\cdot), q(\cdot))$ be the corresponding solution of the following adjoint equation:

$$\begin{cases} -\dot{p} = f_x^*(x,y,u)p, \\ p(T) = -(h_x^*(x(T))h_0 + G_{1x}^*(x(T))h_1), \\ -\dot{q} = f_y^*(x,y,u)p + g_y^*(y,u)q, \\ q(0) = \gamma_y^*(y(0))h_0 + G_{2y}^*(y(0))h_2. \end{cases}$$

Then, the following maximum condition holds

$$H(x(t), y(t), u(t), p(t), q(t), t) = \max_{v \in U} H(x(t), y(t), v, p(t), q(t), t) \quad \text{a.e.} \quad t \in [0, T],$$

where $H(x,y,v,p,q,t) \triangleq \langle p, f(x,y,v)\rangle + \langle q, g(y,v)\rangle$ is the corresponding Hamiltonian function, $h_0 \in \mathbb{R}^1$, $h_1 \in \mathbb{R}^{n_1}$ and $h_2 \in \mathbb{R}^{m_1}$ are constant vectors with $\sum_{i=0}^{2} \| h_i \|^2 = 1$.

4 Sufficiency of the maximum condition for a linear case

We consider a linear forward and backward system

$$\begin{cases} \dot{x} = A(t)x + B(t,v), \quad x(0) = x_0, \\ \dot{y} = C(t)x + D(t)y + E(t,v), \quad y(T) = y_T. \end{cases} \tag{17}$$

Our optimal control problem is to minimize

$$S(v(\cdot)) = c^* x(T) + d^* y(0), \tag{18}$$

over $\mathcal{U}_{ad}$. Where, $A(t) \in \mathbb{R}^{n\times n}$, $B(t,v) \in \mathbb{R}^n$, $C(t) \in \mathbb{R}^{m\times n}$, $D(t) \in \mathbb{R}^{m\times m}$, $E(t,v) \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.

Suppose that A, B, C, D and E are continuous with respect to t, v. We also assume that U is a bounded closed subset of $\mathbb{R}^k$.

For this problem, the maximum condition (4) not only is necessary but also is sufficient. We have the following sufficiency result:

Theorem 3 Let $(x(\cdot), y(\cdot))$ be the trajectory of system (17) corresponding to $u(\cdot) \in \mathcal{U}_{ad}$, $(p(\cdot), q(\cdot))$ be the solution of the following adjoint equation

$$\begin{cases} -\dot{p} = A^*(t)p + C^*(t)q, \quad p(T) = -c, \\ -\dot{q} = D(t)^* q, \quad q(0) = d. \end{cases} \tag{19}$$

If $(u(\cdot), x(\cdot), y(\cdot), p(\cdot), q(\cdot))$ satisfies the maximum condition (4), then $(u(\cdot), x(\cdot), y(\cdot))$ is an optimal solution to problem (17) and (18).

Where the Hamiltonian function is

$$H(x,y,v,p,q,t) \triangleq \langle p, A(t)x + B(t,v)\rangle + \langle q, c(t)x + D(t)y + E(t,v)\rangle.$$

Proof For any $v(\cdot) \in \mathcal{U}_{ad}$, let $\delta x(\cdot) \triangleq x(\cdot, v) - x(\cdot, u)$, $\delta y(\cdot) \triangleq y(\cdot, v) - y(\cdot, u)$, then $(\delta x, \delta y)$ admits

$$\begin{cases} \delta\dot{x} = A(t)\delta x + (B(t,v) - B(t,u)), & \delta x(0) = 0, \\ \delta\dot{y} = C(t)\delta x + D(t)\delta y + (E(t,v) - E(t,u)), & \delta y(T) = 0. \end{cases}$$

From the above equation and (19), one can check that

$$c^*\delta x(T) + d^*\delta y(0) = \int_0^T [H(x(t),y(t),u(t),p(t),q(t),t) - H(x(t),y(t),v(t),p(t),q(t),t)]\mathrm{d}t \geqslant 0.$$

It implies

$$S(v(\cdot)) \geqslant S(u(\cdot)).$$

Thus $u(\cdot)$ is optimal. The proof is complete.

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一类双向确定性系统最优控制问题的最大值原理

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摘要: 本文讨论了一类双向确定性系统的最优控制问题,我们利用 Ekeland 变分原理,推得了最优控制所满足的最大值原理.同时,对线性系统的情况,我们还证明了最大值条件的充分性.

关键词: 双向系统; 最大值原理; 最优控制

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