Optimal Service Rate Allocation Policy of An Unreliable Manufacturing System with Random Demands*

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Abstract: The optimal service rate control of an unreliable manufacturing system with random demands is discussed in this paper. The system can produce many part types and its total capacity is constrained by a fixed constant. The objective is to find an optimal service rate allocation policy by minimizing the expected discounted inventory and backlog cost. It is proved that the optimal policy is of switching structure, and the detailed structural properties are investigated for systems such as producing one part type and two part types. Numeric examples are given to demonstrate the results.

Key words: manufacturing system; dynamic programming; discrete event dynamic system (DEDS); optimal control

1 Introduction

A very interesting issue arising now is how to operate a manufacturing system so that it can meet the demand and keep low in-process and finished-part inventories at the same time. Some remarkable works has been done on the analysis of this problem[1,2,3]. These research works are based on the continuous material flow model. Due to the discrete nature of many manufacturing systems, however, it is more reasonable to deal with them as discrete event dynamic systems (DEDS)[4]. Unfortunately, the optimal policy of most practical DEDS is extremely difficult or impossible to obtain because of the large size of these systems and the presence of discrete stochastic events. On the other hand, the structural properties of the optimal policy isn't so difficult to find, and such properties benefit us a lot to solve or approximate the optimal control. Following this approach, we have investigated the optimal control structure for a two tandem workstations system[5]. In this paper, we consider an unreliable manufacturing system producing many part types with random demands. The control consists of allocation of given constant effort among various part types. Based on uniformization technique and dynamic programming[6], it is proved that the optimal policy is of switching structure. Other structural properties of the optimal policy such as monotonicity and asymptote behavior are also investigated.

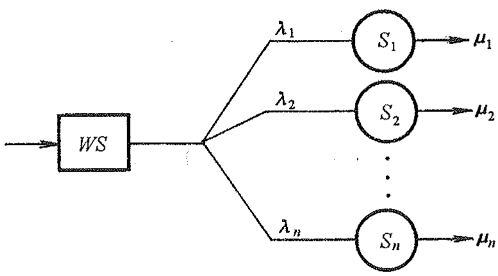

Fig. 1 A manufacturing system producing n part types with random demands

2 Problem Formulation

Consider a manufacturing system producing n part types pictured in Fig. 1. The system

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has infinite part resources. The workstation (WS) is of exponential service, and the exponential service rate for part type i is λ_i, which can be selected to be any value in $[0,\bar{\lambda}]$ and s. t. $0 \leqslant \lambda_1 + \lambda_2 + \cdots + \lambda_n \leqslant \bar{\lambda}$. This means one can control the allocation of effort among various part types. The product demands are random and the time between two demand events for part type i is exponentially distributed with average time $1/\mu_i, i = 1,2,\cdots,n$. The WS is subject to Markovian failure and repair with rate ξ and η respectively. (The assumption that the WS failure process is time dependent rather than operation dependent simplifies the analysis.) Denote $\alpha(t)$ the WS state, $\alpha(t) = 1$ if the WS is up and $\alpha(t) = 0$ if the WS is down. The WS produces nothing if $\alpha(t) = 0$, so the control decision is made only if the WS is up. Let $s_i(t)$ be the production surplus of the part type i at time t. Apparently $s_i(t) \in \mathbb{Z}$. It represents inventory when positive and backlog when negative. Denote $\boldsymbol{s}(t) = (s_1(t), s_2(t), \cdots, s_n(t))$ and $\boldsymbol{u}(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t))$.

Our goal is to operate the system so that it meets the demand without meanwhile overproducing and underproducing for each part type. This can be reflected in the cost function. The problem now is to find the optimal control policy $\boldsymbol{u}^*(t)$ by minimizing the following infinite expected discounted cost

$$V(\boldsymbol{s},\alpha) = \min_{u} E^{u}\int_0^{\infty} \mathrm{e}^{-\beta t} g(\boldsymbol{s}(t),\alpha(t))\mathrm{d}t \tag{1}$$

where $0 < \beta < 1$ is a discount factor and $g(\cdot)$ is a function to penalize both inventory and backlog. Generally, $g(\cdot)$ can be chosen as $g(\boldsymbol{s}(t),\alpha(t)) = \boldsymbol{c}^+ (\boldsymbol{s}^+)^{\mathrm{T}} + \boldsymbol{c}^- (\boldsymbol{s}^-)^{\mathrm{T}}$ where $\boldsymbol{s}^- = (s_1^-, s_2^-, \cdots s_n^-)$, $\boldsymbol{s}^+ = (s_1^+, s_2^+, \cdots, s_n^+)$, $s_i^- = \max\{-s_i, 0\}$, $s_i^+ = \max\{s_i, 0\}$, $i = 1,2,\cdots,n$, $\boldsymbol{c}^- = (c_1^-, c_2^-, \cdots, c_n^-)$, $\boldsymbol{c}^+ = (c_1^+, c_2^+, \cdots, c_n^+)$; c_i^+ and c_i^- are the penalizing coefficients of inventory and backlog for part type i.

Define the following mappings A_i, D_i from Z^n to Z^n and T_0, T_1 from Z_2 to Z_2:

$$A_i s = (s_1, \cdots, s_{i-1}, s_i + 1, s_{i+1}, \cdots, s_n),$$
$$D_i s = (s_1, \cdots, s_{i-1}, s_i - 1, s_{i+1}, \cdots, s_n), \quad i = 1,2,\cdots,n.$$
$$T_0\alpha = 0, \quad \text{if} \quad \alpha = 1; \quad T_1\alpha = 1, \quad \text{if} \quad \alpha = 0.$$

Here A_i, D_i, T_0 and T_1 are called events of the system. A_i reflects a service completion of a product of part type i; D_i reflects a departure of a product of part type i in order to meet the demand; T_0 and T_1 reflect the WS failure and repair respectively. It is always assumed that the initial WS state is up, i. e. $\alpha(0) = 1$.

3 Optimal Control Policy

Let $S = Z^n \times Z_2$ be the state space of the system and $\Omega = \{u = (\lambda_1, \lambda_2 \cdots, \lambda_n) : 0 \leqslant \lambda_1 + \lambda_2 + \cdots + \lambda_n \leqslant \bar{\lambda}, \lambda_i \geqslant 0, i = 1,2,\cdots,n\}$ be the set of admissible controls. Control decision is made if and only if the system state transition happens and the WS is up. The system state transition happens only if the event A_i, D_i, T_0 or T_1 happens on discrete time epoch. It is clear that the occurrence of these events depends on the system state and the control action.

According to the approach called uniformization technique, proposed by Bertsekas[6], we introduce into our system the uniform transition rate $v = \bar{\lambda} + \xi + \eta + \sum_{i \in \underline{n}} \mu_i$, where $\underline{n} = \{1,$

$2,\cdots,n\}$, and transform the original continuous-time Markov chain into an equivalent discrete-time one. For any given $u\in\Omega$, the one-step transition probability function $P(\cdot|\cdot,u)$ is given by

$$P(y|(s,\alpha),u)=\lambda_i/v \quad \text{if} \quad y=(A_is,\alpha),i\in \underline{n}, \quad P(y|(s,\alpha),u)=\mu_i/v \quad \text{if} \quad y=(D_is,\alpha),\ i\in \underline{n},$$

$$P(y|(s,\alpha),u)=\xi/v \quad \text{if} \quad y=(s,T_0\alpha),\alpha=1, \quad P(y|(s,\alpha),u)=\eta/v \quad \text{if} \quad y=(s,T_1\alpha),\ \alpha=0,$$

$$P(y|(s,\alpha),u)=1-(\xi+\sum_{i\in \underline{n}}(\lambda_i+\mu_i))/v \quad \text{if} \quad y=(s,1),$$

$$P(y|(s,\alpha),u)=1-(\eta+\sum_{i\in \underline{n}}(\lambda_i+\mu_i))/v \quad \text{if} \quad y=(s,0),$$

$$P(y|(s,\alpha),u)=0 \quad \text{otherwise}.$$

Let $0=t_0<t_1<\cdots<t_n<\cdots$ be the potential state transition epochs. Denote $z_k:=s(t_k)$, $\alpha_k:=\alpha(t_k)$ and $u_k:=u(t_k)$. Thus, $x(t)=z_k$, $\alpha_k=\alpha(t_k)$ and $u(t)=u_k$, if $t\in[t_k,t_{k+1})$. Compute the objective function under the given control policy $u(t)$, it follows

$$E^u\int_0^\infty e^{-\beta t}g(s(t),\alpha(t))dt=E^u\sum_{k=0}^\infty\int_{t_k}^{t_{k+1}}e^{-\beta t}g(z_k,\alpha_k)dt=\frac{1}{\beta+v}E^u\sum_{k=0}^\infty\theta^k g(z_k,\alpha_k)$$

where $\theta=v/(\beta+v)$. Now the equivalent discrete-time Markov chain problem has been established. It has infinite state space, infinite control actions and unbounded cost per step. For simplicity, assume $\beta+v=1$. It follows from general results on dynamic programming[6] that V_k is well defined if $V_0(\cdot)=0$ and

$$\begin{aligned}V_{k+1}(s,\alpha)=g(s,\alpha)+\min_{u\in\Omega}\{&\sum_{i\in \underline{n}}\mu_iV_k(D_is,\alpha)+\xi V_k(s,T_0\alpha)\cdot 1\{\alpha=1\}\\&+\eta V_k(s,T_1\alpha)\cdot 1\{\alpha=0\}+(\xi\cdot 1\{\alpha=0\}+\eta\cdot 1\{\alpha=1\}+\bar{\lambda})V_k(s,\alpha)\\&+[\sum_{i\in \underline{n}}\lambda_i(V_k(A_is,\alpha)-V_k(s,\alpha))]\cdot 1\{\alpha=1\}\}\end{aligned} \tag{2}$$

where $1\{\cdot\}$ is the indicator function. Further more, equation (2) can be simplified as

$$\begin{aligned}V_{k+1}(s,\alpha)=&\ g(s,\alpha)+\sum_{i\in \underline{n}}\mu_iV_k(D_is,\alpha)+\xi V_k(s,0)+\eta V_k(s,1)\\&+\bar{\lambda}V_k(s,\alpha)\cdot 1\{\alpha=0\}+\bar{\lambda}\min\{V_k(A_1s,\alpha),V_k(A_2s,\alpha),\\&\cdots,V_k(A_ns,\alpha),V_k(s,\alpha)\}\cdot 1\{\alpha=1\}.\end{aligned} \tag{3}$$

Standard results from dynamic programming yield

Theorem 1 Let $V_0(s,\alpha)\equiv 0$, then $\lim_{k\to\infty}V_k(s,\alpha)=V(s,\alpha)$, $\forall\ (s,\alpha)\in Z^n\times Z_2$, where $V(s,\alpha)$ is defined in (1).

Theorem 2 For unreliable manufacturing system pictured in Fig. 1, the optimal control policy $u^*=(\lambda_1^*,\lambda_2^*,\cdots,\lambda_n^*)$ satisfies

$$\lambda_i^*=\begin{cases} & V(A_is,1)<V(A_js,1),j=1,2,\cdots,i-1,\\ \bar{\lambda}, & V(A_is,1)\leqslant V(A_js,1),j=i+1,i+2,\cdots,n,\\ & V(A_is,1)\leqslant V(s,1),\\ 0, & \text{otherwise}.\end{cases}$$

Remark Theorem 1 gives only one type of the optimal policies, called bang-bang type. In fact, if $V(A_is,1)=V(A_js,1)<V(A_ks,1)$, $k\neq i,j$ and $V(A_is,1)\leqslant V(s,1)$, then any

element in $\{u^* \mid \lambda_i^* + \lambda_j^* = 1, \lambda_i^* \geqslant 0, \lambda_j^* \geqslant 0, \lambda_k^* = 0, k \neq i, j\}$ is an optimal policy. However, the optimal policies are of switching structure and multiple optimal choices are made only if the system state on the switching surfaces.

4 Control Structure Properties

In the rest of this paper, we investigate two special cases to illustrate the structural properties of the optimal policy.

4.1 Two Part Type Case

For the two part type manufacturing system ($n=2$), we have, $\boldsymbol{s} = (s_1, s_2)$ and $\boldsymbol{u} = (\lambda_1, \lambda_2)$. We will show that the optimal control policy has a switching structure and can be described by three monotone curves.

Theorem 3 The function $V(\cdot)$ has the following properties:

a) $V(A_1\boldsymbol{s}, \alpha) - V(\boldsymbol{s}, \alpha)$ is nondecreasing in s_1 and s_2.

b) $V(A_2\boldsymbol{s}, \alpha) - V(\boldsymbol{s}, \alpha)$ is nonincreasing in s_1 and s_2.

c) $V(A_1\boldsymbol{s}, \alpha) - V(A_2\boldsymbol{s}, \alpha)$ is nondecreasing in s_1 and nonincreasing in s_2.

Define three switching curves as follows

$$f_1(s_2) = \max_{s_1}\{s_1 \mid V(\boldsymbol{s}, 1) \geqslant V(A_1\boldsymbol{s}, 1)\},$$

$$f_2(s_1) = \max_{s_2}\{s_2 \mid V(\boldsymbol{s}, 1) \geqslant V(A_2\boldsymbol{s}, 1)\},$$

$$f_3(s_2) = \max_{s_1}\{s_1 \mid V(A_2\boldsymbol{s}, 1) \geqslant V(A_1\boldsymbol{s}, 1)\}.$$

Theorem 4 The switching curve $f_1(s_2)$ and $f_2(s_1)$ are nonincreasing; the switching curve $f_3(s_2)$ is nondecreasing.

Theorem 5 a) $f_1(s_2)$ converges to a nonnegative finite asymptote as $s_2 \to \infty$. b) $f_2(s_1)$ also converges to a nonnegative finite asymptote as $s_1 \to \infty$.

Remark Note that $f_1(s_2)$ and $f_2(s_1)$ both are nondecreasing, and s_1, s_2 both are integers, thus, $\exists\ \bar{s}_1, \bar{s}_2$ s.t. $f_1(s_2) = s_1^* \quad \forall\ s_2 \geqslant \bar{s}_2$ and $f_2(s_1) = s_2^* \quad \forall\ s_1 \geqslant \bar{s}_1$.

Theorem 6 The optimal control policy $\boldsymbol{u}^* = (\lambda_1^*, \lambda_2^*)$ has the following region switching structure:

$$(\lambda_1^*, \lambda_2^*) = \begin{cases} (\bar{\lambda}_1, 0), & (s_1, s_2) \in R_1, \\ (0, \bar{\lambda}_2), & (s_1, s_2) \in R_2, \\ (0, 0), & (s_1, s_2) \in R_3 \end{cases}$$

where

$$R_1 = \{(s_1, s_2) \mid s_1 \leqslant f_1(s_2), s_1 \geqslant f_3(s_2)\},$$

$$R_2 = \{(s_1, s_2) \mid s_2 \leqslant f_2(s_1), s_1 < f_3(s_2)\},$$

$$R_3 = \{(s_1, s_2) \mid s_2 > f_2(s_1), s_1 > f_1(s_2)\}.$$

The control regions of R_1, R_2, R_3 are pictured in part (a) of Fig. 2.

Remark From Theorem 6, we know that the optimal control for two part types case is of region switching structure and can be described by three monotone curves. Morever, $f_1(s_2)$ and $f_2(s_1)$ converge to nonnegative finite asympototes. Based on these properties, it is much easier to obtain the optimal or suboptimal policies than original problem. In fact, one

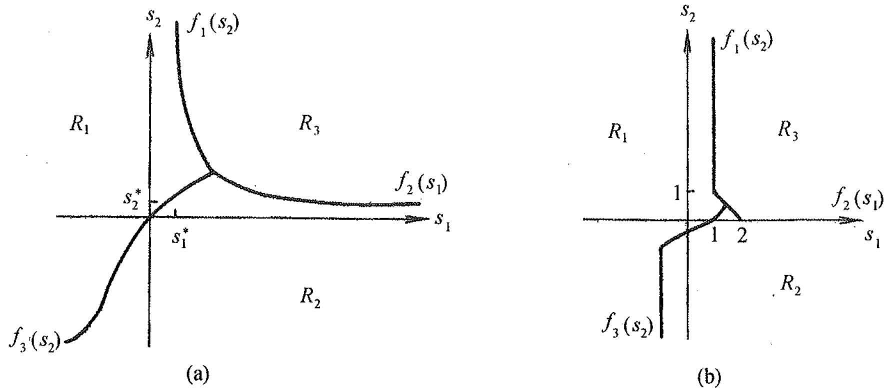

Fig. 2 The optimal control regions on the state space

can derive some simple suboptimal policies which are easy to operate and compute in practical manufacturing systems.

4.2 One Part Type Case

Consider the case $n = 1$. That is, the WS produces only one part type. Therefore $\boldsymbol{s}$ and $\boldsymbol{u}$ both are scalars, rewrite them as s and u. We have the following results.

Theorem 7 For one part type unreliable manufacturing system, we have: a) $V(s+1,\alpha) - V(s,\alpha)$ is nondecreasing in s. b) $V(s+1,\alpha) - V(s,\alpha) \leqslant 0$ if $s < 0$.

Theorem 8 For one part type unreliable manufacturing system, the optimal control policy has the following threshold structure

$$u^* = \begin{cases} \bar{\lambda}, & s \leqslant h \text{ and } \alpha = 1, \\ 0, & \text{otherwise} \end{cases}$$

where h is some nonnegative finite integer.

5 Examples and Conclusion

In the following examples, we use iteration algorithm to find the optimal control policy. Assume the system produce two part types and the state space is limited by $-30 \leqslant s_1 \leqslant 30$ and $-30 \leqslant s_2 \leqslant 30$. Let $\bar{\lambda} = 0.8, \beta = 0.5, \xi = 0.8, \eta = 0.2, \mu_1 = 0.5, \mu_2 = 0.3, c_1^+ = c_2^+ = 1, c_1^- = c_2^- = 8$. By iteration equation (3), we obtain the switching curves as follows

$$f_1(s_2) = \begin{cases} 1, & s_2 \geqslant -1, \\ 2, & s_2 < -1; \end{cases} \quad f_2(s_1) = \begin{cases} 0, & s_1 \geqslant 2, \\ 1, & s_1 < 2; \end{cases} \quad f_3(s_2) = \begin{cases} \cdots & \cdots \\ 2, & s_2 = 1, \\ 1, & s_2 = 0, \\ -1, & s_2 < 0. \end{cases}$$

The optimal control regions are pictured in part (b) of Fig. 2.

In this paper, the optimal service allocation problem of an unreliable manufacturing system is considered. The structural properties of the optimal policy are investigated. For producing two part types case, the optimal control is a region switching policy which can be described by three monotone curves, and the asymptote properties of these curves are discussed. Further interesting questions arise in this field such as: a) how to generalize the results to more complex systems such as multiple-part-type multiple-workstation systems; b) is it possible to generalize the distributions of the service time, the time between demands, the WS up time and the WS down time? etc..

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随机需求不可靠制造系统的最优服务率分配策略

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摘要：本文讨论了一类随机需求不可靠制造系统的最优服务率控制问题.所研究的系统能同时生产多类产品，但生产能力受常数限制.目标是通过最小化库存和欠缺的期望折扣费用，寻找最优服务率分配策略.本文证明了最优策略具有开关结构，并针对生产单类和两类产品的系统详细研究了最优控制策略的结构性质，最后以数值例子验证了本文的结论.

关键词：制造系统；动态规划；离散事件动态系统；最优控制

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