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Robust Root Clustering of Linear Systems with Structured Uncertainties

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Abstract: Using generalized Lyapunov equation approach, this paper studies the problem of root clustering in some subregions of the complex plane for the linear time-invariant systems with structured uncertainties. If all the eigenvalues of the nominal matrix are within a specific region of complex plane, the proposed sufficient conditions will guarantee the root clustering of the perturbed matrix in the same region under structured uncertainties. The criteria presented here are less conservative than the results currently available in the literature.

Key words: linear system; stability; root clustering; structured uncertainty

1 Introduction

In recent years, the problem of robust stability of linear systems with parametric uncertainties has been an active area of research[1,2]. However, most of the literature on robust stabilities is concentrated on the conventional stability regions. Lately, some authors begin to consider the robust D-stability or root clustering problem[3,6], in which the stability region is some specified subregions of the complex plane. In [3], a generalized Lyapunov equation (G. L. E) approach is proposed. This result is extended in [4] to uncertain systems. Quite recently, [5,6] present some criteria for a class of D-regions specified by algebraic inequalities.

Using generalized Lyapunov theory, this paper gives some explicit parameter perturbation bounds that ensure the matrix root clustering in some subregions of complex plane. In contrast with the previous studies, matrix measure is adopted here. This will enable us to reduce the conservatism of the obtained results.

2 Preliminary Results

Consider linear continuous time system

$$\dot{x}(t) = (A + E)x(t), \tag{1}$$

or the linear discrete-time system

$$x(t + 1) = (A + E)x(t), \tag{2}$$

where $x \in \mathrm{R}^n$ is the state vector, A is $n \times n$ system matrix of the "nominal" model, E is the perturbation matrix. further assume that

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$$E := \sum_{i=1}^{q} k_i E_i, \tag{3}$$

where $k_i \geq 0$, i $=1,2,\cdots,q$; are the uncertain parameters, and E_i, $i=1,2,\cdots,q$; are the given real constant matrices.

In the following, we denote $M_n^+ (M_n^-)$ as the set of $n \times n$ positive (negative) definite matrices; $\lambda_{max}(A)$, the maximum eigenvalue of A; $\bar{c}$, the conjugate of c; $\| A \|$, the induced norm of matrix A; $\mu(A)$, the matrix measure of A induced by the matrix norm $\| \cdot \|$, defined by

$$\mu(A) := \lim_{h \to 0_+} \frac{\| I + hA \| - 1}{h}. \tag{4}$$

The properties of matrix measure can be found in [7].

Since for a real matrix, its eigenvalues appear as complex conjugate pairs, we limit our attention here only to the regions Ω_1 and Ω_2, which are symmetric with respect to real axis, and including most subregions interested in control theory.

Lemma 1[3] Let the subregion Ω_1 defined by

$$\Omega_1 := \{\lambda \in C \mid c_{00} + c_{10}\lambda + c_{01}\bar{\lambda} < 0,\ c_{10}^2 + c_{01}^2 \neq 0\} \tag{5}$$

where $c_{10} = c_{01}$. Then all the eigenvalues of A lie in Ω_1, iff for any $Q \in M_n^+$ there exists an unique $P \in M_n^+$ satisfying the following G. L. E:

$$c_{00}P \mp c_{10}A^T P + c_{01}P A = -Q. \tag{6}$$

Lemma 2[3] Let the subregion Ω_2 defined by

$$\Omega_2 := \{\lambda \in C \mid c_{00} + c_{10}(\lambda + \bar{\lambda}) + c_{11}\lambda\bar{\lambda} + c_{20}(\lambda^2 + \bar{\lambda}^2) < 0\}. \tag{7}$$

Then all the eigenvalues of A lie in Ω_2, iff given any $Q \in M_n^+$, there exists a unique $P \in M_n^+$ satisfying

$$c_{00}P + c_{10}(PA + A^T P) + c_{11}A^T PA + c_{20}(PA^2 + (A^T)^2 P) = -Q. \tag{8}$$

Note that all the coefficients c_{ij} in (6) and (8) are real numbers. Some useful regions of Ω_1 include the open left half plane, the α shift subregion of left half plane; while the useful regions of Ω_2 include circle, parabola, hyperbola, ellipse etc., see [3] for details.

3 Unidirectional Perturbations

In this section, we study the problem of matrix root clustering under highly structured unidirectional perturbations, i. e. we implicitly assume that the directional information of the uncertain parameters are available. Let the perturbation matrix is characterized by (3), and define

$$R_i := c_{10}(E_i^T P + PE_i) \quad i = 1, \ldots, q; \tag{9}$$

where P is solution of G. L. E. (6).

Theorem 1 Assume that all the eigenvalues of the matrix A are within Ω_1 specified by (5), then all the eigenvalues of the perturbed matrix $A+E$ remain in Ω_1, if there exist a matrix measure $\mu(\cdot)$, such that

$$\varphi_{\Omega_1} < -\mu(-Q), \tag{10}$$

where $\varphi_{\Omega_1} := \sum_{i=1}^{q} k_i \mu(R_i)$; and R_i is defined by (9).

Proof By Lemma 1, if

$$c_{00}P + c_{10}(A+E)^{\mathrm{T}}P + c_{01}P(A+E) \in M_n^-, \tag{11}$$

then all the eigenvalues of $A+E$ remain in Ω_1, In terms of (6), and since $c_{01}=c_{10}$, and both Q and $E^{\mathrm{T}}P+PE$ being Hermitian, we can rewrite (11) as

$$\lambda_{\max}(c_{10}(E^{\mathrm{T}}P + PE) - Q) < 0. \tag{12}$$

From (3) and the definition of R_i, (12) can be rewritten as

$$\lambda_{\max}(\sum_{i=1}^{q} k_i R_i - Q) < 0. \tag{13}$$

By the properties of matrix measure, (13) will be hold, if

$$\mu(\sum_{i=1}^{q} k_i R_i - Q) < 0. \tag{14}$$

We conclude the proof by noticing that (10) is a sufficient condition for (14) to be hold.

Q. E. D.

Clearly, by the selection of different matrix measure, and different Q, it is possible to find the bounds on k_i with least conservatism. An iterative algorithm to find such a Q is presented in [8].

Now that the subregion Ω_2 will be considered. First define

$$H_i := c_{10}(PE_i + E_i^{\mathrm{T}}P) + c_{11}(A^{\mathrm{T}}PE_i + E_i^{\mathrm{T}}PA) + c_{20}\{(AE_i + E_iA)^{\mathrm{T}}P + P(AE_i + EA_i)\}, \quad i = 1,\cdots,q; \tag{15}$$

$$L_{ij} := c_{11}E_i^{\mathrm{T}}PE_j + c_{20}(PE_jE_i + E_i^{\mathrm{T}}E_j^{\mathrm{T}}P), \qquad i,j = 1,\cdots,q; \tag{16}$$

where P is the solution of G. L. E. (8)

Thorem 2 Assume that all the eigenvalues of A are within Ω_2, difined by (7), then all the eigenvalues of $A+E$ will remain in Ω_2, if there exists a matrix measure, such that

$$\varphi_{\Omega_2} < -\mu(-Q) \tag{17}$$

where $\varphi_{\Omega_2} := \sum_{i=1}^{q} k_i \mu(H_i) + \sum_{i=1}^{q} k_i k_j \mu(L_{ij})$; and Q, H_i, and L_{ij} are defined by (8), (15) and (16), respectively.

Proof As pointed by Lemma 2, if

$$c_{00}P + c_{10}((A+E)^{\mathrm{T}}P + P(A+E)) + c_{11}(A+E)^{\mathrm{T}}P(A+E) + c_{20}(P(A+E)^2 + (A^{\mathrm{T}}+E^{\mathrm{T}})^2P) \in M_n^-, \tag{18}$$

where P is the solution of G. L. E. (8), then all the eigenvalues of $A+E$ will remain in Ω_2. In terms of (8), and since both Q and M being Hermitian, (18) is equivalent to require:

$$\lambda_{\max}\Big(\sum_{i=1}^{q} k_i H_i + \sum_{j=1}^{q} k_i k_j L_{ij} - Q\Big) < 0. \tag{19}$$

A sufficient condition for (19) to be held is

$$\mu\Big\{\sum_{i=1}^{q} k_i H_i + \sum_{j=1}^{q} k_i k_j L_{ij} - Q\Big\} < 0. \tag{20}$$

Clearly ,(20) will be held if (17) holds . Q. E. D.

4 Bidirectional Perturbations

This section will devote to the case of E being bidirectional, i. e. if E is an admissible perturbation, then $-E$ is also an admissible one[2]. To this end ,we still let E defined by (3), but with

$$|k_i| \leq b_i, \quad i = 1, \cdots, q. \tag{21}$$

The following inequality is obvious:

$$\mu(k_i A) \leq \max\{|k_i|\mu(-A), |k_i|\mu(A)\}, \tag{22}$$

where A is any real square matrix, and k_i is a real number.

Theorem 3 Assume that all the eigenvalues of matrix A are within Ω_1 ,defined by (5), and the perturbed matrix E is described by (3) and (21), then all the eigenvalues of perturbed matrix $A+E$ remain in Ω_1 , if

$$v_{\Omega_1} < -\mu(-Q) \tag{23}$$

where $v_{\Omega_1} := \sum_{i=1}^{q} b_i r_i$; r_i: $=\max\{\mu(-R_i), \mu(R_i)\}$and R_i,is defined by (9).

Proof Using inequality (22) ,the proof of this theorem is similar to that of Theorem 1, and is omittd. Q. E. D.

Theorm 4 Assume that all the eigenvalues of A are within Ω_2 defined by (7), and the perturbation matrix E is characterized by (3) and (21), then all the eigenvalues of perturbed matrix $A+E$ will remain in Ω_2 , if

$$v_{\Omega_2} < -\mu(-Q) \tag{24}$$

where $\quad v_{\Omega_2} := \sum_{i=1}^{q} b_i h_i + \sum_{i=1}^{q} b_i b_j n_{ij}$; $h_i := \max\{\mu(-H_i), \mu(H_i)\}$;

and $\quad n_{ij} := \max\{\mu(-L_{ij}), \mu(L_{ij})\}$.

Proof Similar to the proof of Theorem 2.

Although easy to apply numerically, theorem 3 and Theorem 4 are rather conservative. To further reduce the conservatism, we let

$$P_S := \sum_{i=1}^{q} b_1 \hat{R}_i, \quad s = 1, 2, \cdots, 2^q; \tag{25}$$

Where $\hat{R}_i$ is either $-R_i$ or R_i,$i=1,\cdots,q$,and R_i is defined in (9).

Theorem 5 Assume that all the eigenvalues of matrix A are within Ω defined by (5), and the perturbation matrix E are described by (3) and (21). Let P_s,$s=1,2,\cdots,2^q$;defined by (25),denote the 2^q possible combinational sums of $\hat{R}_i$,then all the eigenvalues of perturbed matrix $A+E$ remain in Ω_1,if

$$\max_s \mu(P_s) < -\mu(-Q). \tag{26}$$

Proof Similar to the proof of Theorem 1. Q. E. D.

For the subregion Ω_2 ,we analogously define

$$T_s := \sum_{i=1}^{q} b_i \hat{H}_i + \sum_{i,j=1}^{q} b_i b_j \hat{L}_{ij}; \quad s = 1, 2, \cdots, 2^{q^2}. \tag{27}$$

Where $\hat{H}_i$ is either H_i or $-H_i, i=1,\cdots,q$; $\hat{L}_{ij}$ is either L_{ij} or $-L_{ij}, i,j=1,2,\cdots,q$; and H_i and L_{ij} are defined by (15) and (16), respectively.

Theorem 6 Assume that all the eigenvalues of matrix A are within Ω_2 defined by (7), and the perturbation matrix E are described by (3) and (21). Let $T_s, s=1,\cdots,2^{q^2}$; defined by (27), denote the 2^{q^2} possible combinational sums of $\hat{H}_j$ and $\hat{L}_{ij}$, then all the eigenvalues of perturbed matrix $A+E$ remain in Ω_2, if

$$\max_s \mu(T_s) < -\mu(-Q). \tag{28}$$

Proof Similar to Theorem 2. Q. E. D.

5 Illustrative Example

Consider nominal system matrix

$$A = \begin{bmatrix} -8.7 & -1.0 \\ 1.2 & -1.8 \end{bmatrix},$$

and the perturbation matrix

$$E = 0.28\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.6\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0.08\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume that the subregion Π_2 specified as

$$\Pi_2 := \{(x,y) \mid (x+5.25)^2 + y^2 < 0\}.$$

Clearly, all the eigenvalues of A are within Π_2. From [3], we know that Π_2 is corresponding to Ω_2 defined by (8) with $c_{00}=14.6025$, $c_{10}=5.25$, $c_{11}=1$, $c_{20}=0$, Let $Q = I$, then the solution of G. L. E. (8) is

$$P = \begin{bmatrix} 0.5071 & 0.1467 \\ 0.1467 & 0.4986 \end{bmatrix}.$$

By adopting $\mu_2(\cdot)$, we obtain that $\varphi_{\Omega_2}=0.7847$, and $-\mu(-Q)=1$, by Theorem 2, all the eigenvalues of the perturbed matrix $A+E$ are also within the same region Π_2. However, if the theorem 6 of [6] is used, we have $\rho_{\Omega_2}=2.9568$ and $\sigma_m(Q)=1$, and that Theorem can not be applied. This shows that our results are less conservative.

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具有结构式不确定性线性系统特征根的鲁棒群聚

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摘要：采用广义 Lyapunov 方程，讨论具有结构式不确定性线性系统的特征根在复平面上某些区域内的鲁棒群聚性.如果标称矩阵的所有特征根都在复平面上的特定区域内，本文所给出充分条件将保证当存在有结构式不确定性时矩阵的特征根也在同一区域内.所提出的判据比当前文献中的结论具有较少的保守性.

关键词：线性系统；稳定性；特征根群聚；结构式不确定性

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