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Robust H_∞ Control for Uncertain Linear Discrete-Time Systems with Variance Constraints *

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Abstract: The problem of robust controller design for uncertain linear discrete-time systems with H_∞ norm and variance constraints is considered in this paper. The goal of this problem is to design the robust controller, which stabilizes the plant for all admissible uncertainties, such that the closed-loop transfer function has an H_∞ norm less than a specified scalar and such that the variances of individual states are less than specified constants. An effective, algebraic approach is developed to achieve both H_∞ norm constraint and variance constraints for uncertain linear stochastic discrete systems. A feature of this approach is that no matching condition about uncertainty is needed.

Key words: discrete stochastic systems; uncertain systems; H_∞ contol; constrained variance design; state feedback

1 Introduction

In recent years, approaches to the synthesis of controller with mixed H_2/H_∞ performance criteria have been developed[1~3]. Although H_2/H_∞ control is closely related to many robustness problems such as sensitivity minimization, stabilization of uncertain systems and loop transfer recorvery, it still suffers from non-robustness, a situation to that of LQG control. This has been drawing much attention to robust H_2/H_∞ control; see e. g. [4,5]. However, it is quite common in stochastic control problems to have performance objectives that are naturally described in terms of the acceptable variance values of the system states. Mixed H_2/H_∞ designs may offer a way to minimize an H_2 performance criterion subject to a prespecified H_∞ norm constraint on the closed-loop transfer function, but they are not able to directly accomodate variance constraints that are imposed on individual system states.

The covariance control theory[6,7] has provided a more direct methodology for achieving the individual variance constraints than the LQG control theory. Recently, [8,9] developed covariance control techniques subject to the H_∞ norm constraint on the closed-loop transfer function, [10] investigated the problem of robust covariance control for uncertain linear continuous systems. Moreover, it is significant to study the problem which deals with the H_∞ norm constraint and individual variance constraints for uncertain linear discrete-time systems, simultaneously. Hence, the purpose of the present paper is to develop a technique for choosing a particular controller, which can achieve a specified H_∞ norm upper bound and a

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specified state variance upper bound for the uncertain systems.

In this paper, the robust variance-constrained H_2/H_∞ control problem for a class of linear uncertain systems is considered. The problem addressed is to design the robust controller such that the closed-loop system simultaneously satisfies the prespecified H_∞ norm constraint and the prespecified individual variance constraints. An effective, algebraic, modified Riccati eqution approach is developed to solve the above multiobjective design problem.

2 Problem Formulation and Assumptions

Consider a class of uncertain linear discrete-time stochastic systems described by the state-space equation of the form

$$x(k+1) = [A + \Delta A(\sigma)]x(k) + Bu(k) + Dw(k), \tag{2.1a}$$

$$y(k) = Cx(k). \tag{2.1b}$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$, $w(k) \in \mathbb{R}^{n_w}$, $y(k) \in \mathbb{R}^{n_y}$ and A, B, D, C are constant matrices with appropriate dimensions. $w(k)$ is a zero mean white noise process with covariance I, and $w(t)$ and $x(0)$ are uncorrelated. The notations "$[\cdot]>0$" and "$[\cdot]\geqslant 0$", respectively, denote positive definite and positive semidefinite. σ is the model parameter uncertainty and $\Delta A(\cdot)$ represents the system matrix uncertainty. $\Delta A(\sigma)$ depends on the parameter σ.

Now, we make the following assumptions which are useful in the proof of the main results.

Assumption 1 Uncertain parameter σ belongs to a prescribed compact subset, and $\Delta A(\sigma)$ is an unknown matrix function which is bounded as $\|\Delta A(\sigma)\| \leqslant \alpha$, where $\|\cdot\|$ means the spectral norm and α is a positive constant.

Assumption 2 The pair (A, B) is stabilizable and $DD^T > 0$.

Remark 1 The reason for Assumption 1 and 2 can be found in [10] and [8], which respectively dealed with robust control and covariance control problems. It should be noticed that no matching condition about uncertainty is needed.

Let the state feedback control law be given by $u(k) = Gx(k)$, then the closed-loop system is governed by

$$x(k+1) = [A_c + \Delta A(\sigma)]x(k) + Dw(k), \quad A_c = A + BG, \tag{2.2a}$$

$$y(k) = Cx(k). \tag{2.2b}$$

Furthermore, the closed-loop transfer function $H(z)$ from noise input $w(k)$ to output $y(k)$ may be written as $H(z) = C[zI - (A_c + \Delta A(\sigma))]^{-1}D$. If the closed-loop system (2.4) is asymptotically stable, then the steady-state covariance X defined as $X = \lim_{k\to\infty} E[x(k)x^T(k)]$ exists and satisfies the following discrete Lyapunov equation

$$X = [A_c + \Delta A(\sigma)]X[A_c + \Delta A(\sigma)]^T + DD^T. \tag{2.3}$$

Now, We can formulate the problem under study as follows.

Robust H_∞ control problem with variance constraints: For the uncertain system (2.4), determine the state-feedback gain, G, such that the following performance criteria are simultaneously met.

a) The closed-loop system (2.4) is asymptotically stable, i.e. $A_c + \Delta A(\sigma)$ is asymptotically stable in the face of the existence of uncertainty.

b) The H_∞ norm of the disturbance transfer matrix $H(z)$ from $w(k)$ to $y(k)$ meets the constraint $\|H(z)\|_\infty \leqslant \nu$, where $\|H(z)\|_\infty = \sup_{\theta\in[0,2\pi]} \sigma_{\max}[H(e^{j\theta})]$ and $\sigma_{\max}[\cdot]$ denotes the largest singular value of $[\cdot]$; and ν is a given positive constant.

c) The individual state variance constraints are satisfied, i. e., $[X]_{ii} \leqslant \sigma_i^2, i = 1,2,\cdots,n_x$, where $[X]_{ii}$ is the ith diagonal element of X, and $\sigma_i (i = 1,2,\cdots,n_x)$ denotes the root-mean-squared value constraint for the variance of system state.

3 Main Results and Proofs

In this section, we first establish the conditions for the existence of the feedback gain which achieves both the robust stability constraint and the robust performance constraints. This leads to the modification of an algebraic Riccati equation which enforces the robust H_∞ constraints and the robust variance constraints. The following result expresses the performance in terms of the H_∞ norm of the disturbance transfer matrix and upper bounds for the actual closed-loop steady-state covariance X.

Lemma 1 Let the H_∞ norm upper bound ν and the state-feedback gain G be given. If there exist a positive definite matrix P and a scalar parameter $\varepsilon > 0$ such that

$$Q < \varepsilon I, \tag{3.1}$$

$$P = A_c[Q(\varepsilon I - Q)^{-1}Q + Q]A_c^{\mathrm{T}} + \alpha^2\varepsilon I + DD^{\mathrm{T}}. \tag{3.2}$$

where

$$Q = P + PC^{\mathrm{T}}(\nu^2 I - CPC^{\mathrm{T}})^{-1}CP. \tag{3.3}$$

Then $A_c + \Delta A(\sigma)$ is asymptotically stable for all admissible perturbations and

$$\|H(z)\|_\infty \leqslant \nu. \tag{3.4}$$

and

$$X \leqslant P. \tag{3.5}$$

Proof Note that for the symmetric nonnegative matrix $\Delta A(\sigma)\Delta A(\sigma)^{\mathrm{T}}$, we have

$$\Delta A(\sigma)\Delta A(\sigma)^{\mathrm{T}} \leqslant \|\Delta A(\sigma)\Delta A(\sigma)^{\mathrm{T}}\| I \leqslant \|\Delta A(\sigma)\|^2 I \leqslant \alpha^2 I.$$

Define $R(\sigma) = [A_c Q(\varepsilon I - Q)^{-1/2} - \Delta A(\sigma)(\varepsilon I - Q)^{1/2}]$, then

$$\begin{aligned} 0 &\leqslant R(\sigma)R(\sigma)^{\mathrm{T}} \\ &= A_c Q(\varepsilon I - Q)^{-1} Q A_c^{\mathrm{T}} - A_c Q \Delta A(\sigma)^{\mathrm{T}} - \Delta A(\sigma) Q A_c^{\mathrm{T}} + \Delta A(\sigma)(\varepsilon I - Q)\Delta A(\sigma)^{\mathrm{T}} \\ &\leqslant A_c Q(\varepsilon I - Q)^{-1} Q A_c^{\mathrm{T}} - [(A_c + \Delta A(\sigma))Q(A_c + \Delta A(\sigma))^{\mathrm{T}} - A_c Q(A_c^{\mathrm{T}})] + \alpha^2 \varepsilon I \\ &= \prod \end{aligned}$$

and therefore

$$A_c[Q(\varepsilon I - Q)^{-1}Q + Q]A_c^{\mathrm{T}} + \alpha^2\varepsilon I = \prod + (A_c + \Delta A(\sigma))Q(A_c + \Delta A(\sigma))^{\mathrm{T}}. \tag{3.6}$$

From (3.3) and (3.6), (3.2) can be rewritten as

$$P = (A_c + \Delta A(\sigma))[P + PC^{\mathrm{T}}(\nu^2 I - CPC^{\mathrm{T}})^{-1}CP](A_c + \Delta A(\sigma))^{\mathrm{T}} + \prod + DD^{\mathrm{T}}.$$

The asymptotically stability of the closed-loop matrix $A_c + \Delta A(\sigma)$ can be guaranteed since $\prod + DD^{\mathrm{T}} > 0$. The proof of (3.4) and (3.5) can be easily completed in a manner similar to that of Lemma 2.1 of [3]. This proves the lemma.

Remark 2 Lemma 1 shows that, given a state-feedback gain G for which there exists a positive definite solution to (3. 1)(3. 2), the robust stability constraint and the H_∞ disturbance attenuation are automatically enforced. Furthermore, the actual H_2 performance of the controller is guaranteed to be no worse than the bound given by P.

By using the above lemma, we can appropriately assign P with (3. 1) and

$$[P]_{ii} \leqslant \sigma_i^2 \quad (i = 1, 2, \cdots, n_x) \tag{3.7}$$

where σ_i have been defined in (2. 9), then we seek the set of the state-feedback gain G which satisfies (3. 2) for the specified P. If such a state-feedback gain exists and can be obtained, then from Lemma 1, the following results can be guaranteed: 1) closed-loop robust stability; 2) prespecified H_∞ disturbance attenuation ν; 3) $[X]_{ii} \leqslant [P]_{ii} \leqslant \sigma_i^2 (i = 1, 2, \cdots, n_x)$. Hence, the problem of robust H_∞ control with variance constraints will be solved. To this end, the problem considered in this paper can be converted to the following auxiliary "P-matrix assignment" problem.

"P-matrix assignment" problem: 1) find the conditions under which there exists a state-feedback gain G satisfying (3. 1)(3. 2) for the specified P. In this case, the given positive definite matrix P is called an assignable matrix. 2) find the set of all feedback gains that can achieve the assignable matrix P. In what follows, this auxiliary problem will be solved completely.

Lemma 2[6] Let $M \in \mathbb{R}^{m \times n}$ and $N \in \mathbb{R}^{m \times p} (m \leqslant p)$. There exists a matrix V which simultaneously satisfies $N = MV, VV^{\mathrm{T}} = I$ if and only if $MM^{\mathrm{T}} = NN^{\mathrm{T}}$. In this case, a general soluation for V can be expressed as

$$V = V_M \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_N^{\mathrm{T}}, \quad U \in \mathbb{R}^{(n-r_M) \times (p-r_M)}, \quad UU^{\mathrm{T}} = I. \tag{3.8}$$

where V_M and V_N come from the singular value decomposition of M and N, respectively,

$$M = U_M \begin{bmatrix} Z_M & 0 \\ 0 & 0 \end{bmatrix} V_M^{\mathrm{T}} = [U_{M1} \quad U_{M2}] \begin{bmatrix} Z_M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{M1}^{\mathrm{T}} \\ V_{M2}^{\mathrm{T}} \end{bmatrix},$$

$$N = U_N \begin{bmatrix} Z_N & 0 \\ 0 & 0 \end{bmatrix} V_N^{\mathrm{T}} = [U_{N1} \quad U_{N2}] \begin{bmatrix} Z_N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{N1}^{\mathrm{T}} \\ V_{N2}^{\mathrm{T}} \end{bmatrix}$$

and
$$r_M = \mathrm{rank}(M), \quad U_M = U_N, \quad Z_M = Z_N.$$

Now, we can rewrite (3. 2) as follows:

$$A_c[Q(\varepsilon I - Q)^{-1}Q + Q]A_c^{\mathrm{T}} = P - \alpha^2 \varepsilon I - DD^{\mathrm{T}}. \tag{3.9}$$

Consider (3. 9), since its left-hand side is positive semidefinite, P is required to meet

$$P \geqslant \alpha^2 \varepsilon I + DD^{\mathrm{T}}. \tag{3.10}$$

Hence, we first define $R = Q(\varepsilon I - Q)^{-1}Q + Q$ and $S = P - \alpha^2 \varepsilon I - DD^{\mathrm{T}}$, then take the square roots of R and S

$$R = HH^{\mathrm{T}}, \quad S = TT^{\mathrm{T}}, \quad H, T \in \mathbb{R}^{n_x \times n_x}. \tag{3.11}$$

Since (3. 9) can be rearranged as follows:

$$(A_c H)(A_c H)^{\mathrm{T}} = TT^{\mathrm{T}}, \tag{3.12}$$

then from Lemma 2, (3. 9) is equivalent to

$$BG = TVH^{-1} - A \tag{3.13}$$

where $V \in \mathbb{R}^{n_x \times n_x}$ is some orthogonal matrix.

It follows from [11] that (3.13) has a solution for G if and only if there exists an orthogonal matrix V such that

$$(I - BB^+)AH = (I - BB^+)TV \tag{3.14}$$

where B^+ denotes the Moore-Penrose inverse of B.

By using Lemma 2, (3.14) means

$$[(I - BB^+)AH][(I - BB^+)AH]^{\mathrm{T}} = [(I - BB^+)T][(I - BB^+)T]^{\mathrm{T}}, \tag{3.15a}$$

or equivalently

$$(I - BB^+)(ARA^{\mathrm{T}} - S)(I - BB^+) = 0 \tag{3.15b}$$

which leads to the following result.

Theorem 1 A specified positive definite matrix P satisfying (3.7) is assignable if and only if there exists a scalar parameter $\varepsilon > 0$ such that

$$Q < \varepsilon I, \tag{3.16}$$

$$P \geqslant \alpha^2 \varepsilon I + DD^{\mathrm{T}}, \tag{3.17}$$

$$(I - BB^+)\{A[Q(\varepsilon I - Q)^{-1}Q + Q]A^{\mathrm{T}} - P + \alpha^2 \varepsilon I + DD^{\mathrm{T}}\}(I - BB^+) = 0 \tag{3.18}$$

where $Q = P + PC^{\mathrm{T}}(\nu^2 I - CPC^{\mathrm{T}})^{-1}CP$.

Now, we will characterize the feedback gain guaranteeing the mixed robust, H_∞ and variance constraints. We first take the following singular value decomposition:

$$M = (I - BB^+)T = U_M \begin{bmatrix} Z_M & 0 \\ 0 & 0 \end{bmatrix} V_M^{\mathrm{T}}, \tag{3.19a}$$

$$N = (I - BB^+)AH = U_N \begin{bmatrix} Z_N & 0 \\ 0 & 0 \end{bmatrix} V_N^{\mathrm{T}}. \tag{3.19b}$$

It follows Theorem 1 and [11] that, if the given positive definite matrix is assignable, then a general solution of (3.13) is

$$G = B^+ (TVH^{-1} - A) + (I - B^+ B)Z \tag{3.20}$$

where $Z \in \mathbb{R}^{n_u \times n_x}$ is arbitrary and V is any orthogonal matrix satisfying $MV = N$.

By using Lemma 2, the orthogonal matrix V satisfying $MV = N$ can be expressed as

$$V = V_M \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_N, \quad U \in \mathbb{R}^{(n_x - r_M) \times (n_x - r_M)} \tag{3.21}$$

where matrix U is arbitrary orthogonal.

Substituting (3.21) into (3.20) yields the following theorem.

Theorem 2 Suppose that the given positive definite matrix P satisfying (3.7) is assignable, then the set of all state-feedback gains that assign this P is parameterized as

$$G = B^+ \left(TV_M \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} V_N^{\mathrm{T}} H^{-1} - A\right) + Z - B^+ BZ \tag{3.22}$$

where T, H are defined in (3.11), V_M, V_N are defined in (3.19), and $Z \in \mathbb{R}^{n_u \times n_x}$ is arbitrary, $U \in \mathbb{R}^{(n_x - r_M) \times (n_x - r_M)}$ is arbitrary orthogonal, $r_M = \mathrm{rank} M$.

Finally, the following result is easily accessible.

Theorem 3 Given the desired constant ν and the individual state variance constraints $\sigma_i^2 (i = 1, 2, \cdots, n_x)$. Assume that a specified positive definite matrix P satisfying (3.7) is assignable. Then the solution of the robust H_∞ norm and variance-constrained design problems can be obtained from (3. 22).

Remark 3 In the design of practical control systems, it is required to construct an assignable matrix P and an appropriate constant $\varepsilon > 0$ directly from the assignability conditions (3. 7)(3. 16)(3. 17)(3. 18). Note that (3. 7)(3. 16)(3. 17) are inequalities which are easy to test, the attention is then confined to deal with the nonlinear matrix equation (3. 18) whose type is very similar to the equation (46) of reference[9]. Therefore, the equation (3. 18) can be solved using the same approach adopted in Section 5 of [9] which is suitable for relatively lower order models. It should be pointed out that the proof of convergence of the mentioned algorithm has not been completed yet and is still an open question in covariance control theory[12]. For the relatively higher order model, a possible approach to solving the nonlinear programming problem (3. 7)(3. 16)(3. 17)(3. 18) is to exploit the iterative numerical search method[10]. Also, the influence of scalar $\varepsilon > 0$ upon solutions to (3. 18) can be referred to [13].

4 Conclustions

This paper has introduced a theory for designing robust feedback controllers such that the uncertain closed-loop system meets the prespecified H_∞ norm and variance constraints. A simple, effective, generalized Riccati equation approach has been developed to solve the above problem. It is shown that the above problem can be converted to "P-matrix assignment" problem and this P-matrix assignment problem has been solved completely. The existence conditions of the desired robust controllers and the set of solutions have been introduced in Section 3 of the present paper. It is not difficult to extend the results of this paper to the case of dynamic output feedback. This result will appear in the near future.

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方差约束下不确定线性离散时间系统的鲁棒 H_∞ 控制

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摘要：本文考虑具有 H_∞ 范数及方差约束的不确定线性离散时间系统的鲁棒控制器设计问题，即设计鲁棒控制器，使闭环系统对所有可允的参数扰动保持渐近稳定，同时传递函数满足预先给定的 H_∞ 范数约束，且各状态的稳态方差值不大于预先给定值. 结果表明，一种有效的代数方法可用来使不确定线性随机离散时间系统满足给定的 H_∞ 范数约束及方差约束.

关键词：离散随机系统；不确定系统；H_∞ 控制；约束方差设计；状态反馈

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