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New Necessary and Sufficient Condition for the Stability of Symmetric Interval Matrices

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Abstract: In this present paper the Hurwitz stability of symmetric interval matrices is studied and a simple necessary and sufficient condition is given. Additionally, the stability of linear time-varying interval matrices and nonlinear time-varying interval matrices is also studied.

Key words: symmetric interval matrices ; stability; sign function; linear time-varying interval matrices; nonlinear time-varying interval matrices

Notation $\mathbb{R}$: real-number field; x : a vector; $x = (x_1, x_2, \cdots, x_n)^T$; x^T: transpose of vector x; $\lambda_i(A)$ ith eigenvalue of matrix A ; sgn (s_i) s_i is a scalar, and sgn $(s_i) = 1$ if $s_i > 0$ sgn$(s_i) = 0$ if $s_i = 0$; sgn$(s_i) = -1$ if $s_i < 0$; sgn (s) where $s \in \mathbb{R}^{m\times 1}$, sgn$(s) = (\text{sgn}(s_1), \text{sgn}(s_2), \cdots, \text{sgn}(s_m))^T$.

1 Introduction and Definition

Consider interval matrices

$$G[B,C] = \{A \mid B \leqslant A \leqslant C\},$$

and linear time-varying interval matrices

$$G(t)[B,C] = \{A(t) \mid B \leqslant A(t) \leqslant C\},$$

and nonlinear time-varying interval matrices

$$G(x,t)[B,C] = \{A(x,t) \mid B \leqslant A(x,t) \leqslant C\}.$$

The vertices set of $G[B,C]$, $G(t)[B,C]$ and $G(x,t)[B,C]$ is $H[B,C]$.

$$H[B,C] = \{A \mid A = (a_{ij})_{n\times n}, a_{ij} = b_{ij} \text{ or } c_{ij}\}$$

(Where $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}, C = (c_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, A(t) = (a_{ij}(t))_{n\times n}, A(x,t) = (a_{ij}(x, t))_{n\times n}$).

Definition 1.1 Interval matrices $G[B,C]$ is said to be Hurwitz stable if and only if for any $A \in G[B,C]$, $\text{Re}\,\lambda_j(A) < 0$, $i = 1, 2, \cdots, n$.

Definition 1.2 Linear time-varying interval matrices $G(t)[B,C]$ is said to be stable, if for any $A(t) \in G(t)[B,C]$, the system

$$\dot{x}(t) = A(t)x(t) \tag{1.1}$$

is asymptotic stable.

Definition 1.3 Nonlinear time-varying interval matrices $G(x,t)[B,C]$ is said to be stable, if for any $A(x,t) \in G(x,t)[B,C]$, the system

$$\dot{x}(t) = A(x,t)x(t) \tag{1.2}$$

is global asymptotic stable.

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In 1983, Bailas proposed a sufficient and necessary condition for Hurwitz stability of interval matrices[1], but Barmish, Hollot, William raised examples showing that the result of Bailas was incorrect (see[2],[3]). Before long, Shi Z. C. and Gao W. B. [4], Jiang C. L. [5], M. Mansour[6], Soh, C. B. [7] revised the result of paper [1], and proposed the necessary and sufficient condition for Hurwitz stability of symmetric interval matrices respectively in articles[4~7].

The main result of paper[4~7] is: $G[B,C]$ is Hurwitz stable if and only if $H[B,C]$ is Hurwitz stable. The main weakness of this result is that the calculation process is especially complex as for $n \times n$ interval matrices, need to test stability of $2^{n(n+1)/2}$ matrices of $n \times n$ order. For example, to verify stability of 3×3 symmetric interval matrices, the stability of 64 matrices of 3×3 order should be tested. Therefore, this result is difficult to be appiled.

In this paper, for asymmetric interval matrices $G[B,C]$, we will give a simpler result. With our method, to verify Hurwitz stability of $G[B,C]$, it needs only to test 2^{n-1} matrices. With regard to 3×3 interval matrices, we need only to test the stability of 4 matrices of 3×3 order. We will prove this result in the following section. Additionally, the stability of linear time-varying interval matrices $G(t)[B,C]$ and nonlinear time-varying interval matrices $G(x,t)[B,C]$ is also discussed.

2 Result and Proof

Let $x \in \mathbb{R}^n$, consider the matrix

$$\operatorname{sgn}(x)^{\mathrm{T}}\operatorname{sgn}(x) = \begin{bmatrix} \operatorname{sgn}(x_1)\operatorname{sgn}(x_1) & \operatorname{sng}(x_1)\operatorname{sgn}(x_2) & \cdots & \operatorname{sgn}(x_1)\operatorname{sgn}(x_n) \\ \operatorname{sgn}(x_2)\operatorname{sgn}(x_1) & \operatorname{sgn}(x_2)\operatorname{sgn}(x_2) & \cdots & \operatorname{sgn}(x_2)\operatorname{sgn}(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ \operatorname{sgn}(x_n)\operatorname{sgn}(x_1) & \operatorname{sgn}(x_n)\operatorname{sgn}(x_2) & \cdots & \operatorname{sgn}(x_n)\operatorname{sgn}(x_n) \end{bmatrix}.$$

Let

$$\operatorname{sgn}(\mathbb{R}^n) = \{\operatorname{sgn}(x)^{\mathrm{T}}\operatorname{sgn}(x) \mid x \in \mathbb{R}^n, x_1x_2\cdots x_n \neq 0\}.$$

Obviously, any $S = (s_{ij})_{n\times n} \in \operatorname{sgn}(\mathbb{R}^n)$ satisfies the following properties:

1) $s_{ij} = 1$ or -1;

2) S is a symmetric matrix.

Definition mapping:

$$L: S \to L(S)$$

for any $S = (s_{ij})_{n\times n} \in \operatorname{sgn}(\mathbb{R}^n)$, $L(S) = (l_{ij})_{n\times n}$, where

$$l_{ij} = \begin{cases} c_{ij}, & \text{as} \quad s_{ij} = 1; \\ -b_{ij}, & \text{as} \quad s_{ij} = -1 \end{cases}$$

for convenience, we denote

$$L[B,C] = L(\operatorname{sgn}(\mathbb{R}^n)) = \{L(S) \mid S = (s_{ij})_{n\times n} \in \operatorname{sgn}(\mathbb{R}^n)\}.$$

For example: let $G[B,C] = \begin{bmatrix} [b_{11}, c_{11}] & [b_{12}, c_{12}] \\ [b_{21}, c_{21}] & [b_{22}, c_{22}] \end{bmatrix}$, then

$$L[B,C] = \left\{ \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \begin{bmatrix} c_{11} & -b_{12} \\ -b_{21} & c_{22} \end{bmatrix} \right\}.$$

Theorem 2. 1 The symmetric interval matrices $G[B,C]$ is Hurwitz stable, if and only if $L[B,C]$ is Hurwitz stable.

Proof First, suppose $L[B,C]$ is Hurwits stable, for any $A \in G[B,C]$, considering system

$$\dot{x} = Ax. \tag{2.1}$$

Let $V(x) = x^{\mathrm{T}}x$, we denote $(|x|)^{\mathrm{T}} = (|x_1|, |x_2|, \cdots, |x_n|)^{\mathrm{T}}$, then

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t}\Big|_{(2.1)}$$

$$= 2x^{\mathrm{T}}Ax$$

$$= 2(|x|)^{\mathrm{T}} \begin{pmatrix} \mathrm{sgn}(x_1)\mathrm{sgn}(x_1)a_{11} & \mathrm{sgn}(x_1)\mathrm{sgn}(x_2)a_{12} & \cdots & \mathrm{sgn}(x_1)\mathrm{sgn}(x_n)a_{1n} \\ \mathrm{sgn}(x_2)\mathrm{sgn}(x_1)a_{21} & \mathrm{sgn}(x_2)\mathrm{sgn}(x_2)a_{22} & \cdots & \mathrm{sgn}(x_2)\mathrm{sgn}(x_n)a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{sgn}(x_n)\mathrm{sgn}(x_1)a_{n1} & \mathrm{sgn}(x_n)\mathrm{sgn}(x_2)a_{n2} & \cdots & \mathrm{sgn}(x_n)\mathrm{sgn}(x_n)a_{nn} \end{pmatrix} (|x|)$$

since

$$\mathrm{sgn}(x_i)\mathrm{sgn}(x_j)a_{ij} \leqslant c_{ij}, \quad \text{when} \quad \mathrm{sgn}(x_i)\mathrm{sgn}(x_j) = 1;$$
$$\mathrm{sgn}(x_i)\mathrm{sgn}(x_j)a_{ij} \leqslant -b_{ij}, \quad \text{when} \quad \mathrm{sgn}(x_i)\mathrm{sgn}(x_j) = -1.$$

Therefore, for any $x \in \mathbb{R}^n (x_1x_2\cdots x_n \neq 0)$, there exists $L(S) \in L[B,C](S \in \mathrm{sgn}\ (\mathbb{R}^n))$, such that

$$\frac{\mathrm{d}V(x)}{\mathrm{d}t}\Big|_{(2.1)} \leqslant 2(|x|)^{\mathrm{T}}L(S)(|x|). \tag{2.2}$$

If for any $x \in \mathbb{R}^n$, there exist $i, j(i, j = 1, 2, \cdots, n)$ such that $\mathrm{sgn}(x_i)\mathrm{sgn}(x_j) = 0$, then, we have $\mathrm{sgn}\ (x_i) = 0$ or $\mathrm{sgn}(x_j) = 0$, then $x_i = 0$ or $x_j = 0$, if we suppose $x_i = 0$, then for any $0 \leqslant j \leqslant n$

$$\mathrm{sng}(x_i)\mathrm{sgn}(x_j)a_{ij}|x_i\|x_j| = c_{ij}|x_i\|x_j| = -b_{ij}|x_i\|x_j| = 0.$$

Therefore, there exists $L(S) \in L[B,C](S \in \mathrm{sgn}(\mathbb{R}^n))$ such that (2. 2) is satisfied. So (2. 2) is valid in this case.

Since $L(S)$ is a symmetric matrix, and $L(S)$ is Hurwitz stable, so (2. 1) asymptotic stable.

In other hand, suppose $G[B,C]$ is Hurwitz stable, for any $L \in L[B,C]$, there exist $A = (a_{ij})_{n\times n} \in H[B,C]$ and $y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} \in \mathbb{R}^n$, such that $L = (\mathrm{sgn}(y_i)\mathrm{sgn}(y_j)a_{ij})_{n\times n}$, so

$$x^{\mathrm{T}}Lx = (\mathrm{sgn}(y_1)x_1, \mathrm{sgn}(y_2)x_2, \cdots, \mathrm{sgn}(y_n)x_n)A(\mathrm{sgn}(y_1)x_1, \mathrm{sgn}(y_2)x_2, \cdots, \mathrm{sgn}(y_n)x_n)^{\mathrm{T}}$$
$$= z^{\mathrm{T}}Az$$

since L, A are symmetric matrices, from above formula, we have: A is asymptotic stable implying L is Hurwitz stable.

Remark 2. 1 Since the cardinal number of the set $\mathrm{sgn}(\mathbb{R}^n)$ equals 2^{n-1}, so $L[B,C]$ only has 2^{n-1} matrices. With our result, to test the stability of $G[B,C]$ we need only to verify the stability of 2^{n-1} matrices in $L[B,C]$. However, according to the results given by Shi Zhicheng and Gao Weibin[4], Jiang Chongli[5], M. Mansour[6] and Soh[7], to test the stability of $G[B,C]$ we need to verify the stability of $2^{n(n+1)/2}$ matrices. It is obvious that our results are simpler than the previous.

By the same method as proof of Theorem 2.1, we can prove the following results.

Theorem 2.2 The symmetric linear time-varying interval matrices $G(t)[B,C]$ is stable, if and only if $L[B,C]$ is Hurwitz stable.

Theorem 2.3 The symmetric nonlinear time-varying interval matrices $G(x,t)[B,C]$ is stable, if and only if $L[B,C]$ is Hurwitz stable.

For general nonlinear time-varying interval matrices $G(x,t)[B,C]$, we have the following theorem.

Theorem 2.4 If for any matrices $L \in L[B,C]$, $L^T + L$ is negative definition, then nonlinear time-varying interval matrices $G(x,t)[B,C]$ is stable.

Proof For any $A(x,t) \in G(x,t)[B,C]$, consider the following nonlinear system

$$\dot{x}(t) = A(x,t)x(t). \tag{2.3}$$

Let $V(x) = x^T x$, by the same way as proof of theorem 2.1, we know that for any $x \in \mathbb{R}^n$, there exist matrices $L \in L[B,C]$, such that

$$\frac{dV(x)}{dt}\Big|_{(2.3)} \leqslant 2(|x|)^T L(|x|) \leqslant \lambda_{\max}(L^T + L)(|x|)^T(|x|) = \lambda_{\max}(L^T + L)x^T x.$$

If the condition of this theorem is satisfied, then $\frac{dV(x)}{dt}\Big|_{(2.3)}$ is negative definition. This complete proof of this theorem.

Remark 2.2 Since $G[B,C]$ and $G(t)[B,C]$ are subsets of $G(x,t)[B,C]$, so, if we replace $G(x,t)[B,C]$ with $G[B,C]$ or $G(t)[B,C]$, the result of Theorem 2.4 is also valid.

3 Examples

Example 3.1 Consider

$$G[B,C] = \begin{bmatrix} [b_{11},c_{11}] & [b_{12},c_{12}] & [b_{13},c_{13}] \\ [b_{21},c_{21}] & [b_{22},c_{22}] & [b_{23},c_{23}] \\ [b_{31},c_{31}] & [b_{32},c_{32}] & [b_{33},c_{33}] \end{bmatrix}.$$

Since

$$\mathrm{sgn}(\mathbb{R}^3) = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \right\},$$

so

$$L[B,G] = \left\{ \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \begin{bmatrix} c_{11} & -b_{12} & -b_{13} \\ -b_{21} & c_{22} & c_{23} \\ -b_{31} & c_{32} & c_{33} \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} & -b_{13} \\ c_{21} & c_{22} & -b_{23} \\ -b_{31} & -b_{32} & c_{33} \end{bmatrix}, \begin{bmatrix} c_{11} & -b_{12} & c_{13} \\ -b_{21} & c_{22} & -b_{23} \\ c_{31} & -b_{32} & c_{33} \end{bmatrix} \right\}.$$

Suppose

$$B = \begin{bmatrix} -6 & -2 & 1 \\ -2 & -6 & -2 \\ 1 & -2 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} -4 & 1 & 1.9 \\ 1 & -4 & 1.9 \\ 1.9 & 1.9 & -4 \end{bmatrix},$$

then

$$L[B,C]=\left\{\begin{bmatrix}-4 & 1 & 1.9\\ 1 & -4 & 1.9\\ 1.9 & 1.9 & -4\end{bmatrix},\begin{bmatrix}-4 & 2 & -1\\ 2 & -4 & 1.9\\ -1 & 1.9 & -4\end{bmatrix},\begin{bmatrix}-4 & 1 & -1\\ 1 & -4 & 2\\ -1 & 2 & -4\end{bmatrix},\begin{bmatrix}-4 & 2 & 1.9\\ 2 & -4 & 2\\ 1.9 & 2 & -4\end{bmatrix}\right\}.$$

Since $L[B,C]$ is Hurwitz stable, from Theorem 2.1, we can declare $G[B,C]$ is Hurwitz stable.

Examples 3.2 Consider system

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix}=\begin{pmatrix}-4+\sin(x+t) & 2+\cos(x-t)\\ 2+\cos(x-t) & -5+\sin(x+t)\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}. \tag{3.1}$$

Obviously,

$$\begin{pmatrix}-4+\sin(x+t) & 2+\cos(x-t)\\ 2+\cos(x-t) & -5+\sin(x+t)\end{pmatrix}\in G(x,t)\begin{pmatrix}[-5,-3] & [1,3]\\ [1,3] & [-6,-4]\end{pmatrix},$$

from Theorem 2.3, we can declare that system (3.1) is global asymptotic stable.

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区间矩阵稳定的新充分必要条件

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摘要:本文讨论了对称区间矩阵的稳定性并给出了一个简单的充分必要条件,同时还讨论了线性时变区间矩阵和非线性时变区间矩阵的稳定性.

关键词:对称区间矩阵;稳定性;符号函数;线性时变区间矩阵;非线性时变区间矩阵

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