

Robust Delay-Dependence Stability for Linear Systems with Nonlinear Parameter Perturbations *

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Abstract: The problem of robust stability of delay dependence for both single and composite uncertain delay differential systems are considered. Making use of the delay integral inequality, some sufficient conditions for robust stability, in terms of a bound on the spectral radius of a prescribed nonnegative matrix, are presented. Examples are given to demonstrate the validity of our results.

Key words: robustness; stability; delay differential systems

1 Introduction

In recent years, the robust stability for delay differential systems with uncertainties has received considerable attention^[1,2,3], because in many practical control problems uncertainty may also occur in delay systems due to modelling errors, measurement errors, linearization approximation, and so on. The problem is difficult because the systems with delays are inherently distributed parameter systems, and the delay is assumed to be unknown. In [4], the delay-independent stability is discussed and some effective criteria which do not include information on delay are obtained. It is not quite reasonable to ask a stability criteria for the delay system to have that property, although it is more easier to attack mathematically. More recently, therefore, more effort is devoted to delay-dependent stability criteria^[1,2].

In this paper we use the inequality technique^[3] and obtain some new sufficient conditions for robust stability of delay dependence. In application, better bounds than those reported in [1,2] are given for both the delays and the perturbations.

2 Main Results

Consider the following uncertain dynamical systems with delays

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1(t)) + \Delta A_0(t, x(t)) + \Delta A_1(t, x(t - h_2(t))), \quad (1)$$

where $x(t) \in \mathbb{R}^n$, A_0 and A_1 are known constant matrices of appropriate dimensions. The uncertainties ΔA_0 and ΔA_1 are unknown Caratheodory functions and represent the system parameter perturbations with respect to the current state $x(t)$, and delayed state $x(t - h_1(t))$, respectively. $h_i(t)$ is an unknown time-varying delay with $0 \leq h_i(t) \leq \tau$, $i = 1, 2$, where τ is a constant. The initial function $x(t) = \phi(t)$, $t \in [t_0 - \tau, t_0]$, is continuous and denoted by $\phi \in C$.

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Define $[\cdot]^+$ as the vector (or matrix) obtained by replacing the entries of $[\cdot]$ by its modulus. Further, suppose that there exist $\mathcal{A}_0, \mathcal{A}_1$, whose elements are nonnegative with appropriate dimensions such that the uncertainties satisfy

$$[\Delta A_0(t, x(t))]^+ \leq \mathcal{A}_0 [x(t)]^+, [\Delta A_1(t, x(t - h_2(t)))]^+ \leq \mathcal{A}_1 [x(t - h_2(t))]^+. \tag{2}$$

Theorem 1 Let $\text{Re}\lambda_M(\cdot)$ and $\rho(\cdot)$ denote the maximal real part of all eigenvalues and the spectral radius of a matrix (\cdot) , respectively. Then the system (1) is globally asymptotically stable if $\text{Re}\lambda_M(A_0 + A_1) < 0$ and

$$\rho\left(\sum_{i=1}^6 \int_0^\infty [e^{(A_0+A_1)'B_i}]^+ dt P_i\right) \triangleq \rho(\Pi) < 1, \tag{3}$$

where $B_1 = A_1 A_0, B_2 = A_1^2, B_3 = B_4 = A_1, B_5 = B_6 = I$ (an unity matrix), $P_1 = P_2 = I\tau, P_3 = \mathcal{A}_0\tau, P_4 = \mathcal{A}_1\tau, P_5 = \mathcal{A}_0$ and $P_6 = \mathcal{A}_1$.

Proof Since $x(t - h_1) = x(t) - \int_{t-h_1}^t \dot{x}(s) ds$, the system (1) may be rewritten as follows

$$\begin{aligned} \dot{x}(t) = & (A_0 + A_1)x(t) + \Delta A_0(t, x(t)) + \Delta A_1(t, x(t - h_2)) \\ & - A_1 \int_{t-h_1}^t [A_0 x(s) + A_1 x(s - h_1) + \Delta A_0(s, x(s)) + \Delta A_1(s, x(s - h_2))] ds. \end{aligned} \tag{4}$$

Let $A = A_0 + A_1$. By the variation of parameters formula, (1) can be rewritten as

$$\begin{aligned} x(t) = & e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-s)} \{ \Delta A_0(s, x(s)) + \Delta A_1(s, x(s - h_2)) \\ & - A_1 \int_{s-h_1}^s [A_0 x(u) + A_1 x(u - h_1) + \Delta A_0(u, x(u)) + \Delta A_1(u, x(u - h_2))] du \} ds. \end{aligned} \tag{5}$$

Let $[x_t]_\tau^+ = [\|x_{1t}\|_\tau, \dots, \|x_{nt}\|_\tau]^T, \|x_{it}\|_\tau = \sup_{- \tau \leq s \leq 0} |x_i(t + s)|$ and $x(t_0 + s) = \phi(t_0 - \tau)$ for $s \in [-2\tau, -\tau]$. Thus, by the condition (2),

$$\begin{aligned} \int_{s-h_1}^s [x(u)]^+ du & \leq P_1 [x_s]_{2\tau}^+, \quad \int_{s-h_1}^s [x(u - h_1(u))]^+ du \leq P_2 [x_s]_{2\tau}^+, \\ \int_{s-h_1}^s [\Delta A_0(u, x(u))]^+ du & \leq P_3 [x_t]_\tau^+, \quad \int_{s-h_1}^s [\Delta A_1(u, x(u - h_2))]^+ du \leq P_4 [x_t]_{2\tau}^+, \\ [\Delta A_0(u, x(u))]^+ & \leq P_5 [x_t]_\tau^+, \quad [\Delta A_1(u, x(u - h_2))]^+ \leq P_6 [x_t]_{2\tau}^+. \end{aligned} \tag{6}$$

Applying the absolute value operator $[\cdot]^+$ to (5) where “+” and “ \leq ” are applied element by element to vectors and matrices,

$$[x(t)]^+ \leq [e^{A(t-t_0)}]^+ [x(t_0)]^+ + \int_{t_0}^t \sum_{i=1}^6 [e^{A(t-s)} B_i]^+ P_i [x_s]_{2\tau}^+ ds. \tag{7}$$

Since $\text{Re}\lambda_M(A) < 0$, there is a nonnegative matrix M such that $[e^{At}]^+ \leq M$. Thus

$$[x(t)]^+ \leq M[\phi(t_0)]^+ + \int_{t_0}^t \sum_{i=1}^6 [e^{A(t-s)} B_i]^+ P_i [x_s]_{2\tau}^+ ds. \tag{8}$$

Let $y(t) = \sup_{t_0 - \tau \leq s \leq t} [x(s)]^+$, then $[x(t)]^+ \leq y(t)$ and

$$y(t) \leq M[\phi(t_0)]^+ + \Pi y(t). \tag{9}$$

By Theorem 9.16 of Lasalle^[5] and $\rho(\Pi) < 1$, we have $(I - \Pi)^{-1} \geq 0$ and

$$[x(t)]^+ \leq y(t) \leq [I - \Pi]^{-1} M[\phi(t_0)]^+ \leq \Omega \quad (\Omega \text{ is a constant vector}), \tag{10}$$

which implies the stability of (1).

To prove $\lim_{t \rightarrow \infty} x(t) = 0$ we let $e = [1, \dots, 1]^T \in \mathbb{R}^n$, and for any $\epsilon > 0$ let $\beta \in \mathbb{R}^n$ and

$$\beta = \|(I - \Pi)^{-1}(\Pi + 2I)\|^{-1} \epsilon \geq 0. \tag{11}$$

From $\text{Re} \lambda_M(A) < 0$, we have $[e^{At}]^+ \rightarrow 0$ as $t \rightarrow \infty$. Thus, for given $\phi \in \mathbb{C}$, there is a positive number T such that

$$[e^{A(t-t_0)} \phi(t_0)]^+ < \beta/2, \quad \forall t \geq t_0 + T; \quad \int_T^\infty \sum_{i=1}^6 [e^{As} B_i]^+ P_i \Omega ds < \beta/2. \tag{12}$$

From (7) and (12), we obtain

$$\begin{aligned} [x(t)]^+ &\leq \left\{ \int_{t_0}^{t-T} + \int_{t-T}^t \right\} \sum_{i=1}^6 [e^{A(t-s)} B_i]^+ P_i [x_s]_{2\tau}^+ ds + \beta/2 \\ &\leq \int_T^\infty \sum_{i=1}^6 [e^{Au} B_i]^+ P_i \Omega du + \int_{t-T}^t \sum_{i=1}^6 [e^{A(t-s)} B_i]^+ P_i [x_s]_{2\tau}^+ ds + \beta/2 \\ &\leq \sum_{i=1}^6 \int_{t-T}^t [e^{A(t-s)} B_i]^+ P_i [x_s]_{2\tau}^+ ds + \beta \quad \text{for } t \geq t_0 + T. \end{aligned} \tag{13}$$

Since $[x(t)]^+ \leq \Omega$, we may let $\overline{\lim}_{t \rightarrow \infty} [x(t)]^+ = r \in \mathbb{R}_+^n$. Thus there exists a sequence $\{t_p | t_p \geq t_0 + T, t_{p+1} > t_p, p = 1, 2, \dots\}$ and an integer p^* such that

$$[x_{t_p}]_{2\tau+T}^+ \leq r + \beta, \quad r - \beta \leq [x(t_p)]^+ \quad \text{for all } p \geq p^*,$$

where $[x_{t_p}]_{2\tau+T}^+ = \sup_{-(2\tau+T) \leq s \leq 0} [x(t_p + s)]^+$. Combining (13), we have

$$\begin{aligned} r - \beta &\leq [x(t_p)]^+ \leq \sum_{i=1}^6 \int_{t_p-T}^{t_p} [e^{A(t_p-s)} B_i]^+ P_i [r + \beta] ds + \beta \\ &\leq \Pi[r + \beta] + \beta, \quad \text{for } t_p \geq t_{p^*} + T + 2\tau. \end{aligned} \tag{14}$$

Applying Theorem 9.16 in [5] and using $\rho(\Pi) < 1$, we have $(I - \Pi)^{-1} \geq 0$ and

$$r \leq (I - \Pi)^{-1}(\Pi + 2I)\beta \quad \text{or} \quad \|r\| \leq \epsilon \quad (\text{by (11)}). \tag{15}$$

This implies $\lim_{t \rightarrow \infty} x(t) = 0$ and completes the proof.

For the system (1) with $(A_0 + A_1)$ non-defective (i. e., whose eigenvalues are different each other), we have the following corollary.

Corollary 1 The system (1) is globally asymptotically stable if $(A_0 + A_1)$ is non-defective, $\text{Re} \lambda_M(A_0 + A_1) < 0$ and

$$\rho \left(\sum_{j=1}^n \frac{[v_j w_j^T]^+}{-\text{Re} \lambda_j(A_0 + A_1)} P \right) \triangleq \rho(\mathcal{E}) < 1, \tag{16}$$

where $P = \mathcal{A}_0 + \mathcal{A}_1 + [A_1]^+ ([A_0]^+ + [A_1]^+ + \mathcal{A}_0 + \mathcal{A}_1)\tau$, $\text{Re} \lambda_j(A_0 + A_1)$ is the real part of the j th eigenvalue λ_j of $(A_0 + A_1)$, v_j and w_j are the eigenvectors of $(A_0 + A_1)$ and $(A_0 + A_1)^T$ associated with the eigenvalue λ_j with $w_j^T v_j = 1$, respectively.

Proof It is evident that $\Phi(t) = [e^{\lambda_1 t} v_1, \dots, e^{\lambda_n t} v_n]$ is a fundamental matrix for $\dot{x} = (A_0 + A_1)x$. From $w_j^T v_j = 1$ and (5.1) in [6, § 5], we have

$$e^{(A_0 + A_1)t} = \Phi(t)\Phi^{-1}(0) = \sum_{j=1}^n v_j w_j^T e^{\lambda_j t} \tag{17}$$

and

$$\int_0^\infty [e^{(A_0 + A_1)t} B_i]^+ dt \leq \sum_{j=1}^n [v_j w_j^T]^+ [B_i]^+ / -\text{Re} \lambda_j(A_0 + A_1), \tag{18}$$

which implies $\Pi \leq E$ and $(\Pi - I) \leq (E - I)$ by combining $\sum_{i=1}^6 [B_i]^+ P_i \leq P$. Since $\rho(E) < 1$ is equivalent to $\text{Re}\lambda_M(E - I) < 0$ from Theorem 9.16^[5], we have $\text{Re}\lambda_M(\Pi - I) < 0$ by using Lemma 2 in [7]. Again applying Theorem 9.16 in [5], we obtain $\rho(\Pi) < 1$. Invoking Theorem 1, the result follows.

The above methods are very effective for low-dimensional systems. However, an increase of number of variables may inevitably introduce computational difficulty. Therefore, we had to find effective sufficient conditions for higher dimensional systems or composite systems. Assume that (1) admits a decomposition of the form:

$$\dot{x}_i = A_i x_i + B_i x_i(t - \tau) + f_i(t, x_i), \quad (19)$$

where $x_i \in \mathbb{R}^{n_i}$, $x^T = [x_1^T, \dots, x_N^T]$, $\sum_{i=1}^N n_i = n$. A_i and B_i are $n_i \times n_i$ matrices. The functional $f_i(t, x_i)$ are the interconnections (involving uncertainties) among the isolated subsystems

$$\dot{x} = A_i x_i + B_i x_i(t - \tau)$$

and satisfy

$$[f_i(t, x_i)]^+ \leq \sum_{j=1}^N C_{ij} [x_{j\tau}]^+, \quad (20)$$

where C_{ij} are $n_i \times n_j$ nonnegative matrices.

Similarly, equation (19) may rewrite as

$$\dot{x}_i = (A_i + B_i)x_i - B_i \int_{t-\tau}^t [A_i x_i(s) + B_i x_i(s - \tau) + f_i(s, x_s)] ds + f_i(t, x_i). \quad (21)$$

After applying the variation of parameter formula, combining inequality (20) we take the absolute value operator on both sides to obtain

$$[x_i]^+ \leq [e^{(A_i+B_i)(t-t_0)}]^+ [x_i(t_0)]^+ + \int_{t_0}^t [e^{(A_i+B_i)(t-s)}]^+ \sum_{j=1}^N \{ \delta_{ij} ([B_i A_i]^+ + [B_i^2]^+) \tau + [B_i]^+ C_{ij} \tau + C_{ij} \} [x_{j\tau}]^+ ds, \quad (22)$$

where δ_{ij} is the Kronecher delta, i. e., $\delta_{ii} = 1, \delta_{ij} = 0 (i \neq j)$.

Let $D = (D_{ij})$ be a block matrix and each block be defined as

$$D_{ij} = \delta_{ij} ([B_i A_i]^+ + [B_i^2]^+) \tau + [B_i]^+ C_{ij} \tau + C_{ij}, \quad (23)$$

and $\text{diag} \{ (\cdot)_i \}$ denote a diagonal block matrix whose i th block on its diagonal be $(\cdot)_i$. Then

$$[x]^+ \leq \text{diag} \{ [e^{(A_i+B_i)(t-t_0)}]^+ \} [x(t_0)]^+ + \int_{t_0}^t \text{diag} \{ [e^{(A_i+B_i)(t-s)}]^+ \} D [x_s]_{2\tau}^+ ds. \quad (24)$$

In essentially the same manner as that theorem 1 follows inequality (7), from (24) we can prove the following results.

Theorem 2 The system (19) is globally asymptotically stable if $\text{Re}\lambda_M(A_i + B_i) < 0$ and

$$\rho \left(\int_0^\infty \text{diag} \{ [e^{(A_i+B_i)t}]^+ \} D dt \right) \triangleq \rho(H) < 1. \quad (25)$$

Corollary 2 The system (19) is globally asymptotically stable if $(A_i + B_i)$ for $i = 1, \dots, N$ are stable matrices with $(A_i + B_i)$ non-defective and

$$\rho \left(\text{diag} \left\{ \sum_{j=1}^n \frac{[v_{ij} w_{ij}^T]^+}{-\text{Re}\lambda_j(A_i + B_i)} \right\} D \right) \triangleq \rho(E') < 1, \quad (26)$$

where v_{ij} and w_{ij} are the eigenvectors of $(A_i + B_i)$ and $(A_i + B_i)^T$ associated with the eigenvalue λ_j with $w_{ij}^T v_{ij} = 1$, respectively.

3 Illustrative Examples

Example 1 The following example is reported in Su et al. [1], the system matrices and the perturbations

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad (27)$$

$$\Delta A_0(t, x(t)) = \begin{bmatrix} 0.3 \cos t & 0 \\ 0 & 0.2 \sin t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (28)$$

$$\Delta A_1(t, x(t - h_2)) = \begin{bmatrix} 0.2 \cos t & 0 \\ 0 & 0.3 \sin t \end{bmatrix} \begin{bmatrix} x_1(t - h_2) \\ x_2(t - h_2) \end{bmatrix}. \quad (29)$$

Su et al. [1] gave for the delay the bound $\tau = 0.1614$. By the above Corollary 1, this bound can further be improved. Here $A_0 + A_1 = \begin{bmatrix} -3 & 0 \\ -1 & -2 \end{bmatrix}$ with the eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -2$, and the eigenvectors of $(A_0 + A_1)$ and $(A_0 + A_1)^T$ associated with the eigenvalue λ_j with $w_j^T v_j = 1, j = 1, 2$, are respectively

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad (30)$$

Thus,

$$\sum_{j=1}^2 \frac{[v_j w_j^T]^+}{-\operatorname{Re} \lambda_j(A_0 + A_1)} = \begin{bmatrix} 1/3 & 0 \\ 5/6 & 1/2 \end{bmatrix}$$

and

$$\Xi = \begin{bmatrix} 1/3 & 0 \\ 5/6 & 1/2 \end{bmatrix} \begin{bmatrix} 3.5\tau + 0.5 & 0 \\ 4.5\tau & 2.5\tau + 0.5 \end{bmatrix} = \begin{bmatrix} (3.5\tau + 0.5)/3 & 0 \\ (31\tau + 0.5)/6 & (2.5\tau + 0.5)/2 \end{bmatrix}.$$

Since the maximum eigenvalue of the matrix Ξ is $(2.5\tau + 0.5)/2$, from the condition (16), system (1) is globally asymptotically stable if $\tau < 0.6$ which improves the bound reported in [1] for the delay (by 371%). Similarly, Corollary 1 can also guarantee that Example 2 in [2] is globally asymptotically stable if $\tau < 0.7647$, but Su et al. [2] gave the bound $\tau = 0.3188$.

Example 2 We now consider the composite system (19) with the system matrices

$$A_1 = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \quad (31)$$

and the upper bound matrices of uncertainties

$$\begin{aligned} C_{11} &= \begin{bmatrix} 0.05 & 0.05 \\ 0.05 & 0 \end{bmatrix}, & C_{12} &= \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, \\ C_{21} &= \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, & C_{22} &= \begin{bmatrix} 0 & 0.53 \\ 0.28 & 0 \end{bmatrix}. \end{aligned} \quad (32)$$

Here

$$A_1 + B_1 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \int_0^\infty [e^{(A_1 + B_1)s}]^+ ds = \frac{1}{2} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix},$$

$$A_2 + B_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \int_0^{\infty} [e^{(A_2+B_2)s}]^+ ds = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (33)$$

By matrix multiplication, we can easily obtain matrix H defined by (23) and (25) as follows

$$H = \begin{bmatrix} 10.25\tau + 0.075 & 10.125\tau + 0.025 & 0.5\tau + 0.15 & 0.5\tau + 0.15 \\ 14.35\tau + 0.1 & 14.175\tau + 0.025 & 0.7\tau + 0.2 & 0.7\tau + 0.2 \\ 0.2\tau + 0.2 & 0.3\tau + 0.3 & 1.28\tau + 0.28 & 1.53\tau + 0.53 \\ 0.2\tau + 0.1 & 0.3\tau + 0.1 & 1.28\tau + 0.28 & 2.53\tau \end{bmatrix}. \quad (34)$$

Taking $\|H\| = \max_j \sum_{i=1}^4 h_{ij}$ for nonnegative $H = (h_{ij})$ in (34), we have

$$\|H\| < 1 \quad \text{if} \quad \tau < 0.021.$$

From Theorem 2, we obtain the bound $\tau = 0.021$ to guarantee that this system is robust globally asymptotically stable.

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具有非线性参数振动的线性系统的时滞相关鲁棒稳定性

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摘要: 本文研究不确定时滞单结构与复合系统的时滞相关鲁棒稳定性问题. 利用时滞积分不等式, 一些鲁棒稳定性的充分条件通过一个非负矩阵谱半径的界给出. 文末的例子说明了文中结果的有效性.

关键词: 鲁棒性; 稳定性; 时滞微分系统

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