Algebraic Criteria for Stability of Linear Singular Systems with Time Delay

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Abstract: In this paper, the stability of linear singular systems with time delay is discussed, some algebraic criteria are proposed.

Key words: singular system; delay; stability; algebraic criteria

1 Introduction

The theory of singular systems as an appropriate tool dealing with the complicate systems with the construction of different levels is followed with increasing interest[1]. There is inevitably the effects of time delay in some complicate systems. Substituting for such systems, the investigation of stability in practice is made usually by using the systems in which the time delay is omitted. But it is well known that the omission for regular systems can cause the disastrous consequence. Therefore, it is necessary to investigate the conditions under which the stability can be discussed by the systems in which the time delay is omitted for more general singular systems. The problem solving involves two aspects, one is whether the given conditions ensure that the stability is independent of the delay, another is that if the stability depends upon the delay, it is needed to give an estimate of bound for time delay such that, when time delay is within the bound, the system remains its stability provided related system without time delay is of the stability.

In this paper, some algebraic criteria to the stability of linear singular systems with time delay are given. The criteria enable us to determine whether the stability is delay-independent and an interval of the delay for stability if the stability is delay-dependent.

The symbol convention used subsequently are in following. $\|\cdot\|$ is Euclidean vector norm which is also denoted the matrix norm derived by the vector norm, $\mu(\cdot)$ is the matrix measure derived by the vector norm.

Consider linear singular systems with time delay described by following equation

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau), t \geqslant 0 \tag{1.1}$$

where $E, A, B \in \mathbb{R}^{n\times n}$ are constant matrices, $x(t)$ is a variable vector of dimension n, $\tau \geqslant 0$ is a constant, E may be singular with rank $E = r \leqslant n$. In this paper, $[E, A]$ is considered regular, that is, $\det(sE - A) \not\equiv 0$, $s \in \mathbb{C}$. In addition, assume that the index of related system without time delay

$$E\dot{x}(t) = Ax(t)$$

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is one so as to avoid the impulsive terms included in the solutions of system (1.1) which have the effects of amplification for the initial disturbance.

For the matrices in system (1.1), we can choose appropriate nonsingular matrices P, Q such that

$$PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad PBQ = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{1.2}$$

where I_r is $r \times r$ identity matrix and the other matrices are of appropriate dimensions, The assumption of the index ensures that matrix A_{22} is nonsingular. Let $\hat{E} = PEQ, \hat{A} = PAQ, \hat{B} = PBQ$, then system (1.1) is equivalent to the following system.

$$\hat{E}\dot{x}(t) = \hat{A}x(t) + \hat{B}x(t-\tau), t \geqslant 0. \tag{1.3}$$

In terms of the discussion in [4], if $\phi(t) \in \mathbb{C}[[-\tau,0],\mathbb{R}^n]$ astisfies

$$A_{21}\phi_1(0) + A_{22}\phi_2(0) + B_{21}\phi_1(-\tau) + B_{22}\phi_2(-\tau) = 0$$

where $\phi_1(t) \in \mathbb{R}^r, \phi_2(t) \in \mathbb{R}^{n-r}, [(\phi_2(t))^{\mathrm{T}}, (\phi_1(t))^{\mathrm{T}}]^{\mathrm{T}} = \phi(t)$, then $\phi(t)$ is compatible initial function, the solution $x(t)$ of system (1.3) satisfying

$$x(t) = \phi(t), \quad t \in [-\tau, 0]$$

is unique existence in $[0, \infty)$, so do the related solution of system (1.1). In subsequent discussion, solution means that the solution is determined by the compatible initial function.

Definition 1.1 System (1.1) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0$$

for every solution $x(t)$ of system (1.1).

Definition 1.2 The equation

$$\det[\lambda E - A - Be^{-\lambda\tau}] = 0 \tag{1.4}$$

is called the generalized characteristic equation of system (1.1), and the roots of (1.4) are called the generalized eigenvalues of system (1.1).

Theorem 1.3[5] Suppose that every root of the equation

$$\det[\sigma A_{22} + B_{22}] = 0$$

is in the interior of the unit circle. If there exists a positive constant λ_0, Such that

$$\mathrm{Re}\lambda \leqslant -\lambda_0$$

for every generalized eigenvalue λ of system (1.1), then system (1.1)is asymptotically stable.

Lemma 1.4 Suppose that the roots of equation (1.5) are all in the interior of the unit circle and all generalized eigenvalues of system (1.1) have negative real parts. Then there exists a positive number λ_0 such that

$$\mathrm{Re}\lambda \leqslant -\lambda_0$$

for every generalized eigenvalues λ of system (1.1).

Proof Since the roots of equation (1.5) are all in the interior of the unit circle, there exists constant $\sigma_0, 0 < \sigma_0 < 1$, such that every root σ of equation (1.5) satisfies

$$|\sigma| < \sigma_0.$$

Let $\lambda_1 = \dfrac{1}{\tau}\ln\sigma_0$, then $\lambda_1 < 0$ and every root λ of the equation

$$\det [A_{22} + B_{22}e^{-\lambda\tau}] = 0$$

satisfies $\mathrm{Re}\lambda < \lambda_1$. Hence $M(\lambda), M^{-1}(\lambda)$ are bounded in the strip $S : \{\lambda \in \mathbb{C} \mid \lambda_1 \leqslant \mathrm{Re}\lambda \leqslant 0\}$, where $M(\lambda) = - A_{22} - B_{22}e^{-\lambda\tau}$.

If $\lambda \in \mathbb{C}$ such that

$$\det[\lambda I - A_{11} - B_{22}e^{-\lambda\tau} - (A_{12} + B_{12}e^{-\lambda\tau})M^{-1}(\lambda)(A_{21} + B_{21}e^{-\lambda\tau})] = 0, \quad (1.6)$$

then there exists a unit vector $\xi \in \mathbb{C}^r$, such that

$$\lambda < \xi, \xi > - < A_{11}\xi, \xi > - < B_{11}\xi, \xi > e^{-\lambda\tau}$$
$$- < (A_{12} + B_{21}e^{-\lambda\tau})M^{-1}(\lambda)(A_{21} + B_{21}e^{-\lambda\tau})\xi, \xi > = 0.$$

Let

$$a = < A_{11}\xi, \xi >, \quad b = < B_{11}\xi, \xi >,$$

$$b_1(\lambda) = < (A_{12} + B_{21}e^{-\lambda\tau})M^{-1}(\lambda)(A_{21} + B_{21}e^{-\lambda\tau})\xi, \xi >,$$

then $| a | \leqslant \| A_{11} \|$, $| b | \leqslant \| B_{11} \|$, $| b_1(\lambda) | \leqslant \| A_{12} + B_{21}e^{-\lambda\tau} \| \| M^{-1}(\lambda) \| \| A_{21} + B_{21}e^{-\lambda\tau} \|$. Note that the function

$$h(\lambda) = \lambda - a - be^{-\lambda\tau} - b_1(\lambda)$$

is analytic in S. Therefore, for every given ε, $0 < \varepsilon < 1$, there exists a positive L, such that, for $\omega = u + jv \in S$,

$$\frac{1}{|\omega|}[\, \| A_{11} \| + \| B_{11} \| \, | e^{-\omega\tau} | + \| A_{12} + B_{21}e^{-\omega\tau} \| \, \| M^{-1}(\omega) \| \, \| A_{21} + B_{21}e^{-\omega\tau} \| \,] \leqslant \varepsilon,$$

when $| v | > L$. Note that the right member of above inequality tends to infinity as $| v | \to \infty$. It shows that $u \pm j\infty$, $\lambda_1 \leqslant u \leqslant 0$ are not the limiting points of the zeros of $h(\lambda)$. Hence $h(\lambda) = 0$ has at most finite number of roots in S and so does equation (1.6). Also note that equation (1.4) can be written as

$$\det\{\mathrm{diag}[\lambda I_r - A_{11} - B_{22}e^{-\lambda\tau} - (A_{12} + B_{12}e^{-\lambda\tau})M^{-1}(\lambda)(A_{21} + B_{21}e^{-\lambda\tau}), M]\} = 0.$$

Thus equation (1.4) has at most finite number of roots $\omega_1, \cdots, \omega_m$ in S. Let

$$- \lambda_0 = \max\{\lambda_1, \mathrm{Re}\omega_1, \cdots, \mathrm{Re}\omega_m\},$$

then $\lambda_0 > 0$, and every root λ of equation (1.4) satisfies

$$\mathrm{Re}\lambda \leqslant - \lambda_0.$$

The proof is complete.

It is easy to prove the following theorem from Theorem 1.3 and Lemma 1.4.

Theorem 1.5 Suppose that the roots of equation (1.5) are all in the interior of the unit circle. If all generalized eigenvalues of system (1.1) have negative real parts, then system (1.1) is asymptotically stable.

Remark 1.6 It is obvious that the condition that the roots of equation (1.5) are all in the interior of the unit circle is equivalent to that $\rho(B_{22}A_{22}^{-1}) < 1$ from the proof of Lemma 1.4.

Remark 1.7 The condition $\rho(B_{22}A_{22}^{-1}) < 1$ is necessary. If the inequality does not holds, then system (1.1) may be unstable though all generalized eigenvalues of system (1.1) have negative real parts.

Example 1.8 Consider the system described by

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \dot{y}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} y(t) + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} y(t-1) \tag{1.7}$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_{11} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $B_{11} = B_{12} = B_{21} = 0$, $B_{22} = -I_2$.

It is obvious that all roots of the generalized characteristic equation

$$[\lambda(e^{-\lambda} - 1) - 1]^2 = 0 \tag{1.8}$$

have negative real parts. But $\rho(B_{22}A_{22}^{-1}) = 1$. It is not difficult to see that system (1.7) is equivalent to the equation

$$\frac{d^2}{dt^2}[x(t) - 2x(t-1) + x(t-2)] + \frac{d}{dt}[x(t) - x(t-1)] + x(t) = 0 \tag{1.9}$$

which has an unbounded solution by the discussion in [6]. Thus equation (1.9) is unstable. Therefore, system (1.7) is unstable.

In the following discussion, the generalized eigenvalue λ is called unstable if λ satisfies that $\mathrm{Re}\lambda \geqslant 0$. It is obvious that system (1.1) is not asymptotically stable if it has unstable generalized eigenvalues.

2 Algebraic Criteria of Stability

Theorem 1.5 gives a stability condition. But this condition is transcendental so that it is inconvenient to use. In this section, some algebraic criteria with simple form are proposed for system (1.1). We first establish a estimate of the upper bound of the real parts and the bound of the imaginary parts for unstable generalized eigenvalues.

Lemma 2.1 Suppose that $\rho(B_{22}A_{22}^{-1}) < 1$. If λ is an unstable generalized eigenvalue, and the vector $\xi \in \mathbb{C}^n$, $\xi \neq 0$ satisfies

$$(\lambda\hat{E} - \hat{A} - \hat{B}e^{-\lambda r})\xi = 0, \tag{2.1}$$

then $\hat{E}\xi \neq 0$.

Proof It is easy to see that

$$(I - \hat{E})(\hat{A} + \hat{B}e^{-\lambda r})\xi = 0$$

from formula (2.1). If $\hat{E}\xi = 0$, then

$$(I - \hat{E})(\hat{A} + \hat{B}e^{-\lambda r})(I - \hat{E})\xi = 0.$$

Notice that $(I - \hat{E})\xi \neq 0$, we have that $\det(A_{22} + B_{22}e^{-\lambda r}) = 0$, which contradict with $\rho(B_{22}A_{22}^{-1}) < 1$. It shows that $\hat{E}\xi \neq 0$. The proof is complete.

Lemma 2.2 Suppose that $\rho(B_{22}A_{22}^{-1}) < 1$. Let $D(\sigma) = \{\xi \in \mathbb{C}^n : (I - \hat{E})(\hat{A} + \hat{B}\sigma)\xi = 0\}$ and $\overline{N} = S_0 \cap (\bigcup_{|\sigma| \leqslant 1} D(\sigma))$ where $S_0 = \{\xi \in \mathbb{C}^n : \|\xi\| = 1\}$. Then $m_0 = \inf_{\xi \in \overline{N}} \{\|\hat{E}\xi\|\} > 0$.

The proof is given in Appendix A.

Lemma 2.3 Suppose that $\rho(B_{22}A_{22}^{-1}) < 1$. If λ is an unstable generalized eigenvalue, then

$$\mathrm{Re}\lambda \leqslant m_0^{-1}\mu(\hat{A} + \hat{B}e^{-\lambda r}), \tag{2.2}$$

$$-m_0^{-1}\mu(j\hat{A} + j\hat{B}e^{-\lambda r}) \leqslant I_m\lambda \leqslant m_0^{-1}\mu(-j\hat{A} - j\hat{B}e^{-\lambda r}). \tag{2.3}$$

Proof Taking a unit vector ξ such that

$$(\lambda\hat{E} - \hat{A} - \hat{B}e^{-\lambda\tau})\xi = 0,$$

we have that $\lambda\hat{E}\xi = (\hat{A} + \hat{B}e^{-\lambda\tau})\xi$. Thus

$$\|(I + h\lambda\hat{E})\xi\| = \|[I + h(\hat{A} + \hat{B}e^{-\lambda\tau})]\xi\| \tag{2.4}$$

for any positive number h. Thus

$$\|(I + h\lambda\hat{E})\xi\| - 1 \leqslant \|I + h(\hat{A} + \hat{B}e^{-\lambda\tau})\| - 1,$$

that is

$$\|[(1 + h\lambda)\xi_1^T, \xi_2^T]^T\| - 1 \leqslant \|I + \lambda(\tilde{A} + \tilde{B}e^{-\lambda\tau})\| - 1$$

where $\hat{E}\xi = [\xi_1^T, 0]^T$ and $(I - \hat{E})\xi = [0, \xi_2^T]^T$. It implies that

$$|1 + h\lambda| \cdot \|\xi_1\| - 1 \leqslant \|I + \lambda(\tilde{A} + \tilde{B}e^{-\lambda\tau})\| - 1.$$

Multiplying the inequality by h^{-1}, letting $h \to 0$ and taking limit, we obtain that

$$\mathrm{Re}\lambda \|\hat{E}\xi\| \leqslant \mu(\hat{A} + \hat{B}e^{-\lambda\tau}),$$

from the definition of the matrix measure and L'Hospital rule. Formula (2.2) consequently holds.

Similarly, using

$$(-j\lambda\hat{E} + j\hat{A} + j\hat{B}e^{-\lambda\tau})\xi = 0$$

and

$$(j\lambda\hat{E} - j\hat{A} - j\hat{B}e^{-\lambda\tau})\xi = 0,$$

we can obtain Formula (2.3). The proof is complete.

Theorem 2.4 Suppose that $\rho(B_{22}A_{22}^{-1}) < 1$. If λ is an unstable generalized eigenvalue, then

$$\mathrm{Re}\lambda \leqslant l_1, \tag{2.5}$$

$$|\mathrm{Im}\lambda| \leqslant l_2, \tag{2.6}$$

where $l_1 = (a + b)m_0^{-1}$, $l_2 = m_0^{-1}\max\{|a_1 + b|, |a_2 + b|\}$ and $a = \mu(\hat{A})$, $b = \|\hat{B}\|$, $a_1 = \mu(-j\hat{A})$, $a_2 = \mu(j\hat{A})$.

Proof From Lemma 2.3, using the property of the matrix measure, we obtain that

$$\mathrm{Re}\lambda \leqslant m_0^{-1}\mu(\hat{A} + \hat{B}e^{-\lambda\tau}) \leqslant m_0^{-1}[\mu(\hat{A}) + \mu(\hat{B}e^{-\lambda\tau})]$$
$$\leqslant m_0^{-1}(\mu(\hat{A}) + \|\hat{B}\|) \leqslant m_0^{-1}(a + b).$$

Similarly, using formula (2.3), we can obtain

$$-m_0^{-1}(a_2 + b) \leqslant \mathrm{Im}\lambda \leqslant m_0^{-1}(a_1 + b).$$

Consequently, (2.6) holds. The proof is complete.

It is easy to see the following result from Theorem 2.4.

Theorem 2.5 Suppose that $\rho(B_{22}A_{22}^{-1}) < 1$. If

$$\mu(\hat{A}) + \|\hat{B}\| < 0, \tag{2.7}$$

then system (1.1) is asymptotically stable for any $\tau > 0$.

Note that

$$\|I_{n-r} + hA_{22}\| \leqslant \left\|\begin{bmatrix}0 & 0\\ 0 & I_{n-r}\end{bmatrix} + h\begin{bmatrix}0 & 0\\ 0 & A_{22}\end{bmatrix}\right\|$$
$$\leqslant \|P_0\| \|I + h\hat{A}\| \|P_0\| = \|I + hA\|$$

where $P_0 = \begin{bmatrix}0 & 0\\ 0 & I_{n-r}\end{bmatrix}$, $h > 0$ is a constant. Therefore, if $A_{22} = -I_{n-r}$, then (2.7) implies that

$$\mu(A_{22}) + \|B_{22}\| < 0.$$

Thus

$$\|B_{22}^{(3)}\| = \|B_{22}(A_{22})^{-1}\| < 1.$$

Therefore, $\rho(B_{22}A_{22}^{-1}) < 1$. It shows the following corollary holds.

Corollary 2.6 If $A_{22} = -I_{n-r}$, and

$$\mu(\hat{A}) + \| \hat{B} \| < 0,$$

then the zero solution of system (1.1) is asymptotically stable for any $\tau > 0$.

Lemma 2.7 Suppose that $\rho(B_{22}A_{22}^{-1}) < 1$. Then the generalized eigenvalues depend continuously upon time delay τ in the right half-plane.

The proof is given in Appendix B.

Theorem 2.8 Suppose that $\rho(B_{22}A_{22}^{-1}) < 1$. and $\mu(\hat{A} + \hat{B}) < 0$. Let

$$c = (-\mu(\hat{A} + \hat{B}) / \| \hat{B} \|)^2.$$

If $c > 4$, then system (1.1) is asymptotically stable independent of delay. Otherwise system (1.1) keeps its asymptotic stability if

$$\tau < \begin{cases} l_2^{-1} \arccos(1 - \frac{c}{2}), & l_2 \neq 0, \\ \infty, & l_2 = 0. \end{cases}$$

Proof Assume that system (1.1) has an unstable generalized eigenvalue λ_0 for a time delay τ . Since the unstable generalized eigenvalue is dependent continuously of time delay, and when $\tau = 0$, the generalized eigenvalues have negative real parts, there exists an unstable generalized eigenvalue $j\theta_0$ which real parts is zero for some $\tau > 0$. From Lemma 2.3, we have

$$\mu(\hat{A} + \hat{B}e^{-j\theta_0\tau}) \geqslant 0.$$

Consequently, from the property of the matrix measure, we have

$$f(\theta_0\tau) = \mu(\hat{A} + \hat{B}) + | 1 - e^{-j\theta_0\tau} | \, \| \hat{B} \| \geqslant 0.$$

But

$$f(0) = \mu(\hat{A} + \hat{B}) < 0.$$

It implies that $f(\theta\tau) = \mu(\hat{A} + \hat{B}) + |1 - e^{-j\theta\tau}| \, \| \hat{B} \|$ has a zero in $[-\pi, \pi]$. Note that $f(-\theta\tau) = f(\theta\tau)$, $f(\theta\tau)$ has a zero $\theta_1\tau \in (0, \pi)$. It is obvious that $0 < \theta_1 \leqslant \theta_0 \leqslant l_2$, which is impossible if $l_2 = 0$.

If $l_2 > 0$, then θ_1 satisfies

$$1 - \cos(\theta_1\tau) = c/2.$$

Thus $c \leqslant 4$ and

$$\theta_1\tau = \arccos(1 - c/2).$$

Therefore, $\tau > l_2^{-1} \arccos(1 - c/2)$, it is impossible from the conditions of the theorem and the results of above discussion. It completes the proof.

3 Conclusion

In this paper, the asymptotic stability of linear singular systems with time delay are considered. Some stability criteria are established, which can fall two categories. One is delay-independent which can be used to test the stability independent of the time delay for the systems. Another depend upon the delay and gives an estimate of the bound for the delay time such that if the system is asymptotically stable without time delay, the system retains its asymptotic stability when the delay time is within the bound. The criteria are algebraic so that are convenient to use.

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Appendix A

Proof of Lemma 2.2 Otherwise, assume $m_0 = 0$. There is a convergent sequence $\{\xi_k\} \subset N$ and a sequence $\{\sigma_k\} \subset \{z \in \mathbb{C} : |z| \leqslant 1\}$ such that

$$(I - \hat{E})(\hat{A} + \hat{B}\sigma_k)\xi_k = 0, k = 1, 2, \cdots, \quad (A1)$$

$\xi_0 = \lim_{k\to\infty} \xi_k$ with $\| \xi_0 \| = 1, \hat{E}\xi_0 = 0$. In addition, $\{z \in \mathbb{C} : |z| \leqslant 1\}$ is a bounded closed set, hence we can assume that $\{\sigma_k\}$ is convergent. Let $\sigma_0 = \lim_{k\to\infty} \sigma_k$. Therefore,

$$(I - \hat{E})(\hat{A} + \hat{B}\sigma_k)(I - \hat{E})\xi_k + (I - \hat{E})(\hat{A} + \hat{B}\sigma_k)\hat{E}\xi_k = 0$$

from (A_1). Let k tend to infinity, take the limit for Formula (A_2), we obtain $(I - \hat{E})(\hat{A} + \hat{B}\sigma)(I - \hat{E})\xi_0 = 0$. Thus $(I - \hat{E})\xi_0 = 0$. This is a contradictory. Hence $m_0 > 0$.

Appendix B

Let $G = \hat{A} + \hat{B}e^{-\lambda\tau_1}, H = \hat{A} + \hat{B}e^{-\lambda\tau_2}, \tau_1, \tau_2 \geqslant 0$, and denote $G = (g_{ij}), H = (h_{ij})$.

Lemma B.1 Let

$$m_a = \max_{i,j}\{|a_{i,j}| + |b_{i,j}|\},$$

$$\| G - H \|_a = \frac{1}{n}\sum_{i,j=1}^{n} |g_{ij} - h_{ij}| = \frac{1}{n}\sum_{i,j=1}^{n} |b_{ij}|\,|e^{-\lambda\tau_1} - e^{-\lambda\tau_1}|,$$

$$\delta = (n + 2)m_a^{1-\frac{1}{n}} \| G - H \|_a.$$

Then

$$|\phi(\lambda) - \psi(\lambda)| < \delta^n, \quad \forall \lambda \in \{z \in \mathbb{C}^+_1 \mid |z| \leqslant nm_a\} \quad (B1)$$

where $\phi(\lambda) = \det(\lambda\hat{E} - G), \psi(\lambda) = \det(\lambda\hat{E} - H), \mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Re}z > 0\}$.

Proof Putting the elements of G and H in same order $g_1, \cdots, g_n^2, h_1, \cdots, h_n^2$, there is a n^2-variable polynomials p, such that

$$\phi(\lambda) = p(g_1, \cdots, g_n^2), \quad \psi(\lambda) = p(h_1, \cdots, h_n^2).$$

Let $\Delta_k = p(g_1, \cdots, g_{k-1}, h_k, h_{k+1}, \cdots, h_n{}^2), k = 1, \cdots, n^2$, then

$$\phi(\lambda) - \psi(\lambda) = \sum_{k=1}^{n^2} \Delta_k.$$

And we obtain

$$\Delta_k = \pm (g_k - h_k)\det T_k$$

from the expansion of the determinant, where $T_k \in \mathbb{C}^{(n-1)\times(n-1)}$ which has at most one $\lambda - g_i$ or $\lambda - h_j$, and others elements of which are g_k or h_l. Thus the Euclid norm of every row vector of T_k is not large than $[(n - 2)m_a^2 + (|\lambda| + m_a)^2]^{1/2} < (n + 2)m_a$, provided $|\lambda| \leqslant nm_a$ and $\mathrm{Re}\lambda > 0$. Consequently, we have

$$|\det T_k| \leqslant ((n+2)m_a)^{n-1}.$$

From Hadamard Inequality. Hence

$$|\phi(\lambda) - \psi(\lambda)| \leqslant (n+2)m_a)^{n-1} \sum_{k=1}^{n^2} | gk = h_k | \leqslant (n+2)^n m_a^{n-1} \| G - H \|_a = \delta''.$$

The proof is complete.

Lemma B.2 Let

$$\chi_t(z) = \det(z\hat{E} - (1-t)G - tH), \quad 0 \leqslant t \leqslant 1. \tag{B2}$$

Suppose that D is a close region in $\mathbb{C}$ with the boundary D consisted of finite smooth curves. If $\chi_t(z) \neq 0$ for every $t \in [0,1]$, $z \in \partial D$, then the number of the zeros of $\chi_t(z)$ in D is independent of $t \in [0,1]$.

Proof This is a immediate corollary of Rouché Theorem. The proof is complete.

Using Lemma 2.3, we know that the number of zeros of the analytic function $\phi(\lambda)$ is finite in $\mathbb{C}^+$. Let zeros of $\phi(\lambda)$ in $\mathbb{C}^+$ are $\lambda_1, \cdots, \lambda_r$. Then $\phi(\lambda)$ can be written as

$$\phi(\lambda) = \phi_0 \prod_{i=1}^{r} (\lambda - \lambda_i)$$

where $\phi_0(\lambda)$ is analytic and $\phi_0(\lambda) \neq 0$, $\lambda \in \mathbb{C}^+$. Let

$$D_1 = \bigcup_{i=1}^{r} D(\lambda_i, \eta_1)$$

where $D(\lambda, R) = \{z \in \mathbb{C} : |z - \lambda| \leqslant R\}$, $\lambda \in \mathbb{C}$, $R > 0$ denotes the disc with center λ, radius R.

Take η_1 is small sufficiently such that $D_1 \subset \mathbb{C}^+$. Let

$$m_\phi = \min\{|\phi(\lambda)|, \lambda \in \overline{D_1}\}.$$

Lemma B.3 Under the assumption of Lemma B.1, $\psi(\lambda)$ has same number of zeros as $\phi(\lambda)$ in $\overline{D_1}$ and the inequality

$$\min\{\max_{1\leqslant j\leqslant r} |\lambda_j - \mu_{\iota(j)}|\} \leqslant (2r-1)\eta \tag{B3}$$

holds, where $\eta = \min\{(\delta''/m_\phi)^r, \eta_1\}$, ι is a permutation of $1, \cdots, r$; $\mu_1, \cdots, \mu_r$ are the roots of $\psi(\lambda) = 0$.

Proof If λ is a root of $\phi(\lambda) = 0$, and $\mathrm{Re}\lambda > 0$, then there exists a unit vector $\xi \in \mathbb{C}^n$, such that

$$\lambda \hat{E} \xi = G\xi.$$

Therefore,

$$|\lambda| \leqslant \| G \|_\infty \leqslant n m_a.$$

The root λ_i and λ_j in $\mathbb{C}^n$ are called in a link if there are roots $\lambda_{\nu_1}, \cdots, \lambda_{\nu_k}$ of $\phi(\lambda) = 0$ in $\mathbb{C}^n$, such that the distance between every point and its neighbouring point(s) in the sequence $\lambda_i, \lambda_{\nu_1}, \cdots, \lambda_{\nu_k}, \lambda_j$ is less 2η. Decompose the set D_0 of the roots of $\phi(\lambda) = 0$ into several subset $S_1, \cdots, S_s$ such that all roots in a link are in same subset and the roots not in a link are included in different subsets. Let

$$\partial \mathscr{G}_k = \bigcup_{\lambda \in S_k} D(\lambda_i, \eta), \quad k = 1, \cdots, s$$

Let $\partial \mathscr{G}_k$ denotes the boundary of $\mathscr{G}_k$ which consists of finite arcs. If $\chi_t(z)(0 \leqslant t \leqslant 1)$ has not zeros in the union $\bigcup \partial \mathscr{G}_k$. The number $n_k(t)$ of the zeros of $\chi_t(z)$ in every $\mathscr{G}_k$ does not depend on $t \in [0, 1]$, $k = 1, \cdots, s$. Consequently, $\phi(\lambda)$ and $\psi(\lambda)$ have same number of zeros in every $\mathscr{G}_k$ since $\chi_0(\lambda) = \phi(\lambda)$ and $\chi_1(\lambda) = \psi(\lambda)$. Suppose that the number of the zeros of them in $\mathscr{G}_1$ in n_1, then the distance between every zero λ_i of $\phi(\lambda)$ and any zero μ_j of $\psi(\lambda)$ in $\mathscr{G}_1$ satisfies

$$|\lambda_i - \mu_j| \leqslant (2n_1 - 1)\eta \leqslant (2r - 1)\eta.$$

The inequality keeps holds in others ζ_k. Thus, there is a permutation ζ of $1, \cdots, r$ such that

$$|\lambda_i - \mu_{\zeta(i)}| \leqslant (2r - 1)\eta.$$

It implies that (B3) holds. It remains to prove that $\chi_t(z)(0 \leqslant t \leqslant 1)$ have not any zero in $\bigcup_{k=1}^{s} \partial \mathscr{G}_k$. In the fact, the conclusion is truth when $t = 0$, If there is a $t \in (0, 1]$ such that $\chi_t(\lambda') = 0$, $\lambda' \in \mathbb{C}^+$. Let $\tilde{H} = (1 - t)G + tH = (\tilde{h}_{ij})$, and $\tilde{\psi}(\lambda) = \det(\lambda \hat{E} - \tilde{H})$. Then

$$\phi_0(\lambda')\prod_{i=1}^{r}|\lambda'-\lambda_i| = |\phi(\lambda')| = |\phi(\lambda')-\hat{\psi}(\lambda')|$$

since $\hat{\phi}(\lambda')=0$. But $|\hat{h}_{ij}|< m_a$ from the definition of $\hat{H}$, hence the zero λ' of $\hat{\psi}(\lambda)$ satisfies $|\lambda'|\leqslant\|H\|_\infty\leqslant nm_a$. Therefore,

$$\prod_{i=1}^{r}|\lambda'-\lambda_i| = |\phi(\lambda')-\hat{\psi}(\lambda')|/\phi_0(\lambda') < \delta^n/m_\phi = \eta^r$$

from Lemma B.1. There exists consequently $\lambda_{i'}$, such that $|\lambda'-\lambda_i'|<\eta$, it implies that there exists $k, 1\leqslant k\leqslant r$ such that λ' is included in a $\mathscr{G}_k$, thus $\chi_t(z)(0\leqslant t\leqslant 1)$ have not any zero in $\bigcup_{k=1}^{t}\partial\mathscr{G}_k$. It completes the proof.

Lemma B.4 Let $g(\lambda\tau)=e^{-\lambda\tau}, \lambda\in\overline{D}_1, \tau\geqslant 0$. Then

$$|g(\lambda r_1)-g(\lambda r_2)|< K|r_1-r_2|$$

for any $\tau_1,\tau_2\geqslant 0$ and $\lambda\in\overline{D}_1$.

Proof In the fact, in terms of L'Hospital rule

$$\lim_{r_1\to r_2}\frac{e^{-\lambda r_1}-e^{-\lambda r_2}}{r_1-r_2} = -\lambda e^{-\lambda r_2},$$

thus $|e^{-\lambda r_1}-e^{-\lambda r_2}|<K|\tau_1-\tau_2|$, where $K=\max_{\lambda\in\overline{D}_1}|\lambda e^{-\lambda r_2}|+1$. The proof is complete.

It follows Lemma 2.7 immediately from Lemma B.3 and Lemma B.4.

具时滞的线性奇异系统稳定性的代数判据

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摘要:本文讨论了具时滞的线性奇异系统的稳定性,建立了几个稳定性的代数判据.

关键词:奇异系统;时滞;稳定性;代数判据

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