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Stability of High Dimension Interval Dynamic Systems

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Abstract: In this paper, method of vector Lyapunov function is used to analyse the asymptotic stability of large-scale interval dynamic systems, some suffuicent conditions for the asymptotic stability of interval dynamic systems are obtained.

Key words: interval matrix; vector Lyapunov function; asymptotic stability

1 Intrduction and Lamma

Recently, there have been a number of significant advances in the stability of interval dynamic systems[1~6], however, most of these results are applicable to lower dimension systems, the present paper presents some sufficient conditions for stability of interval systems, especially for high orders.

Let $R^{m\times n}$ be real $m\times n$ matrices space, $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{m\times n}$, $C=(c_{ij})_{m\times n}\in R^{m\times n}$, the matrices set

$$G[B,C]=\{A\,|\,b_{ij}\leqslant a_{ij}\leqslant c_{ij};i=1,2,\cdots,m;j=1,2,\cdots,n\}$$

is called $m\times n$ interval matrix, set

$$H[B,C]=\{A\,|\,a_{ij}=b_{ij}\ or\ a_{ij}=c_{ij};i=1,2,\cdots,m;j=1,2,\cdots,n\}$$
$$=\{A_1,A_2,\cdots,A_k\},\quad (K\leqslant 2^{mn})$$

is called the verteces set of $G[B,C]$.

Let G be a set of $m\times n$ matrices, denote matrices

$$S(G)=(s_{ij})_{m\times n},\quad T(G)=(t_{ij})_{m\times n},$$

where

$$s_{ij}=\max\{|a_{ij}|\,|\,A=(a_{ij})_{m\times n}\in G\};\quad i=1,2,\cdots,m,\quad j=1,2\cdots,n,$$

$$t_{ij}=\begin{cases}\max\{a_{ij}\,|\,A=(a_{ij})_{m\times n}\in G;\ \text{while } i=j\},\\ \max\{|a_{ij}|\,|\,A=(a_{ij})_{m\times n}\in G;\ \text{while } i\neq j\},\end{cases}\quad i=1,2,\cdots,m;\ j=1,2,\cdots,n.$$

Let $x_1=\mathrm{col}(x_1^{(1)},x_2^{(1)},\cdots,x_m^{(1)})$, $x_2=\mathrm{col}(x_1^{(2)},x_2^{(2)},\cdots,x_n^{(2)})$, $(|x_1|)=\mathrm{col}(|x_1^{(1)}|,|x_2^{(1)}|,\cdots,|x_m^{(1)}|)$, $(|x_2|)=\mathrm{col}(|x_1^{(2)}|,|x_2^{(2)}|,\cdots,|x_n^{(2)}|)$, then we have

Lemma 1 Suppose $A\in G$ is a $m\times n$ matrix, then

$$x_1^{\mathrm T}Ax_2\leqslant(|x_1|)^{\mathrm T}S(G)(|x_2|).$$

Proof

$$x_1^{\mathrm T}Ax_2=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_i^{(1)}x_j^{(2)}\leqslant\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|\,|x_i^{(1)}|\,|x_j^{(2)}|\leqslant\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij}|x_i^{(1)}|\,|x_j^{(2)}|$$

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$$= (|x_1|)^T S(G)(|x_2|).$$

Lemma 2 Suppose $A \in G$ is a $m \times m$ matrix, then

$$x_1^T A x_1 \leqslant (|x_1|)^T T(G)(|x_1|).$$

Proof $x_1^T A x_1 \leqslant \sum_{i=1}^{m} a_{ii} x_i^{(1)^2} + \sum_{i=1}^{m} \sum_{i \neq j=1}^{m} |a_{ij}| |x_i^{(1)}| |x_j^{(1)}|$

$$\leqslant \sum_{i=1}^{m} t_{ii} |x_i^{(1)}|^2 + \sum_{i=1}^{m} \sum_{i \neq j j=1}^{m} t_{ij} |x_i^{(1)}| |x_j^{(1)}|$$

$$= (|x_1|)^T T(G)(|x_1|).$$

Let $P = (p_{ij})_{m \times m}$ be a $m \times m$ symmetric matrix, consider sets

$G_P[B,C] = \{PA | A \in G[B,C]\}, H_P[B,C] = \{PA | A \in H[B,C]\},$

$G_P{}^*[B,C] = \{A^T P + PA | A \in G[B,C]\}, H_P{}^*[B,C] = \{A^T P + PA | A \in H[B,C]\}.$

Lemma 3 Suppose $G[B,C]$ is $m \times n$ interval matrix, then

$$S(G_P[B,C]) = S(H_P[B,C]).$$

Proof On the one hand, since $H_P[B,C]) \subseteq G_P[B,C])$, therefore

$$S(G_P[B,C]) \geqslant S(H_P[B,C]) \tag{1.1}$$

on the other hand, by property of closed convex polygon[6], for any $A \in G[B,C]$, there exist numbers $\alpha_i \geqslant 0 (i = 1, 2, \cdots, k)$, such that

$$\sum_{i=1}^{k} \alpha_i A_i = A, \quad \sum_{i=1}^{k} \alpha_i = 1.$$

Let $PA = (a_{ij}{}^*)_{m \times n}, A_l = (a_{ij}{}^{(l)})_{m \times n}, PA_l = (a_{ij}{}^{*(l)})_{m \times n}$, then

$$|a_{ij}{}^*| = |\sum_{l=1}^{k} \alpha_l (\sum_{s=1}^{n} p_{is} a_{sj}^{(l)}| \leqslant \sum_{l=1}^{k} \alpha_l |(\sum_{s=1}^{n} p_{is} a_{sj}^{(l)}|$$

$$\leqslant \sum_{l=1}^{k} \alpha_l \max\{|\sum_{s=1}^{k} p_{is} a_{sj}^{(l)})| \,|\, l = 1, 2, \cdots, k\}$$

$$= \max\{|\sum_{s=1}^{k} p_{is} a_{sj}^{(l)})| \,|\, l = 1, 2, \cdots, k\} = \max\{|a_{ij}^{*(l)}| \,|\, l = 1, 2, \cdots, k\},$$

therefore

$$S(G_P[B,C]) \leqslant S(H_P[B,C]). \tag{1.2}$$

From (1.1), (1.2), we have

$$S(G_P[B,C]) = S(H_P[B,C]).$$

Lemma 4 Suppose $G[B,C]$ is a $m \times m$ interval matrix, then

$$T(G_P^*[B,C]) = T(H_P^*[B,C]).$$

This Lemma can be proved in a similar way as Lemma 3. As a matter of fact, Lemma 3, Lemma 4 have shown method of calculation $S(G_P^*[B,C])$ and $T(G_P^*[B,C])$.

Lemma 5[7] If B is $n \times n$ matrix, such that $b_{ii} < 0, b_{ij} \geqslant 0, i \neq j$; assume that $x(t; x_0, t_0), y(t; y_0, t_0)$ are solutions of $\dot{x} \leqslant Bx$ and $\dot{y} = By$, there $x_0 = y_0$, then for any $t \geqslant t_0$,

$$x(t; x_0, t_0) \leqslant y(t; y_0, t_0).$$

2 Stability of General Interval Dynamic Systems

Consider interval dynamic system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax \tag{2.1}$$

where $x=\mathrm{col}(x_1,x_2,\cdots,x_n)$ $G[B,C]$ is a $n\times n$ interval matrix, $A\in G[B,C]$ is a $n\times n$ matrix.

Theorem 2. 1 Suppose P is a $n\times n$ real postive definite symmetric matrix, such that $A_l^{\mathrm{T}}P+PA_l$ is negative definite, $A_l\in H[B,C]$; $l=1,2,\cdots,k$; then, (2. 1) is asymptotic stable and there exists a real number $c>0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\mathrm{T}}Px]|_{(2.1)} \leqslant -c(\sum_{i=1}^{n}x_i^2).$$

Proof Let $V(x)=x^{\mathrm{T}}Px$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\mathrm{T}}Px]|_{(2.1)} = x^{\mathrm{T}}[A^{\mathrm{T}}P+PA]x$$
$$= x^{\mathrm{T}}[(\sum_{l=1}^{k}\alpha_lA_l)P+P(\sum_{l=1}^{k}\alpha_lA_l)]x$$
$$= \sum_{l=1}^{k}\alpha_l[x^{\mathrm{T}}(A_l^{\mathrm{T}}P+PA_l)x],$$

thus, if $A_l^{\mathrm{T}}+PA_l$ is negative definite, there exist real numbers $c_l>0$; $l=1,2,\cdots,k$; such that

$$x^{\mathrm{T}}(A_l^{\mathrm{T}}P+PA_l)x \leqslant -c_l(\sum_{i=1}^{n}x_i^2); \quad l=1,2,\cdots,k.$$

Let $c=\min\{c_1,c_2,\cdots,c_k\}$, then

$$\left.\frac{\mathrm{d}V(x)}{\mathrm{d}x}\right|_{(2.1)} \leqslant -c(\sum_{i=1}^{n}x_i^2),$$

therefore, (2. 1) is asymptotic stable.

Theorem 2. 2 Suppose P is $n\times n$ real postive definite symmetric matrix, such that $T(H_P^*[B,C])$ is negative definite, then, (2. 1) is asymptotic stable and there exists a real number $c>0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathrm{T}}Px]|_{(2.1)} \leqslant -c(\sum_{i=1}^{n})x_i^2.$$

This theorem can be proved in a similar way as theorem 2. 1.

Remark If we select $A_0\in G[B,C]$ properly, then the matrix P mentioned in Theorem 2. 1, Theorem 2. 2 is determined by matrix equation $A_0^{\mathrm{T}}P+PA_0=-2I$.

3 Stability of Large-Scale Interval Dynamic Systems

At first, we decompose system (2. 1) as follow

$$\frac{\mathrm{d}x^{(i)}}{\mathrm{d}t} = \sum_{j=1}^{r}A_{ij}x^{(j)}, \tag{3.1}$$

where $x^{(i)}=\mathrm{col}(x_1^{(i)},x_2^{(i)},\cdots,x_{n_i}^{(i)})$, $i=1,2,\cdots,r$; $n_1+n_2+\cdots+n_r=n$, $A_{ij}\in G[B_{ij},C_{ij}]$,

$G[B_{ij},C_{ij}]$ is an $n_i \times n_j$ interval matrix.

Consider subsystems

$$\frac{dx^{(i)}}{dt} = A_{ii}x^{(i)}, \tag{3.2}$$

$A_{ii} \in G[B_{ii},C_{ii}] (i=1,2,\cdots,r)$ are $n_i \times n_i$ in terval matrix, suppose there exist postive definite functions $V_i(x^{(i)}) = x^{(i)^T} P_i x^{(i)}$, where P_i are $n_i \times n_i$ symmetric matrices, such that

1) $$a_i(\sum_{j=1}^{n_i} x_j^{(i)^2}) \leqslant \nu_i(x^{(i)}) \leqslant b_i(\sum_{j=1}^{n_i} x_j^{(i)^2})],$$

2) $$\frac{d\nu_i(x^{(i)})}{dt}|_{(3.2)} \leqslant -c_i[\sum_{j=1}^{n_i} x_j^{(i)^2}],$$

where a_j, b_j, c_j can be determined by the way of introduced in 2.

Consider vector Lyapunov function $V(x) = \mathrm{col}(\nu_1,\nu_2,\cdots,\nu_r,)$

$$\begin{aligned}\frac{d\nu_1}{dt}|_{(3.1)} &= (A_{11}x^{(1)} + A_{12}x^{(2)} + \cdots + A_{1r}x^{(r)})^T P_1 x^{(1)} \\ &\quad + x^{T(1)} P_1 (A_{11}x^{(1)} + A_{12}x^{(2)} + \cdots + A_{1r}x^{(r)}) \\ &\leqslant -c_1(\sum_{j=1}^{n_i} x_i^{(1)^2}) + x^{T(2)} A_{12}^T P_1 x^{(1)} + x^{T(1)} P_1 A_{12}^T x^{(2)} \\ &\quad + \cdots + x^{T(r)} A_{1r}^T P_1 x^{(1)} + x^{T(1)} P_1 A_{1r} x^{(r)}\end{aligned}$$

since P_1 is a symmetric matrix, we have $x^{T(k)} A_{1k}^T P_1 x^{(1)} = x^{T(1)} P_1 A_{1k} x^{(k)}, k=1,2,\cdots,r$,

$$\frac{d\nu_1}{dt}|_{(3.1)} \leqslant -c_1(\sum_{j=1}^{n_i} x_i^{(1)^2}) + 2x^{T(r)} A_{12}^T P_1 x^{(1)} + \cdots + 2x^{T(r)} A_{1r}^T P_1 x^{(1)}$$

form Lemma 3

$$\begin{aligned}\frac{d\nu_1}{dt}|_{(3.1)} &\leqslant -c_1(\sum_{j=1}^{n_i} x_i^{(1)^2}) + 2(|x^{(1)}|)^T S(G_P[B_{12},C_{12}])(|x^{(2)}|) + \cdots \\ &\quad + 2(|x^{(1)}|)^T S(G_P[B_{1r},C_{1r}])(|x^{(r)}|) \\ &= -c_1(\sum_{i=1}^{n_1} x_i^{(1)^2}) + 2\sum_{j=1}^{n_2} S_{1j}^{(12)} |x_1^{(1)}||x_j^{(2)}| + \cdots \\ &\quad + 2\sum_{j=1}^{n_2} S_{n_1 j}^{(12)} |x_{n_1}^{(1)}||x_j^{(2)}| + \cdots \\ &\quad + 2\sum_{j=1}^{n_r} S_{1j}^{(1r)} |x_1^{(1)}||x_j^{(r)}| + \cdots + 2\sum_{j=1}^{n_2} S_{n_j}^{(1r)} |x_{n_1}^{(1)}||x_j^{(r)}|\end{aligned}$$

(where $S(G_{P_1}[B_{1k},C_{1k}]) = (S_{ij}^{(1k)})_{n_1 \times n_k}$)

$$\begin{aligned}&\leqslant -\frac{c_1 n_2}{n-n_1} x_1^{(1)^2} + 2\sum_{j=1}^{n_2} S_{1j}^{(12)} |x_1^{(1)}||x_j^{(2)}| + \cdots \\ &\quad -\frac{c_2 n_2}{n-n_1} x_{n_1}^{(1)^2} + 2\sum_{j=1}^{n_2} S_{n_1 j}^{(12)} |x_{n_1}^{(1)}||x_j^{(2)}| + \cdots\end{aligned}$$

$$
\begin{aligned}
&-\frac{c_1 n_r}{n-n_1}x_1^{(1)^2}+2\sum_{j=1}^{n_1}S_{1j}^{(1r)}|x_1^{(1)}||x_j^{(r)}|+\cdots\\
&-\frac{c_1 n_r}{n-n_1}x_1^{(1)^2}+2\sum_{j=1}^{n_r}S_{n_1 j}^{(1r)}|x_{n_1}^{(1)}||x_j^{(r)}|\\
\leqslant&-\frac{c_1}{2(n-n_1)}x_1^{(1)^2}+\frac{n-n_1}{c_1}S_{11}^{(12)^2}x_1^{(2)^2}\\
&-\frac{c_1}{2(n-n_1)}x_1^{(1)^2}+\frac{n-n_1}{c_1}S_{12}^{(12)^2}x_2^{(2)^2}+\cdots\\
&-\frac{c_1}{2(n-n_1)}x_1^{(1)^2}+\frac{n-n_1}{c_1}S_{1n_1}^{(12)^2}x_{n_1}^{(2)^2}+\cdots,\\
&-\frac{c_1}{2(n-n_1)}x_{n_1}^{(1)^2}+\frac{n-n_1}{c_1}S_{n_1 1}^{(12)^2}x_1^{(2)^2}\\
&-\frac{c_1}{2(n-n_1)}x_{n_1}^{(1)^2}+\frac{n-n_1}{c_1}S_{n_1 2}^{(12)^2}x_2^{(2)^2}+\cdots\\
&-\frac{c_1}{2(n-n_1)}x_{n_1}^{(1)^2}+\frac{n-n_1}{c_1}S_{n_1 n_2}^{(12)^2}x_{n_2}^{(2)^2}+\cdots\\
&-\frac{c_1}{2(n-n_1)}x_1^{(1)^2}+\frac{n-n_1}{c_1}S_{11}^{(1r)^2}x_1^{(r)^2}\\
&-\frac{c_1}{2(n-n_1)}x_1^{(1)^2}+\frac{n-n_1}{c_1}S_{12}^{(1r)^2}x_1^{(r)^2}+\cdots\\
&-\frac{c_1}{2(n-n_1)}x_1^{(1)^2}+\frac{n-n_1}{c_1}S_{1n_r}^{(1r)^2}x_{n_r}^{(r)^2}+\cdots\\
&-\frac{c_1}{2(n-n_1)}x_{n_1}^{(1)^2}+\frac{n-n_1}{c_1}S_{n_1 1}^{(1r)^2}x_1^{(r)2}\\
&-\frac{c_1}{2(n-n_1)}x_{n_1}^{(1)^2}+\frac{n-n_1}{c_1}S_{n_1 2}^{(1r)^2}x_2^{(2)^2}+\cdots\\
&-\frac{c_1}{2(n-n_1)}x_{n_1}^{(1)^2}+\frac{n-n_1}{c_1}S_{n_1 n_r}^{(1r)^2}x_{n_r}^{(r)^2}+\cdots\\
=&-\frac{c_1}{2}(x_1^{(1)^2}+x_2^{(1)^2}+\cdots+x_{n_1}^{(1)^2}+\frac{n-n_1}{c_1}[\sum_{i=1}^{n_1}S_{i1}^{(12)^2}x_1^{(2)^2}\\
&+\sum_{i=1}^{n_1}S_{i2}^{(12)^2}x_2^{(2)^2}+\cdots+\sum_{i=1}^{n_1}S_{in_2}^{(12)^2}x_{n_2}^{(2)^2}]+\cdots\\
&+\frac{n-n_1}{c_1}[\sum_{i=1}^{n_1}S_{i1}^{(1r)^2}x_1^{(r)^2}+\sum_{i=1}^{n_1}S_{i2}^{(1r)^2}x_2^{(r)^2}+\cdots+\sum_{i=1}^{n_1}S_{in_r}^{(1r)^2}x_{n_r}^{(r)^2}]\\
\leqslant&-\frac{c_1}{2}(\sum_{i=1}^{n_1}x_i^{(1)^2})+L_{12}(\sum_{i=1}^{n_2}x_i^{(2)^2})+\cdots+L_{1r}(\sum_{i=1}^{n_r}x_i^{(r)^2})
\end{aligned}
$$

where

$$
L_{12}=\max\{\frac{n-n_1}{c_1}(\sum_{i=1}^{n_1}S_{i1}^{(12)^2}),\frac{n-n_1}{c_1}(\sum_{i=1}^{n_1}S_{i2}^{(12)^2}),\cdots,\frac{n-n_1}{c_1}(\sum_{i=1}^{n_1}S_{in_2}^{(12)^2})\cdots\},
$$

$$\vdots$$

$$L_{1r} = \max\{\frac{n-n_1}{c_1}(\sum_{i=1}^{n_1} S_{i1}^{(1r)^2}), \frac{n-n_1}{c_1}(\sum_{i=1}^{n_1} S_{i2}^{(1r)^2}), \cdots, \frac{n-n_1}{c_1}(\sum_{i=1}^{n_1} S_{in_r}^{(1r)^2})\cdots\},$$

we have

$$\frac{dv_1}{dt}|_{(3.1)} \leqslant -\frac{c_1}{2b_1}v_1 + \frac{L_{12}}{a_2}v_2 + \cdots + \frac{L_{1r}}{a_r}v_r,$$

by same way, we can obtain:

$$\frac{dv_2}{dt}|_{(3.1)} \leqslant \frac{L_{21}}{a_1}v_1 - \frac{c_2}{2b_2}v_2 + \cdots + \frac{L_{2r}}{a_r}v_r,$$

$$\cdots$$

$$\frac{dv_r}{dt}|_{(3.1)} \leqslant \frac{L_{r1}}{a_1}v_1 + \frac{L_{r2}}{a_2}v_2 + \cdots - \frac{c_r}{2b_r}v_r$$

denote matrix

$$M = \begin{bmatrix} -\frac{c_1}{2b_1} & \frac{L_{12}}{a_2} & \cdots & \frac{L_{1r}}{a_r} \\ \frac{L_{21}}{a_1} & -\frac{c_2}{2b_2} & \cdots & \frac{L_{2r}}{a_r} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{L_{r1}}{a_1} & \frac{L_{r2}}{a_2} & \cdots & -\frac{c_r}{2b_r} \end{bmatrix}.$$

Consider the following equation

$$\frac{dv^*}{dt} = Mv^*$$

(where $v^* = (v_1^*, v_2^*, \cdots, v_r^*)^T$. This is an ordinary differential equation, if it's coefficent matrix is asymptotic stable, then from Lemma 5 we can have:

$$\lim_{t\to\infty} v_1^* = \lim_{t\to\infty} v_2^* = \cdots = \lim_{t\to\infty} v_r^* = 0$$

so we have:

Theorem 3.1 if there exist postive definive quadric $v_k (k=1,2,\cdots,n)$, such that

1) $a_k(\sum_{i=1}^{n_k} x_i^{(k)^2}) \leqslant v_k \leqslant b_k(\sum_{i=1}^{n_k} x_i^{(k)^2})$,

2) $\frac{dv_k}{dt}|_{(3.2)} \leqslant -c_k(a_k(\sum_{i=1}^{n_k})x_i^{(k)^2}$,

3) matrix M is asymptotic stable

then, system (3.1) is asymptotic stable.

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高维区间动力系统的稳定性

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摘要: 本文用向量 Lyapunov 函数方法讨论了高维区间动力大系统的渐近稳定性,得到了区间动力系统渐近稳定的若干充分条件.

关键词: 区间矩阵; 向量 Lyapunov 函数; 渐近稳定性

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