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Remarks on Reachable Solutions of One Predator and Two Preys System

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Abstract: In this paper, a general Lotka-Volterra system of one predator and two preys is investigated to solve the open problems in [4]. By constructing domains in R^3, some interesting results are obtained: there are a number of periodic solutions which can be approached as $t \to \infty$. We also give the analysis of the Hopf bifurcation of the system.

Key words: Lotka-Volterra predator-prey system; reachable solution

1 Introduction

Consider the ordinary differential equations

$$\begin{cases} x' = ax - dxy - exz, \\ y' = by + dxy - fyz, \\ z' = -cz + exz + fyz, \end{cases} \tag{1}$$

where a, b, c, d, e and f are positive constants. We will focus on the solutions of (1) with positive initial conditions, i.e., $x(0), y(0), z(0) > 0$. It is easy to show that under such conditions, $x(t), y(t), z(t)$ are all positive, for all $t > 0$.

From the ecological viewpoint, to settle the question for these classes would be interesting, cf. [1]. It is easy to see that all solutions of (1) in the xz-plane are periodic except the critical points. That means the prey dies out and the predator z and the prey x persist. It follows from the paper that if $be + dc < af$, not all solutions of (1) tend to critical points, that is: there exists one periodic solution which would be approached. Further, two distinct periodic solutions which could be approached are found. At last, a number of periodic solutions which can be approached is proved. If $be + dc > af$, a number of periodic solutions which can be approached is proved. To some extent, the method of constructing a domain on R^3 for (1) is new and mathematically interesting.

Some results for special cases of (1) have already been established, cf. [2~4]. In [4], we know that each solution of (1) may tend to critical point as $t \to \infty$, or approach a periodic solution on coordinate plane. An open question is: in the case $be + dc \neq af$, if given any periodic solutions of (1) in either of the coordinate planes, there exists a unique solution which appoaches it orbitally as $t \to \infty$ from the positive orthant $x > 0, y > 0, z > 0$. Theorems 1~4 of the paper answered this question to some extent.

Definition: A periodic solution of (1) is said to be reachable if there exists a distinct solution which approaches it orbitally as $t \to \infty$ from the positive orthant $x > 0, y > 0, z > 0$.

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The purpose of this note is to prove that a number of the periodic solutions of (1) in either of the coordinate planes are reachable.

2 Results in case $be + dc < af$

Case 1 $be + dc < af$.

Let $x_0 = c/e, y_0 = 0, z_0 = a/e$, and $(x(t), y(t), z(t))$ be a solution of (1). Put

$x_1(t) = x(t) - x_0$, $x_2(t) = y(t) - y_0$, $x_3(t) = z(t) - z_0$, $\sigma = (dc - af)/e + b$.

and (1) becomes

$$\begin{cases} x_1' = -ex_3x_1 - dx_2x_1 - ex_3x_0 - dx_2x_0, \\ x_2' = dx_1x_2 - fx_3x_2 + \sigma x_2, \\ x_3' = ex_1x_3 + fx_2x_3 + ex_1z_0 + fx_2z_0. \end{cases} \tag{2}$$

Let $F(x_1) = x_1 - x_0\log(x_1/x_0 + 1), H(x_3) = x_3 - z_0\log(x_3/z_0 + 1)$, and $V(x_1, x_2, x_3) = F(x_1) + x_2 + H(x_3)$, then it follows from (2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_1(t), x_2(t), x_3(t)) = \sigma x_2(t) \leqslant 0. \tag{3}$$

Lemma 1[4] For any $c > 0, V(x_1, x_2, x_3) = c$ defines a closed bounded strictly convex 2-surface, here strictly convex means that any line in (x_1, x_2, x_3) space intersects it in at most 2 points.

Lemma 2[4] $V(x_1(t), x_2(t), x_3(t)) \to L_0$ as $t \to \infty$, where $0 \leqslant L_0 < \infty$. So, $x_1(t), x_2(t), x_3(t)$ are bounded for $t \geqslant 0$.

Lemma 3[4] For any periodic solution $(\bar{x}_1(t), 0, \bar{x}_3(t))$ of (2) with the least period $T > 0$, if it is the ω-limit set of $(x_1(t), x_2(t), x_3(t))$, there exists $\tau, T \geqslant \tau \geqslant 0$, such that

$$(x_1(t) - \bar{x}_1(t + \tau))^2 + x_2^2(t) + (x_3(t) - \bar{x}_3(t + \tau))^2 \to 0 \text{ as } t \to \infty.$$

Lemma 4 For any sloution $(x_1(t), x_2(t), x_3(t))$ of (2), $x_2(t) \to 0$ as $t \to \infty$.

Proof It follows from (3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_1(t), x_2(t), x_3(t)) \leqslant 0, \tag{4}$$

$$V(x_1(t), x_2(t), x_3(t)) - V(x_1(0), x_2(0), x_3(0)) = \sigma\int_0^t x_2(s)\mathrm{d}s. \tag{5}$$

So $\int_0^t x_2(s)\mathrm{d}s < \infty$, that is: for any $\varepsilon > 0$, there exists an $N > 0$, such that $\int_n^\infty x_2(t)\mathrm{d}t < \varepsilon$ as $n > N$.

It follows from Lemma 2 and (2) that there exists an $M > 0$ such that $|x_2'(t)| < M$ as $t \geqslant 0$.

Suppose there exists $\{t_n\}, t_n \to \infty$ as $n \to \infty$, such that $x_2(t_n) \to c$ as $n \to \infty$ where $c > 0$. So for each t_n, there exists $n_0 > 0$, as $n > n_0$, there exist t_n', t_n'', such that

$$t_{n-1} < t_n'' < t_n' < t_n, \quad x(t_n'') = c/3, x(t_n') = c/2,$$

and
$$t_n' - t_n'' \to 0 \text{ as } n \to \infty, \quad x_2(t) > c/3 \text{ as } t \in (t_n'', t_n'). \tag{6}$$

There exists $n_1 > n_0, t_n' - t_n'' < c/(6M)$ as $n > n_1$. Let $\varepsilon_0 = c^2/(18M)$, there exists $T_0 > t_{n_1}$, such that $\int_{T_0}^\infty x_2(t)\mathrm{d}t < \varepsilon_0$. It follows from (6) that $c(t_n' - t_n'')/3 \leqslant \int_{t_n''}^{t_n} x_2(t)\mathrm{d}t < \varepsilon_0$, so there exists $t \in (t_n'', t_n')$, such that $x_2'(t) = (c/2 - c/3)/(t_n' - t_n'') > M$,this constradicts $M > x_2'(t)$.

Q. E. D.

For $N > 1$, let

$$G_N = \{(x_1, 0, x_3) \mid x_1 > -x_0, x_3 > -z_0, |dx_1 + fx_3| \leqslant -\sigma/N, |dx_1 - fx_3| \leqslant -\sigma/N\}.$$

For each N, there exists $C_N > 0$, such that

$$S_{C_N} = \{(x_1, 0, x_3) \mid V(x_1, 0, x_3) \leqslant C_N\} \subset G_N, \quad S_{C_N} \cap \partial G_N = \varnothing. \tag{7}$$

For such C_N, we have

Lemma 5 If $V(x_1(0), x_2(0), x_3(0)) = C_N$, then the solution $(x_1(t), x_2(t), x_3(t))$ of (2) satisfies:

$$|dx_1(t) + fx_3(t)| \leqslant -\sigma/N, \quad |dx_1(t) - fx_3(t)| \leqslant -\sigma/N \text{ as } t \geqslant 0.$$

Proof It follows from (3) that

$$V(x_1(t), x_2(t), x_3(t)) \leqslant V(x_1(0), x_2(0), x_3(0)) = C_N \text{ as } t \geqslant 0.$$

$$V(x_1(t), 0, x_3(t)) \leqslant V(x_1(t), x_2(t), x_3(t)) \leqslant C_N \text{ as } x_2(t) \geqslant 0.$$

that is, $(x_1(t), 0, x_3(t)) \in S_{C_N}$, and

$$|dx_1(t) + fx_3(t)| \leqslant -\sigma/N, \quad |dx_1(t) - fx_3(t)| \leqslant -\sigma/N.$$

Q. E. D.

It is easy to see that:

Lemma 6 In the sense of Liapunov stability, the critical point (0,0,0) of (2) is stable.

Theorem 1 There exists a periodic solution of (1) in x_1x_3 plane which can be approached.

Proof Let $N > (1 + \sqrt{5})/2$. Suppose solution $(x_1(t), x_2(t), x_3(t))$ of (1) satisfies $V(x_1(0), x_2(0), x_3(0)) = C_N$, here C_N is chosen in (7).

It follows from Lemma 5 that $|dx_1(t) - fx_3(t)| \leqslant -\sigma/N$ for all $t \geqslant 0$, that is:

$$(1 + 1/N)\sigma x_2 \leqslant \frac{dx_2}{dt} \leqslant (1 - 1/N)\sigma x_2, \tag{8}$$

$$\sigma\int_0^t x_2(s)ds \geqslant x_2(0)\{\exp[(1 - 1/N)\sigma t] - 1\}N/(N - 1), \tag{9}$$

$$\sigma\int_0^t x_2(s)ds \leqslant x_2(0)\{\exp[(1 + 1/N)\sigma t] - 1\}N/(N + 1), \tag{10}$$

$$\sigma\int_0^t x_2(s)ds \geqslant -x_2(0)N/(N - 1),$$

$$V(x_1(t), x_2(t), x_3(t)) \geqslant F(x_1(0)) + H(x_3(0)) + x_2(0)/(1 - N). \tag{11}$$

Let $x_3(0) = 0, x_2(0) = C_N/(N + 1), F(x_1(0)) = NC_N/(N + 1)$, then

$$V(x_1(t), x_2(t), x_3(t)) \geqslant (N^2 - N - 1)/(N - 1) > 0 \text{ as } t \geqslant 0.$$

It follows from (3) that for this solution there exists $C > 0$,

$$V(x_1(t), x_2(t), x_3(t)) \to C \text{ as } t \to \infty.$$

That is, the approached periodic solution is: $L_C: V(x_1, x_2, x_3) = C$ and $x_2 = 0$. Q. E. D.

From Theorem 1, we can choose N_1, $(x_1(0), x_2(0), x_3(0))$, such that $V(x_1(0), x_2(0), x_3(0)) = C_{N_1}$, and $V(x_1(t), x_2(t), x_3(t)) \to C$. Here the existence of C is shown in Theorem 1. Then we can choose $N_2 > N_1$, such that $G_{N_2} \subset S_C$. From the proof of Theorem 1, we can choose $(\bar{x}_1(0), \bar{x}_2(0), \bar{x}_3(0))$, $V(\bar{x}_1(0)), \bar{x}_2(0), \bar{x}_3(0)) = C_{N_2}$, and

$$V(\bar{x}_1(t),\bar{x}_2(t),\bar{x}_3(t)) \geqslant (N_2^2 - N_2 - 1)/(N_2 - 1) > 0,$$
$$V(\bar{x}_1(t),\bar{x}_2(t),\bar{x}_3(t)) \to \bar{C} > 0,$$

and it is easy to see that $C > \bar{C} > 0$.

That is: an instinct periodic solution can be approached. Then we have:

Theorem 2 If the periodic solution $L_{C_1}(C_1 > 0)$ can be approached, there exists $C_2, 0 < C_2 < C_1$, such that L_{C_2} can be approached.

Theorem 3 If $L_{C_1}, L_{C_2} \subset G_N (0 < C_1 < C_2)$ can be approached, there exists $C > 0, C_1 < C < C_2$, such that L_C can be approached.

Proof Let $D > 0, C_1 < D < C_2$. Suppose the solution $(x_1(t),x_2(t),x_3(t))$ satisfies $V(x_1(0),x_2(0),x_3(0)) = D$, and $x_2(0) = (D - C_1)(N-1)/(2N)$, from (9), we have

$$V(x_1(t),x_2(t),x_3(t)) \geqslant D + \{\exp[(1-1/N)\sigma t] - 1\}Nx_2(0)/(N-1)$$
$$\geqslant D - Nx_2(0)/(N-1) \geqslant (D + C_1)/2 > C_1.$$

Suppose $V(x_1(t),x_2(t),x_3(t)) \to C$ as $t \to \infty$, then L_C is approached and $C_2 > D > C > C_1$. Q.E.D.

From theorems 1,2,3, we can conclude that for any $N > (1+\sqrt{5})/2$, and the related S_{C_N}, the set of periodic solutions which can be approached is dense in S_{C_N}.

3 Results in case $be + dc > af$ and some discussions

Case 2 $be + dc > af$.

Let $x_0 = 0, y_0 = c/f, z_0 = b/f$, and $(x(t),y(t),z(t))$ be a solutin of (1). Put

$$x_1(t) = x(t) - x_0, x_2(t) = y(t) - y_0, x_3(t) = z(t) - z_0,$$

and (1) becomes

$$\begin{cases} x_1' = x_1(\sigma - dx_2 - ex_3), \\ x_2' = (x_2 + y_0)(dx_1 - fx_3), \\ x_3' = (x_3 + z_0)(ex_1 + fx_2), \end{cases} \tag{12}$$

here $$\sigma = a - (dc + be)/f > 0.$$

By using $V(x_1,x_2,x_3) = x_1 + x_2 - y_0\log(x_2/y_0 + 1) + x_3 - z_0\log(x_3/z_0 + 1)$ in the place of $V(x_1,x_2,x_3)$ of Case 1.

Let $G_N = \{(0,x_2,x_3) \mid x_2 > -y_0, x_3 > -z_0, |dx_2 + ex_3| \leqslant \sigma/N, |dx_2 - ex_3| \leqslant \sigma/N\}$.

Let $N > (1+\sqrt{5})/2$, choose $C_N > 0, S_{C_N} = \{(0,x_2,x_3) \mid V(0,x_2,x_3) \leqslant C_N\} \subset G_N$ and $S_{C_N} \cap \partial G_N = \varnothing$.

Let $L_C: V(x_1,x_2,x_3) = C$ and $x_1 = 0$.

Then it follows that:

Theorem 4 1) There exists a periodic solution of (12) in x_2x_3 plane which can be approached.

2) If periodic solution $L_{C_1}(C_1 > 0)$ can be approached, there exists $C_2, 0 < C_2 < C_1$, such that L_{C_2} can be approached.

3) If $L_{C_1}, L_{C_2} \subset G_N (0 < C_1 < C_2)$ can be approached, there exists $C > 0, C_1 < C < C_2$, such that L_C can be approached.

The proof is omitted.

From Theorem 4, we can conclude that for any $N > (1+\sqrt{5})/2$, and the related S_{C_N}, the set of periodic solutions of (12) which can be approached is dense in S_{C_N} on yz-plane.

Here the result of Hopf bifurcation of the system is stated but the proof is omitted:

In case $be + dc < af$, all the solution phases from R^3_+ tend to the critical point or closed phases on xz-plane.

In case $be + dc = af$, all ω-sets of the solution phases from R^3_+ are themselves.

In case $be + dc > af$, all the solution phases from R^3_+ tend to the critical points or closed phases on yz-phane.

Remarks An open question arised from the paper is: Can the set of periodic solutions which can be approached be extended to the related R^2_+ plane (on the xz-plane or on the yz-plane)?

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关于一类含二个被食者的三维掠俘系统可达解的注记

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摘要:本文研究了一类一个捕食者二个被食者的三维 Lotka-Volterra 系统,在更一般的情形下回答了文[4]提出的开问题,即通过在三维空间中构造环域,证明了在坐标平面上某个区域内,有无穷多个可达解.对该三维系统的分支现象,也得到了较好的结果.

关键词:Lotka-Volterra 掠俘系统;可达解

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