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Study on Stability of Hybrid Systems via Multiple Lyapunov Functions *

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Abstract: The hybrid systems considered here consist of the continuous-valued systems under the supervision of discrete event. We first analyze the prior results of stability using multiple Lyapunov approach and present that the stability can not be guaranteed if only the method of multiple Lyapunov functions is used when the switching hypersurface becomes sliding mode. Based on Filipov theory, the result of viable Lyapunov stability is obtained. When the subsystems of hybrid systems are linear time-invariant, quadratic stabilization condition of LTI hybrid systems is studied. Finally an example is given to illustrate the proposed method.

Key words: hybrid systems; multiple Lyapunov functions; viable stability
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采用多 Lyapunov 函数的混杂系统稳定性研究

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摘要: 针对一类由离散事件监控的连续动态子系统组成的混杂动态系统,首先分析利用多 Lyapunov 函数方法已有成果,指出切换超平面为滑动模时,利用这种方法不能确保混杂系统的稳定.基于 Filipov 理论给出了能活稳定性结果.对于混杂系统的连续动态子系统为线性时不变情况下,研究了混杂系统二次镇定条件.最后给出一个例子来说明本文方法.

关键词: 混杂系统; 多 Lyapunov 函数; 能活稳定性

1 Introduction

Hybrid systems are dynamical systems that inherently combine logical and continuous process, usually coupled finite automata and differential equations. In recent years there has been considerable interest in the modeling, analysis and the design of hybrid control systems. Many prior results identifying sufficient conditions for hybrid systems to be Lyapunov stability were reported. In Ye et al[1], a model suitable for qualitative analysis of hybrid systems was presented, the notion of an invariant set and several types of Lyapunov-like stability concepts for the invariant set were defined and finally the sufficient conditions for Lyapunov stability of hybrid systems were established. Peleties[2] and Savkin[3] proposed a single positive definite function as the Lyapunov function for all controlled systems. Branicky[4] and Hou[5] presented multiple Lyapunov function approaches that should be applicable to a larger set of systems than the single Lyapunov function method.

Although these prior results have provided deep insight into the Lyapunov stability problem of hybrid systems, most of them did not consider the actual switching law used by the system, which can cause the emergence of sliding mode such that the system can not run safely. Pettersson[6] studied this problem and assumed that there is no sliding mode in hybrid systems. He presented the sufficient condition of Lyapunov stability, but this sufficient condition does not contain the case that the guard of hybrid systems, that is, switching hypersurface, can be the sliding mode.

In this paper, the problem of Lyapunov stability for hybrid systems is further studied. Wicks[7] proposed a

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method to solve the problem that the switching hypersurface is sliding mode, but the difficulty of that work lies in constructing a stable convex combination of multiple systems matrices. Based on the Filipov theory[8] of existence condition for sliding mode and prior stability results for hybrid systems, we proposed a sufficient condition of Lyapunov stability which contains the case that sliding mode can occur when subsystem is switched to the next. Then the problem of quadratic stabilization is studied by using the sufficient condition and these problems can be transformed into the LMI's problems.

2 Modeling hybrid systems

In this section we consider controlled hybrid systems of the form

$$\begin{aligned} \dot{x} &= f_{q(t)}(x(t), u(t)), \\ q(t) &= v(x(t), q(t^-)), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n, q(t) \in Q \subset \{1, \cdots, N\}; f_{q(t)}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, each locally Lipschitz vector fields; $q(t^-)$ refers to the left-hand limit of the function $q(t)$ at point t. The trajectory of hybrid systems is the ordered pair, (x, q), where $x: \mathbb{R} \rightarrow \mathbb{R}^n$ and $q: \mathbb{R} \rightarrow Q$ which solves the system equation. The value taken by the trajectory at time $t \in \mathbb{R}$ is denoted by $(x(t), q(t))$. Therefore, (x, q) solves the system equation if and only if the equations are satisfied by $x(t)$ and $q(t)$ for all $t \in \mathbb{R}$.

A finite automaton associated with the hybrid system is the directed graph (V, A) where $V = I$ is a set of vertices and $A \subset V \times V$ is a set of directed arcs. By definition, the automaton associates a subsystem $\dot{x} = f_i$ with each vertex of the (V, A). It is possible to express the change of discrete states by defining a number of switch sets as $\Omega_{ij} = \{x \in \mathbb{R}^n \mid q_j = v(x, q_i)\}$. Typically, the set Ω_{ij} is given by hypersurface $h_{ij} = 0$. The hybrid systems (1) evolves from the initial conditions (x_0, q_0). The order pair (i, j) is an arc of A if and only if $\Omega_{ij} \neq \phi$. The guard therefore represents a subset of the hybrid system's state space in which a switch can occur. The set $\Omega_i^{\mp}$ represents the set in which subsystem f_i remains active.

3 Stability analysis via multiple Lyapunov functions

In this section, we discuss Lyapunov stability of hybrid systems via multiple Lyapunov functions. The sufficient condition presented in [4,5] and used in [9,10] to compute candidate Lyapunov functions provide an approach for testing switched and hybrid system stability. In the first place, the stability theorems in [4,5] require that V_j be Lyapunov-like. Those papers have not considered that switching between subsystems can lead to sliding mode so that the whole hybrid systems can not run safely. Here is an example to show that the hybrid system can not run safely when the sliding mode occurs even if it satisfies the requirements of Branicky multiple Lyapunov stability.

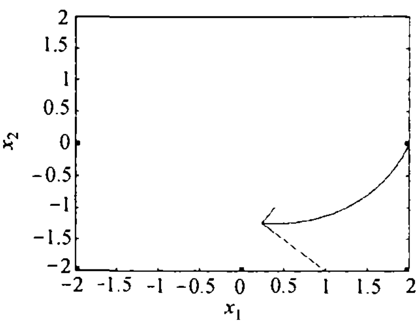

Fig. 1 Trajectory of hybrid systems

Example 1 Consider $\dot{x} = f_q(x) = A_q x(t)$ where $A_1 = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & -3 \\ 6 & -4 \end{bmatrix}$. And suppose that the first switching hypersurface is $h_{12} = x_1(t) + x_2(t) + 1$, $x_0 = (2, 0)$.

Then $\dot{x} = f_q(x)$ is globally stable for $q = 1, 2$. But the hybrid system using f_1 when $h_{12} > 0$, and f_2 when $h_{12} < 0$, is used. From Fig. 1, we can see that the switching event occurs when the trajectory of hybrid system from initial point (2, 0) clockwise reaches the hypersurface $h_{12} = 0$, then the subsystem is switched to $x(t) = A_2 x(t)$. But the trajectory evolves through the hypersurface $h_{12} = 0$ counter-clockwise. If the switching hypersurface is the surface of invariant set of subsystem, it will violate the safety requirements. In addition, it can cause other switching event so that the system can not evolve stable if the switching event generated by switching hypersurface is "two-sided"; that is, a switching event occur when the hypersurface is crossed in either direction. Therefore such a hybrid system can not be considered stable in the sense of Lyapunov. In the real industry process, there are many hybrid systems like Example 1. So it is necessary to analyze the stability of these systems and study how to design the controller that can stabilize the hybrid systems such as Example 1.

3.1 Prior results

Before we analyze and design the stabilizing controller of hybrid system, the prior results on the switched and hybrid system stability are to be reviewed. Let (x, q) be any trajectory generated by the hybrid systems. Assume that $f_q(0) = 0$ for all $f_q, q \in Q$. Then equilibrium point $x = 0$ is said to be stable in the sense of Lyapunov if and only if all $\varepsilon > 0$ there exists $\delta > 0$ such that $\| x(t_0) \| > \delta$ implies $\| x(t) \| < \varepsilon$ for all $t \geqslant t_0$.

Definition 1 Given a strictly increasing sequence of times $T = t_0, t_1, \cdots, t_N, \cdots$ in $\mathbb{R}$, V is Lyapunov-like for function f and trajectory $x(\cdot)$ over T if:

· $\dot{V}(x(t)) \leqslant 0$ for all $t \in I(T)$ which is the interval completion;

· V is monotonically non-increasing on the even sequence of T.

Using this definition, a sufficient condition of Lyapunov stability was proved in [11]. But this sufficient condition does not include the case that the guard of hybrid systems can become the sliding mode when the subsystem is switched to another subsystem. So the hybrid systems can not run according to the requirement of safety constraints and switching logic generated by a finite discrete event transition system, such as finite automata or Petri net. Then the definition of viable stability can be given as follows.

Definition 2 If the guard of hybrid systems is not the sliding mode and at the same time hybrid systems satisfy the sufficient condition of Lyapunov stability in [11], then the system is viably stable.

3.2 Main result

In the sequel we present and prove the main result of the research in this paper.

Theorem 1 Suppose we have candidate Lyapunov functions $V_i, i = 1, \cdots, N$ and vector fields $\dot{x} = f_q(x(t))$ with $f_q(0) = 0$ for all q. Let S be the set of all switching sequences associated with the system. If for each S we have that for all q, $V_i, i = 1, \cdots, N$ is Lyapunov-like for f_q and $x(\cdot)$ over S/q, and for each switching guard, we have

$$\forall x \in \tilde{\Omega}_{qr}, \lim_{h_{qr} \to 0^+} \frac{d}{dt} h_{qr} \cdot \lim_{h_{qr} \to 0^-} \frac{d}{dt} h_{qr} > 0, \quad (2)$$

then the hybrid system is viably stable in the sense of Lyapunov.

Proof For each switching guard that should be reached and then be left, according to the idea of existence condition of sliding mode in Filipov theory[8], we have

$$\lim_{h_{qr} \to 0^+} \frac{d}{dt} h_{qr} > 0 \text{ and } \lim_{h_{qr} \to 0^-} \frac{d}{dt} h_{qr} > 0$$

or

$$\lim_{h_{qr} \to 0^+} \frac{d}{dt} h_{qr} < 0 \text{ and } \lim_{h_{qr} \to 0^-} \frac{d}{dt} h_{qr} < 0,$$

so we can get $\forall x \in \tilde{\Omega}_{qr}, \lim_{h_{qr} \to 0^+} \frac{d}{dt} h_{qr} \cdot \lim_{h_{qr} \to 0^-} \frac{d}{dt} h_{qr} > 0$, so that the sliding mode can be avoided. The remainder of proof is similar to [11].

In order to apply this theorem to the analysis and design of hybrid systems easily, a stronger condition can be got which may be computed using LMI's through non-increasing condition in [11]. At the same time the general form of formula (2) is adopted, then the following corollary can be obtained.

Corollary 1 If there exist scalar function $V_q : \Omega_q^x \to \mathbb{R}$, each $V_q(x)$ differential in x, $\forall x \in \Omega_q^x$, and class K function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ and $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\forall x \in \Omega_q^x, \ \alpha(x) \leqslant V_q(x) \leqslant \beta(x),$$
$$\forall x \in \Omega_q^x, \ \dot{V}_q \leqslant 0,$$
$$\forall x \in \tilde{\Omega}_{qr}, \ V_r(x) \leqslant V_q(x),$$
$$\forall x \in \tilde{\Omega}_{qr}, \ \frac{d}{dt} h_{qr} < 0 (\text{or } \frac{d}{dt} h_{qr} > 0),$$

then hybrid systems are viably stable in the sense of Lyapunov.

4 Quadratic stabilization of LTI hybrid systems

In this section, we will derive the condition of stability of linear time-invariant (LTI) hybrid systems using Theorem 1. The LTI hybrid systems are described by the following equations:

$$\begin{aligned} \dot{x}(t) &= A_q x(t) + B_q u(t), \\ q(t) &= v(x(t), q(t^-)), \end{aligned} \quad (3)$$

the switching can be autonomous or can be controlled by some external factors. We want to find a stabilizing state feedback controller, $u = K_q x(t)$, in discrete state q. The overall control law will be defined by using each of these controllers for the corresponding discrete state. In order to satisfy the conditions of Corollary 1, we would like to determine the K matrices so that in each discrete state, the closed loop system has the same form of Lyapunov function $V(x(t)) = x^T(t)Px(t)$. If these matrices can be found, then Corollary 1 can ensure that the LTI hybrid systems are stable in the sense of Lyapunov. In this section we assume that the guard sets can be bounded by the conic sectors parameterized by symmetric matrix R_{qr}. In other words, consider sets, $\Omega_{qr} \subseteq \{x \in \mathbb{R}^n; x^T R_{qr} x \leqslant 0\}$ where $q \neq r$, and Ω_{qr} represent the set in which the guard set for transition between the q-th and r-th vertices.

Definition 3 For dynamic system (3), if there exists a symmetrical positive definition matrix $P > 0$, subject to Lyapunov functions $V_q(x(t)) = x^T(t)P_q x(t)$ satisfy

$$\frac{dV_q(x(t))}{dt} = (A_q x(t) + B_q u_q(t))^T P_q x(t) + x^T(t) P_q (A_q x(t) + B_q u_q(t)) \leqslant 0,$$

$$V_q - V_r \leqslant 0 \text{ and } \forall x \in \tilde{\Omega}_{qr}, \frac{dh_{qr}}{dt} < 0 (\text{or } \frac{dh_{qr}}{dt} > 0),$$

then the system (3) is quadratically stable in the sense of Lyapunov. In the following, applying Corollary 1, we give the condition of quadratic stabilization in this section.

Theorem 2 If there exist positive constants $\alpha_q > 0$, $\alpha_{qr} > 0, \beta_k > 0$ and positive definite symmetric matrices $Q_q, Q_{qr}, \hat{Q}_k, P_q$, such that the following matrix inequalities

$$\begin{bmatrix} A_k^T P_k + P_k A_k + \alpha_k Q_k & P_k B_k \\ B_k^T P_k & \frac{\bar{k}_k^{-1}}{2} I \end{bmatrix} \leqslant 0,$$

$$P_r - P_q + \alpha_{qr} Q_{qr} \leqslant 0,$$

$$\tilde{A}_k^T R_{qr} + R_{qr} \tilde{A}_k + \beta_k \hat{Q}_k < 0 (\text{or } > 0),$$

where $k = q, r$ and $\tilde{A}_q = A_q - B_q K_q$, are satisfied, then LTI hybrid systems (3) are quadratically stabilizable by using control law $u_q = -K_q x(t)$ where $K_q = \bar{k}_q B_q^T P_q$ and $\bar{k}_q$ is a constant.

Proof Let $u_k = -K_k x(t), K_k = \bar{k}_k B_k^T P_k, \bar{k}_k$ be a constant, $k = q, r$ under the condition of theorem. Since

$$\frac{dV_k(x(t))}{dt} = \frac{d}{dt}[x^T(t) P_k x(t)] = (A_k x(t) + B_k u_k(t))^T P_k x(t) + x^T(t) P_k (A_k x(t) + B_k u_k(t)) \leqslant 0,$$

there exists the positive definite symmetric matrix Q_q so that

$$A_k^T P_k + P_k A_k - 2\bar{k}_k P_k B_k B_k^T P_k + \alpha_k Q_k \leqslant 0,$$

which is equivalent to

$$\begin{bmatrix} A_k^T P_k + P_k A_k + \alpha_k Q_k & P_k B_k \\ B_k^T P_k & \frac{\bar{k}_k^{-1}}{2} I \end{bmatrix} \leqslant 0.$$

Since $V_Q - V_r \leqslant 0$, we have $P_r - P_q + \alpha_{qr} Q_{qr} \leqslant 0$. In addition,

$x \in \tilde{\Omega}_{qr}$,

$$\frac{dh_{qr}}{dt} = \frac{d}{dt}(x^T R_{qr} x) = x^T (A_k - B_k K_k)^T R_{qr} x + x^T R_{qr} (A_k - B_k K_k) x < 0 (\text{or } > 0),$$

so $\tilde{A}_k^T R_{qr} + R_{qr} \tilde{A}_k + \beta_k \hat{Q}_k < 0$ (or > 0) can be got where $\tilde{A}_k = A_k - B_k K_k$.

Remark This condition is more restrictive, but it can be reformulated as linear matrix inequalities (LMI's) that can be solved easily using interior point methods for convex optimization.

5 Example

For simplification, we only consider a numerical example of hybrid systems composed of two controlled subsystems. We suppose that the hybrid systems are switched from discrete state $q = 1$ to discrete state $q = 2$, then lie in discrete state $q = 2$, and hybrid systems reach stable state.

Let

$$A_1 = \begin{bmatrix} -1 & 1 \\ 3 & -4 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & -3 \\ 6 & -4 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, R_{12} = \begin{bmatrix} 3.2629 & -1.2089 \\ -1.2089 & 1.5466 \end{bmatrix}.$$

Stating the stability conditions and solving the corresponding LMI's problem results in a solution

$$P_1 = \begin{bmatrix} 0.6047 & -0.3793 \\ -0.3793 & 0.4472 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.1791 & -0.1012 \\ -0.1012 & 0.0993 \end{bmatrix}.$$

Hence, the hybrid systems are stable and the control laws $u_1 = [0.3793 \quad -0.4472]x(t)$ and $u_2 = [-0.1791 \quad 0.1012]x(t)$ can be got.

6 Conclusion

In this paper, we discuss the problem of stability for hybrid systems based on multiple Lyapunov functions. At first the viable Lyapunov stability is studied. When the subsystems of hybrid systems are linear time-invariant, the quadratic stability is discussed and stabilization conditions that can be computed by LMI's are obtained.

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