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Some new results on chaos synchronization

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Abstract: Some new criteria on chaos synchronization are derived, which further improves the results obtained in the literature. It offers some fairly simple algebraic conditions that are very easy to verify for chaos synchronization.

Key words: chaos; synchronization; algebraic condition

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混沌同步的一些新结果

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摘要: 给出了混沌同步的一些新判据，改进了当前文献的结果，并提供了一些非常容易检验的关于混沌同步的代数条件.

关键词: 混沌; 同步; 代数条件

1 Introduction

Chaos synchronization has been the topic of intensive research in the past decade (see, e.g., [1 ~ 12] and the references therein). One common approach to chaos synchronization is to reformulate it as a (generalized) Lur'e system and then discuss the absolute stability of its error dynamics [4,7 ~ 10].

This paper aims to improve the results of [4], and offers some fairly simple algebraic conditions that are very easy to verify, on the basis of the authors' recent investigations [9, 10]. The basic idea supporting this study is based on the fundamental theory and techniques developed in the nonlinear systems literature [13,14].

2 Chaos synchronization as stabilization

Consider a uni-directional feedback-controlled chaos synchronization system in the following form:

$$\begin{cases} \dfrac{dx}{dt} = Ax + f(q_1^T x, \cdots, q_r^T x, t)(bc^T - cb^T)x \text{ (drive)}, \\ \dfrac{dy}{dt} = Ay + f(q_1^T x, \cdots, q_r^T x, t)(bc^T - cb^T)y - K(x-y) \text{ (response)}, \end{cases} \tag{1}$$

where $x(t), y(t), q_i, b, c \in \mathbb{R}^n$, $i = 1, \cdots, r \leqslant n$, $A, K \in \mathbb{R}^{n\times n}$ and $f \in C[\mathbb{R} \times [t_0, \infty), \mathbb{R}]$. Let the synchronization error be

$$e = x - y. \tag{2}$$

Then, system (1) can be reformulated as

$$\frac{de}{dt} = Ae + f(q_1^T x, \cdots, q_r^T x, t)(bc^T - cb^T)e + Ke. \tag{3}$$

The objective is to choose the constant feedback gain matrix K, such that $e(t) \to 0$ as $t \to \infty$, thereby achieving synchronization $y(t) \to x(t)$ as $t \to \infty$.

The following is main result of [4]:

Theorem 1 Suppose that f is bounded for any bounded variables, and that there exists a symetric and positive definite matrix P such that

$$\begin{aligned} &(A+K)^T P + P(A+K) < 0, \\ &B^T P + PB = 0, \end{aligned} \tag{4}$$

where $B = bc^T - cb^T$, in which b and c are linearly independent. Then, the zero solution of system (3) is globally asymptotically stable, so that system (1) synchronizes.

A few remarks about Theorem 1 are in order.

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First, for the purpose of synchronization, under conditions (4), the boundedness of the function $f(\cdot)$ is redundant as can be seen from the detailed discussion given in the next section.

Second, the matrix $B = bc^{\mathrm{T}} - cb^{\mathrm{T}}$ is a very special asymmetric matrix, so that the second quation in (4) always has a solution (e.g., the identity matrix or a block-diagonal matrix). As a result, by selecting an appropriate gain matrix K, the matrix inequality in (4) is always solvable.

3 Main results of the present paper

Two questions are addressed in this section:

1) How to simplify the conditions given in (4) of Theorem 1 above?

2) If conditions (4) are not satisfied, how to select a suitable feedback gain matrix K to ensure the synchronization?

Theorem 2 Suppose that vectors b and c are linearly independent. Then

i) If every column of matrix B has a nonzero element $b_{ij} \neq 0$, then the matrix question in (4) can only have solutions of the form $P = \lambda I_n$ with $\lambda > 0$.

ii) If

$$B = \begin{bmatrix} 0_{r\times r} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & B_{(n-r)\times(n-r)} \end{bmatrix}$$

and every column of the submatrix $B_{(n-r)\times(n-r)}$ has a nonzero element $b_{ij} \neq 0$, then a solution of the matrix equestion in (4) can be simply chosen as

$$P\begin{bmatrix} P_{r\times r} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & I_{(n-r)\times(n-r)} \end{bmatrix},$$

in which $P_{r\times r}$ is a symmetric and positive definite constant matrix.

Proof Since $B = bc^{\mathrm{T}} - cb^{\mathrm{T}}$, one has

$$B^{\mathrm{T}} = -bc^{\mathrm{T}} + cb^{\mathrm{T}} = -(bc^{\mathrm{T}} - cb^{\mathrm{T}}) = -B,$$

implying that B is a skew-symmetric matrix satisfying $B + B^{\mathrm{T}} = 0$.

In applying the mathematical induction argument, start from $n = 2$:

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}[c_1 \quad c_2] - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}[b_1 \quad b_2] =$$

$$\begin{bmatrix} 0 & b_1c_2 - c_1b_2 \\ c_1b_2 - b_1c_2 & 0 \end{bmatrix} := \begin{bmatrix} 0 & b_{12} \\ -b_{12} & 0 \end{bmatrix}.$$

Since b and c are linearly independent,

$$\det\begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix} \neq 0 \Leftrightarrow$$

$$b_1c_2 - c_1b_2 \neq 0 \Leftrightarrow b_{12} \neq 0.$$

Consequently,

$$B^{\mathrm{T}}P + PB = 0 \Leftrightarrow$$

$$-BP + PB = 0 \Leftrightarrow BP = PB.$$

The last equality gives

$$-p_{12}b_{12} = b_{12}p_{12} \Rightarrow p_{12} = 0,$$

$$b_{12}p_{11} = b_{12}p_{22} \Rightarrow p_{11} = p_{22} = \lambda > 0.$$

Without loss of generality, take $\lambda = 1$ in the following. Thus, when $n = 2$, one has $P = I_2$.

Next, assume that the claim is true for $n - 1$, and consider the case of n. In this case, $BP = PB$, with their special forms obtained in the case of $n - 1$, yields

$$\begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1n} \\ -b_{12} & 0 & b_{23} & \cdots & b_{2n} \\ -b_{13} & -b_{23} & 0 & & b_{3n} \\ \vdots & \vdots & & \ddots & \\ -b_{1n} & -b_{2n} & 0 & \cdots & 0 \end{bmatrix}\cdot$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{12} & 1 & 0 & \cdots & 0 \\ p_{13} & 0 & 0 & & 0 \\ \vdots & \vdots & & \ddots & \\ p_{1n} & 0 & 0 & \cdots & 1 \end{bmatrix} =$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{12} & 1 & 0 & \cdots & 0 \\ p_{13} & 0 & 0 & & 0 \\ \vdots & \vdots & & \ddots & \\ p_{1n} & 0 & 0 & \cdots & 1 \end{bmatrix}\cdot$$

$$\begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1n} \\ -b_{12} & 0 & b_{23} & \cdots & b_{2n} \\ -b_{13} & -b_{23} & 0 & & b_{3n} \\ \vdots & \vdots & & \ddots & \\ -b_{1n} & -b_{2n} & 0 & \cdots & 0 \end{bmatrix}.$$

By comparing corresponding elements, one obtains $P = I_n$, as claimed in Part i) of the theorem.

Part ii) of the theorem can be similarly verified, completing the proof of the theorem.

Note that since $P = I_n$ is always a solution of the second Eq. of (4), conditions in (4) can be simplified to be

$$(A+K)^{\mathrm{T}}+(A+K)<0. \qquad (5)$$

Theorem 3 In system (1), let $\lambda_{\max}$ be the largest eigenvalue of the matrix $[A+A^{\mathrm{T}}]$. If $\lambda_{\max}<0$ then one can choose $K=0$, namely, no feedback is needed in order to achieve synchronization. If $\lambda_{\max}\geqslant 0$, then one may choose $K=\mu I_n$ with $2\mu<-\lambda_{\max}$ to achieve synchronization.

Proof When $\lambda_{\max}<0$, let $K=0$. Then

$$(A+K)^{\mathrm{T}}+(A+K)=A^{\mathrm{T}}+A<0.$$

When $\lambda_{\max}\geqslant 0$, let $K=\mu I_n$. Then,

$$x^{\mathrm{T}}[(A+K)^{\mathrm{T}}+(A+K)]x = x^{\mathrm{T}}(A^{\mathrm{T}}+A)x+x^{\mathrm{T}}(K^{\mathrm{T}}+K)x \leqslant \lambda_{\max}x^{\mathrm{T}}x+2\mu x^{\mathrm{T}}x<0,$$

for $x\neq 0$ if $2\mu<-\lambda_{\max}$. This completes the proof of the theorem.

It is well known that for $n\gg 1$, computing the eigenvalues is a very difficult task. The following criterion is easier to use.

Theorem 4 Let $H=A+A^{\mathrm{T}}=[h_{ij}]_{n\times n}$. Define

$$L=\max_{1\leqslant i\leqslant n}\sum_{j=1,j\neq i}^{n}|h_{ij}|.$$

If $K=\mu I_n$ with $2\mu+h_{ii}<-L(i=1,\cdots,n)$, then conditions (4) must be satisfied, implying synchronization of system (1).

Proof Since

$$(A+K)+(A+K)^{\mathrm{T}}=(A+A^{\mathrm{T}})+2\mu I_n,$$

one has

$$2\mu+h_{ii}=2\mu+2a_{ii}<-L=-\max_{1\leqslant i\leqslant n}\sum_{j=1,j\neq i}^{n}|h_{ij}|,\ i=1,\cdots,n.$$

According to the Gershgorin Theorem, the eigenvalues of the matrix $(A+K)+(A+K)^{\mathrm{T}}$ are located on the left-half of the complex plane, so that the symmetric matrix $(A+K)+(A+K)^{\mathrm{T}}$ is negative definite. Therefore, conditions (4) are satisfied, implying synchronization of system (1). This completes the proof of the theorem.

4 Multi-variable systems

As a further generalization of the above study, consider the following synchronization problem of a multi-variable system:

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=Ax+\sum_{i=1}^{m}f_i(q_1^{\mathrm{T}}x,\cdots,q_r^{\mathrm{T}}x,t)\cdot\\ \qquad (b_ic_i^{\mathrm{T}}-c_ib_i^{\mathrm{T}})x\ (\text{drive}),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=Ay+\sum_{i=1}^{m}f_i(q_1^{\mathrm{T}}x,\cdots,q_r^{\mathrm{T}}x,t)(b_ic_i^{\mathrm{T}}-c_ib_i^{\mathrm{T}})y-\\ \qquad K(x-y)\ (\text{response}),\end{cases} \qquad (6)$$

where $x(t),y(t),q_i,b_i,c_i\in\mathbb{R}^n$, $i=1,\cdots,r\leqslant n$, $A,K\in\mathbb{R}^{n\times n}$, and $f_j\in\mathbb{C}[\mathbb{R}\times[t_0,\infty),\mathbb{R}]$, $j=1,\cdots,m$.

As before, let the synchronization error be $e=x-y$ and then consider the synchronization error dynamics

$$\frac{\mathrm{d}e}{\mathrm{d}t}=(A+K)e+\sum_{i=1}^{m}f_i(q_1^{\mathrm{T}}x,\cdots,q_r^{\mathrm{T}}x,t)B_ie, \qquad (7)$$

where $B_i=b_ic_i^{\mathrm{T}}-c_ib_i^{\mathrm{T}}$, $i=1,\cdots,m$. The following result was obtained in [4], with slight corrections given here:

Theorem 5 A sufficient condition for synchronization of system (6) is that for any functions f_i there exists an $n\times n$ constant matrix $P=P^{\mathrm{T}}>0$ such that

$$\begin{aligned}&(A+K)^{\mathrm{T}}P+P(A+K)<0,\\ &PB_i+B_i^{\mathrm{T}}P=0.\end{aligned} \qquad (8)$$

Note that according to Theorem 3 above, the second Eq. in (8) always has a solution, e.g., $P=I_n$. By using this solution and suitably choosing a K, the matrix Ineq. in (8) can always be satisfied. The following gives a much simpler criterion under which condition (8) is satisfied.

Theorem 6 In system (6), let $\lambda_{\max}$ be the largest eigenvalue of the matrix $[A+A^{\mathrm{T}}]$. If $\lambda_{\max}<0$ then one can choose $K=0$, namely, no feedback is needed in order to achieve synchronization. If $\lambda_{\max}\geqslant 0$, then one may choose $K=\mu I_n$ with $2\mu<-\lambda_{\max}$ to achieve synchronization.

Proof Consider the Lyapunov function $V=e^{\mathrm{T}}e$. When $\lambda_{\max}<0$, let $K=0$. Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7)}=e^{\mathrm{T}}[(A+K)^{\mathrm{T}}+(A+K)]e+\sum_{i=1}^{m}f_i(q_1^{\mathrm{T}}x,\cdots,q_r^{\mathrm{T}}x,t)e^{\mathrm{T}}(B_i+B_i^{\mathrm{T}})e=e^{\mathrm{T}}[(A+K)^{\mathrm{T}}+(A+K)]e=e^{\mathrm{T}}(A+A^{\mathrm{T}})e<0,$$

for all $e \neq 0$.

When $\lambda_{\max} \geqslant 0$, let $K = \mu I_n$ with $2\mu < -\lambda_{\max}$. Then, the largest eigenvalue of $[(A+K)^T + (A+K)]$ is strictly negative, so that

$$\left.\frac{dV}{dt}\right|_{(7)} = e^T(A+K)^T + (A+K)e < 0,$$

for all $e \neq 0$. This completes the proof of the theorem.

Theorem 7 Let $H = A + A^T = [h_{ij}]_{n\times n}$. Define

$$L = \max_{1\leqslant i\leqslant n} \sum_{j=1,j\neq i}^{n} | h_{ij} |.$$

If $K = \mu I_n$ with $2\mu + h_{ii} < -L(i = 1,\cdots,n)$, then conditions (8) must be satisfied, implying synchronization of system (6).

Proof It follows from the properties of the M-matrix that if

$$2\mu + h_{ii} = 2\mu + 2a_{ii} < -L = -\max_{1\leqslant i\leqslant n} \sum_{j=1,j\neq i}^{n} | h_{ij} |,\ i = 1,\cdots,n,$$

then the matrix $(A + A^T) + 2\mu I_n < 0$, implying that conditions (8) are satisfied. Thus, system (6) synchronizes, and the proof of the theorem is completed.

5 Some illustrative examples

Example 1 Consider the following system:

$$\frac{dx}{dt} = \begin{bmatrix} -2 & 3 & 4 & 1 \\ 3 & -2 & 4 & 6 \\ -3 & 2 & 1 & 3 \\ 6 & -2 & 2 & 0 \end{bmatrix} x + f(q_1^T x, q_2^T x, t)(bc^T - cb^T)x,$$

where $f(\cdot)$ and b, c, q_1, q_2 are not important in the present discussion. Since

$$H: = A + A^T = \begin{bmatrix} -4 & 6 & 1 & 7 \\ 6 & -4 & 6 & 4 \\ 1 & 6 & 2 & 5 \\ 7 & 4 & 5 & 0 \end{bmatrix}$$

is not negative definite, according to Theorem 4,

$$L = \max\left\{\sum_{j=2}^{4} | h_{ij} |, \sum_{j=1,j\neq 2}^{4} | h_{ij} |, \sum_{j=1,j\neq 3}^{4} | h_{ij} |\right\} = \max\{14,16,12,16\} = 16.$$

When $2\mu + 2 < -16$ namely $\mu < -9$, one has $2 + \mu + h_{ii} < -16, i = 1,2,3,4$. Therefore, if one takes $\mu = -9.1$ so that $K = -9.1 I_4$, then the matrix $(A + K) + (A + K)^T$ is negative definite, implying that the coupled system synchronizes.

Example 2 Consider a system with

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 4 & 1 & 2 \\ -4 & 2 & -1 \end{bmatrix},\ b = [0\ \ 1\ \ 1]^T,$$

$$c = [0\ \ 1\ \ 0]^T,\ f(\cdot, t) = 2u(t),$$

where $u(t)$ is a control input signal. With $P = I_3$ and $K = -2.5 I_3$, the matrix $(A + K) + (A + K)^T$ is negative definite, so as result, the coupled system of this model will synchronize.

Example 3 Consider the following system:

$$\frac{dx}{dt} = \begin{bmatrix} -2 & 3 & -4 \\ 4 & 2 & 2 \\ -4 & 2 & 0 \end{bmatrix} x + f_1(q_1^T x, q_2^T x, t)\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} x + f_2(q_1^T x, q_2^T x, t)\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix} x.$$

Since

$$H: = A + A^T = \begin{bmatrix} -4 & 7 & -8 \\ 7 & 4 & 4 \\ -8 & 4 & 0 \end{bmatrix}$$

is not negative definite, and both B_1 and B_2 are skew symmetric, and

$$L = \max\left\{\sum_{j=2}^{3} | h_{ij} |, \sum_{j=1,j\neq 2}^{3} | h_{ij} |, \sum_{j=1}^{2} | h_{ij} |\right\} = \max\{15,11,12\} = 15.$$

If $2\mu + h_{22} < -15$ then $\mu < -19/2$. So, if one takes $\mu = -10$ so that $K = -10 I_3$, then the matrix $(A + K) + (A + K)^T$ is negative definite. Hence, Theorem 7 implies that the coupled system synchronizes.

6 Concluding remarks

A common philosophy in the study of chaos control and synchronization today is trying to either construct a special Lyapunov function or to solve a system of linear matrix inequalities. As has been widely shown, both can be quite implicit and indeed technically difficult. This paper improves the results of [4], and offers some fairly simple algebraic conditions that are very easy to verify. It is believed that more efforts should be devoted to simpler design so as to obtain more feasible structures and conditions in the future investigation of chaos control and synchronization, as has been tried in [9,10].

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