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Deadbeat Covariance Controllers for Discrete Systems *

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Abstract: It is quite common in stochastic control problems to have performance objectives that are expressed in terms of the covariances of the system states. In this research the following control problem will be solved: How to design the linear feedback controller such that the covariance of the system state achieves its steady-state value during finite beats, and this steady-state covariance achieves the prespecified value simultaneously? The existence conditions and the design methods of this class of feedback controllers are discussed in this paper. The straightforward design steps are also given.

Key words: stochastic discrete systems; covariance control; state feedback

1 Introduction

Many robustness properties of linear time-invariant systems are naturally described in terms of the state covariance matrix[1,2]. Recently, the state covariance assignment (SCA) theory has been the subject of research of many authors[3~7]. The main idea of this theory is to specify a state covariance matrix X to the different requirements on the system robustness and performance, and then design a controller such that the state covariance of the resulting closed-loop system is equal to this specified X. Therefore, this closed-loop will possess the desired properties, such as robustness.

However, much of the SCA literature focuses on the steady-state behavior of the system and the transient properties are seldom considered. For example, the SCA theory can not ensure that the closed-loop system achieves the specified state covariance in finite beats. The too long time for the state covariance to achieve its steady-state value will influence the realization of the design goals seriously. It is a disadvantage for the SCA theory to be applied in practical engineering system.

In this paper, the problem of designing deadbeat covariance controller is considered. The goal of this problem is to design linear feedback controller such that the state covariance of the resulting closed-loop system achieves its steady-state value during finite beats, and this steady-state covariance achieves the prespecified value simultaneously. Such a controller is called a deadbeat covariance controller. If the deadbeat covariance controller exists, the closed-loop system will possess fine behavior.

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2 Problem Formulation and Preliminaries

Consider the stationary vector process x generated by

$$x(k+1) = Ax(k) + Bu(k) + Dw(k), \tag{1a}$$

$$u(k) = Gx(k). \tag{1b}$$

Here $x \in \mathrm{R}^{n_x}, u \in \mathrm{R}^{n_u}, w \in \mathrm{R}^{n_w}$; w is a zero mean white noise process with covariance $W > 0$; and $w(k)$ and $x(0)$ are uncorrelated. The notation "$[\cdot] > 0$" and "$[\cdot] \geqslant 0$" denote positive definite and positive semidefinite, respectively. The pairs (A,B) and (A,D) are, respectively, assumed to be stabilizable and controllable.

The steady-state covariance X of the closed-loop system defined as

$$X = \lim_{k\to\infty} E[x(k)x^{\mathrm{T}}(k)]. \tag{2}$$

is the solution to the following discrete Lyapunov equation:

$$X = (A+BG)X(A+BG)^{\mathrm{T}} + DWD^{\mathrm{T}}. \tag{3}$$

It is required to determine all state feedback controllers G such that the steady-state covariance X achieves a specified value $\bar{X}$, where $\bar{X} = \bar{X}^{\mathrm{T}} > 0$. Such a problem requires two steps: ① finding the necessary and sufficient conditions for the solvability of G from eqn. (3), and ② finding all solutions G for this equation. This problem is referred to as the SCA problem.

A specified $\bar{X}$ (for simplicity, we use X for $\bar{X}$) is said to be assignable if a controller exists such that the steady-state covariance of the closed-loop system is equal to this X. The solution of the SCA problem is presented in the following two lemmas.

Lemma 1[4] For the system (1), a specified X is assignable if and only if

$$X = X^{\mathrm{T}} > 0, \tag{4a}$$

$$X \geqslant DWD^{\mathrm{T}}, \tag{4b}$$

$$(I - BB^{+})(AXA^{\mathrm{T}} - X + DWD^{\mathrm{T}})(I - BB^{+}) = 0 \tag{4c}$$

where B^{-} denotes the Moore-Penrose inverse of B.

Lemma 2[4] For the system (1), let X be assignable and T be a square factor of X (i.e., $TT^{\mathrm{T}} = X$). The matrices N and P defined by

$$N = (I - BB^{+})(\sqrt{X - DWD^{\mathrm{T}}}), \tag{5a}$$

$$P = (I - BB^{+})AT. \tag{5b}$$

both have rank r and the singular value decompositions

$$N = M\Lambda E^{\mathrm{T}}, \tag{6a}$$

$$P = M\Lambda F^{\mathrm{T}}. \tag{6b}$$

where M, E, and F are orthonormal, $\Lambda = \mathrm{diag}(\sigma_1, \cdots, \sigma_{n_x}), \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_{n_x}$. The system (1) is assigned the X if and only if G is in the set $G(X)$ defined by

$$G(X) = \left\{G: G = B^{+}\left(\sqrt{X - DWD^{\mathrm{T}}}E\begin{bmatrix} I_r & 0 \\ 0 & U_0 \end{bmatrix}F^{\mathrm{T}}T^{-1} - A\right) + (I_{n_u} - B^{+}B)Y\right\}. \tag{7}$$

where $U_0 \in \mathrm{R}^{(n_x - r)\times(n_x - r)}$ is arbitrary orthonormal, $Y \in \mathrm{R}^{n_u \times n_x}$ is arbitrary.

By analysing lemma 1 and 2, we discover that the appropriate choice of X and U_0, Y in (7) will generate an element of the set $G(X)$, because the SCA problem is a multiobjective design task. The freedom in U_0, Y does not influence the steady-state covariance, but influences the transient properties, input and output covariances, robustness of the system. Hence, the freedom can be exploited to achieve the desired closed-loop properties. For example, it has been used to minimize the input variance in [4]. In this paper, the freedom will be utilized to achieve the deadbeat property via state feedback.

Let $X(k) = E[x(k)x^{\mathrm{T}}(k)]$ and $X = \lim_{k\to\infty} X(k)$, we give the following definitions.

Definition 1 The closed-loop system (1) is said to be ε-n-DB(deadbeat) controllable if, for prespecified $\varepsilon > 0$ and prespecified positive integer n, there exists a feedback controller such that $\|X(k) - X\| < \varepsilon$ for all $k \geqslant n$.

Remark In fact, Definition 1 offers a further constraint on the general covariance controller. In the design of the practical engineering system, if the system is ε-n-DB controllable and if the error precision ε is chose to be small enough, the design goal can be effectively achieved.

To this end, the problem considered in this paper can be expressed as follows: For the system (1), find all the feedback controllers G such that the system is ε-n-DB controllable and the steady-state covariance X achieves the prespecified value $\overline{X}$.

3 Main Results and Proofs

Lemma 3[8] For the system (1), the state covariance $X(k)$ can be expressed as

$$X(k) = (A + BG)^k R_0[(A + BG)^{\mathrm{T}}]^k + \sum_{s=0}^{k-1}(A + BG)^s R_1[(A + BG)^{\mathrm{T}}]^s. \tag{8}$$

where $R_0 = E[x(0)x^{\mathrm{T}}(0)], R_1 = DWD^{\mathrm{T}}$.

Theorem 1 For prespecified $\varepsilon > 0$ and a positive integer n, the system (1) is ε-n-DB controllable if there exists a G such that the maximum singular value of $H(H = A + BG)$ is $\sqrt{\lambda}$, here $\lambda(0 < \lambda < 1)$ is determined by

$$\lambda^n/(1 - \lambda) \leqslant \varepsilon/\max\{\|R_0 - R_1\|, \|R_1\|\}. \tag{9}$$

Proof It is clear that $\|H\|^2 = \lambda < 1$ and $H^\infty = 0$. Using Lemma 3, we have

$$\begin{aligned}
\|X(n) - X\| &= \|X(n) - X(\infty)\| \\
&= \|H^n R_0 (H^{\mathrm{T}})^n - H^\infty R_0 (H^{\mathrm{T}})^\infty - \sum_{s=n}^{\infty} H^s R_1 (H^{\mathrm{T}})^s\| \\
&= \|H^n R_0 (H^{\mathrm{T}})^n - \sum_{s=n}^{\infty} H^s R_1 (H^{\mathrm{T}})^s\| \\
&\leqslant \|H\|^{2n}\|R_0 - R_1\| + \sum_{s=n+1}^{\infty} \|H\|^{2s}\|R_1\| \\
&= \lambda^n\|R_0 - R_1\| + \sum_{s=n+1}^{\infty} \lambda^s\|R_1\| \\
&= \lambda^n\|R_0 - R_1\| + [\lambda^{n+1}/(1 - \lambda)]\|R_1\|
\end{aligned}$$

$$\leqslant [\lambda^n/(1-\lambda)]\max\{\|R_0 - R_1\|, \|R_1\|\}.$$

If condition (9) is satisfied, then $\|X(n) - X\| \leqslant \varepsilon$ and, therefore, $\|X(k) - X\| < \varepsilon$ when $k \geqslant n$. This proves the theorem.

From Theorem 1, we can conclude the above results as follows: For the system (1), the problem of designing ε-n-DB controllers can be converted to the problem of assigning desired poles (singular values) to the closed-loop system. Hence, our task is to find the feedback controller G such that steady-state covariance achieves the specified value and the equation $A + BG = H$ can be satisfied, where the desired closed-loop matrix H is defined as the matrix whose maximum singular value is $\sqrt{\lambda}$ in the case of designing ε-n-DB controllers.

In what follows, we will focus on the existence conditions and expression of the desired feedback controller G.

Theorem 2 Let H be the desired closed-loop matrix with the dimension $n_x \times n_x$. Suppose that X is assignable. Then there exists G such that system (1) assigns the specified state covariance X and $A + BG = H$, if and only if

$$[E^{\mathrm{T}}LHTF]_{11} = I_r, \tag{10a}$$

$$[E^{\mathrm{T}}LHTF]_{12} = 0, \tag{10b}$$

$$[E^{\mathrm{T}}LHTF]_{21} = 0, \tag{10c}$$

$$[E^{\mathrm{T}}LHTF]_{22}[E^{\mathrm{T}}LHTF]_{22}^{\mathrm{T}} = I_{n_u} \tag{10d}$$

where $L = (\sqrt{X - DWD^{\mathrm{T}}})^{-1}$ and T, E, F are defined as in Lemma 2, the dimension of $[\cdot]_{11}, [\cdot]_{12}, [\cdot]_{21}, [\cdot]_{22}$ is $r \times r, r \times n_u, n_u \times r$ and $n_u \times n_u$, respectively.

Proof of sufficiency:

Assume that conditions (10a)~(10d) are satisfied, then

$$E^{\mathrm{T}}LHTF = \begin{bmatrix} I_r & 0 \\ 0 & U_0 \end{bmatrix}.$$

where U_0 is an orthonormal matrix with the dimension of $n_u \times n_u$. Since $EE^{\mathrm{T}} = I, FF^{\mathrm{T}} = I$, $L = (\sqrt{X - DWD^{\mathrm{T}}})^{-1}$, we have

$$H = \sqrt{X - DWD^{\mathrm{T}}}\, E \begin{bmatrix} I_r & 0 \\ 0 & U_0 \end{bmatrix} F^{\mathrm{T}}T^{-1}.$$

Define

$$V = E\begin{bmatrix} I_r & 0 \\ 0 & U_0 \end{bmatrix} F^{\mathrm{T}},$$

$$G = B^{+}(\sqrt{X - DWD^{\mathrm{T}}}\,VT^{-1} - A) + (I_{n_u} - B^{+}B)Y.$$

where $Y \in \mathrm{R}^{n_x \times n_x}$ is arbitrary, then $A + BG = H$. By Lemma 2, we know that G is also the controller such that the closed-loop system assigns the specified covariance X.

Proof of necessity If there exists G such that the system (1) assigns the specified state covariance X and meet $A + BG = H$, then from lemma 2, we have

$$G = B^{+}\left\{\sqrt{X - DWD^{\mathrm{T}}}\, E\begin{bmatrix} I_r & 0 \\ 0 & U_0 \end{bmatrix} F^{\mathrm{T}} T^{-1} - A\right\} + (I_{n_u} - B^{+}B)Y.$$

where U_0 is orthonormal and $Y \in \mathrm{R}^{n_x \times n_x}$ is arbitrary. Hence we easily get

$$H = \sqrt{X - DWD^{\mathrm{T}}}\, E\begin{bmatrix} I_r & 0 \\ 0 & U_0 \end{bmatrix} F^{\mathrm{T}} T^{-1},$$

and $E^{\mathrm{T}} LHTF = \begin{bmatrix} I_r & 0 \\ 0 & U_0 \end{bmatrix}$, then obtain conditions (10a)～(10d).

Theorem 3 If the conditions of Theorem 2 are satisfied, then the feedback controller G, which can make the system (1) achieve the assignable matrix X and have expected closed-loop matrix H, can be expressed as:

$$G = B^{+}\left\{\sqrt{X - DWD^{\mathrm{T}}}\, E\begin{bmatrix} I_r & 0 \\ 0 & [E^{\mathrm{T}} LHTF]_{22} \end{bmatrix} F^{\mathrm{T}} T^{-1} - A\right\} + (I_{n_u} - B^{+}B)Y. \quad (11)$$

where E, F, T, L, Y are defined as in Theorem 2.

Proof The result follows from Lemma 2 and Theorem 2 immediately.

Algorithm to controller design

In this subsection we shall present the design steps of the ε-n-DB controller.

Consider system (1), for specified state covariance X, error precision ε and beat number n, we want to design the controller G.

Step 1 Compute B^{+}, verify conditions (4a)(4b)(4c) are correct or not. If not, the specified X is not assignable and the desired controller does not exist.

Step 2 Choose λ by (9) and v_i satisfying $0 \leqslant v_i \leqslant \lambda$ $(i = 1, 2, \cdots, n_x)$.

Step 3 By reference [9], obtain positive definite matrix K (a set) which has poles $v_1, \cdots, v_{n_x}$, and the square root H of K (a set).

Step 4 Verify H satisfy (10a)～(10d) or not. If not, go to Step 2.

Step 5 Obtain the controller from Theorem 3.

Remark The design has much freedom which can be used to achieve other desired properties. For example, we can consider the robustness.

4 Conclusion

This paper has introduced a theory for designing feedback controllers that assign a specified state covariance to the closed-loop system in finite beats. The theory is restricted to linear time-invariant systems with constant gain state feedback and may be applied in practical control system with computers.

The primary contributions of this paper are summarized in Definition 1 and Theorem 2, 3. Definition 1 defines the notion of dead-beat controllable, and Theorem 2 characterizes the desired closed-loop matrix that may be assigned to the system. Theorem 3 identifies the set of all constant state feedback matrices that achieve the assigned covariance and closed-loop matrix simultaneously.

The results of this paper may be extended to the case of state-estimate feedback control, dynamic feedback control and the case of bilinear discrete-time system. These results will appear at later date.

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关于离散系统的有限拍协方差控制器

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摘要：在随机控制问题中，性能指标常常表示为系统状态协方差的形式. 本文将讨论如下问题，即设计线性反馈控制器，使系统的状态协方差在有限拍内达到其稳态值，同时该稳态值达到预先指定值. 本文讨论了这类反馈控制器的存在条件及设计方法，并给出了直接的设计步骤.

关键词：随机离散系统；协方差控制；状态反馈

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