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Fast algorithm for constrained linear system control via geometric techniques

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Abstract: A new method was proposed to solve the control problem for linear system with constrained states and control constraints. Polytope and Ellipsoidal techniques had been used for constrained system control, but they were either too complex or too conservative for many applications. Based on the mathematic analysis, the algorithm with simple computations was proposed by optimizing the computation of the level sets. Simulations show that the computation of this new method is much simpler than before and all the initial points can be steered to the given set properly.

Key words: constraints; time-optimal control; linear system

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基于几何方法的约束线性系统控制快速算法

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摘要: 针对存在状态量和控制量约束的线性系统控制问题,提出了新快速算法.已有的处理上述约束系统控制问题的多面体方法和椭球方法在实际应用过程中分别存在计算繁琐和计算保守的问题.在数学分析的基础上,通过对上述算法的水平集计算过程的优化,提出了一种计算简单的约束控制算法.仿真结果表明,这种算法计算简单,且可以满足系统控制的要求.

关键词: 约束; 时间最优控制; 线性系统

1 Introduction

In most process control applications there is a need for controllers, which are able to take into account constraints both on input and state variables. For such a control problem, Gilbert & Tan[1] defined the concept of maximal output admissible set and employed it to construct controllers. Mayne & Schroeder[2] designed nonlinear controllers to stabilize the system and steer all trajectories emanating a prescribed set to a control invariant set in minimum time. But it cannot be used for the system with higher dimensions in state spaces because of the computation difficulties. J. Chen & S.M. Veres[3] use ellipsoids to compute the level sets instead of polytopes. It can be used for the system that does not suitable to the polytope techniques because of the computation complexity. But it is conservative and the solution cannot be ensured for some special points.

A new algorithm is suggested in this paper. With this algorithm, the computations for any system will be easy enough to be conducted on line. In the worst case, the time to the terminal set will increase. The new algorithm for searching the control inputs is proposed solving this problem.

2 Minimum time problem

Suppose the controllable linear system of the form

$$x[k+1] = Ax[k] + Bu[k] + w, \tag{1}$$

where $x[k], [k], w \in \mathbb{R}^n$. The states sequence, control inputs sequence and disturbance are subject to the following hard constraints

$$x[k] \in E,\ u[k] \in \Omega \text{ and } w \in M, \tag{2}$$

where $E \in \mathbb{R}^n$ is convex compact set, $\Omega \in \mathbb{R}^n$ is convex closed set and $M \in \mathbb{R}^n$ is convex compact set, each set containing the origin in its interior.

Let V denote the value function for the discrete-time,

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minimum time problem, and $X_k, k \in N$ the associated level sets are defined by[2] as

$$V(x) = \min \{k \mid x[k;x,0] \in X_0\}, \tag{3}$$

$$X_k = \{x \mid V(x) \leqslant k\}. \tag{4}$$

Because of the disturbance, we cannot steer the states to the origin but to a small terminal set X_0. X_k is the set of states of the given system that can be steered to the terminal set in no more than k steps. These level sets can be recursively generated as Algorithm 1 in [2], where it is supposed that Ω and M are polytopes and E is a polyhedral.

Algorithm 1

Data Given A, B, E, Ω, M and N.

Step 1 Compute terminal set X_0.

Step 2 Set $k = 0$.

Step 3 Compute the set $X_k' = X_k \succ M$.

Step 4 Compute the set

$X_{k+1} = \{A^{-1}X_k' - A^{-1}B\Omega\} \cap E$.

Step 5 Set $k = k + 1$. If $k = N$, stop.

Else, go to Step 3.

Notation Given two sets X, Y be non-empty, convex subsets of $\mathbb{R}^n$,

$$A + B = \{a + b \mid a \in A, b \in B\}, \tag{5}$$

$$A \succ B = \{x \mid x + B \in A\}. \tag{6}$$

Definition 1[1] A set $X \subset E$ is said to be control invariant for the discrete-time system (A, B, E, Ω, M) if, for every $x \in X$, there exists a $u \in \Omega$ such that $Ax + Bu + w \in X$.

If X_0 is a robust control invariant polytope, then the non-decreasing sequence $\{X_k\}$ generated by Algorithm 1 is a sequence of control invariant polytopes for the discrete-time system (A, B, E, Ω, W). For all $k > 0$ and every $x[k] \in X_k$, there exists a $u \in \Omega$ such that

$$Ax[k] + Bu[k] + w \in X_{k-1} \subset X_k. \tag{7}$$

And there exists a control law which robustly steers the discrete time system from any initial state in X_k to the terminal sets in k steps, then maintains the state in the terminal sets.

Although the algorithm as above is simple to express, there are considerable computational difficulties. A major concern is the increasing complexity of the sets X_k as n and k increase. Mayne has given a table to show the total number of vertices, simplices, and inequalities arising in various example problems in [2]. We can see that for a system with state dimension 5, the vertices may be 566, simplices 8122 and inequalities 3329. It is hoped to simplify the computation.

3 Simplified algorithm

For later use, the following elementary results are required. Let $A, B \in \mathbb{R}^{n\times n}$ be arbitrary, and X, Y and Z be non-empty, convex subsets of $\mathbb{R}^n$. Then we can get

$$(A + B)X \subset AX + BX, \tag{8}$$

$$A(X + Y) = AX + AY, \tag{9}$$

$$(X \succ Y) + Z \subset (X + Z) \succ Y, \tag{10}$$

$$(X \cap Y) \succ Z = (X \succ Z) \cap (Y \succ Z). \tag{11}$$

According to Algorithm 1 we can see that if $M = \{0\}$ and $E = \mathbb{R}^n$, then we get

$$X_0 = \{0\}, \quad X_k = -\sum_{i=1}^{k} A^{-i}B\Omega. \tag{12}$$

We can take

$$Y_k = -(\sum_{i=1}^{k} A^{-i}B)\Omega \subset X_k \tag{13}$$

instead of X_k as the level set. The computation is much simpler.

The new algorithm will be discussed in three computation steps: the terminal set, the level sets and the control inputs.

3.1 Terminal set

It is hoped that the terminal set is a small set in accordance with the disturbance. And from Algorithm 1, we can see that, the terminal set X_0 must be a control invariant polytope and the existence of $X_0' = X_0 \succ M$ must be ensured.

Theorem 1 For the system above,

$$X_0 = -A^{-1}B\Omega \cap E \tag{14}$$

is a robust control invariant set for (A, B, E, Ω, M), if $M \subset -A^{-1}B\Omega$ holds.

Proof For all $x[k] \in X_0$, we take

$$u[k] = -B^{-1}Ax[k],$$

so that

$$u[k] \in -B^{-1}A \times (-A^{-1}B\Omega \cap E) \subset \Omega,$$

and $AX[k] + Bu[k] + w = w \in M \subset X_0$.

For the system is controllable, we can get $M \subset -B\Omega$, and $A^{-1} \geqslant I$. So $M \subset -A^{-1}B\Omega$ holds.

We can say that the terminal set as above is a small set in accordance with the disturbance.

The computation of the terminal set as shown above is much more easier than those in [2].

3.2 Level sets

First, we consider the system with $E = \mathbb{R}^n$, that means there are no states constraints.

Theorem 2 Suppose $E = \mathbb{R}^n$, for all k,

$$Y_k = (\sum_{i=0}^{k} A^{-i}) X'_0 \subset X'_k. \qquad (15)$$

Proof

As $X_0 = -A^{-1}B\Omega \cap E = -A^{-1}B\Omega$,

we get

$$X'_0 = X_0 \succ M = -A^{-1}B\Omega \succ M,$$

$$X'_{k+1} = (A^{-1}X'_k - A^{-1}B\Omega) \succ M \supset A^{-1}X'_k + (-A^{-1}B\Omega \succ M) = A^{-1}X'_k + X'_0.$$

Take $Y_{k+1} = A^{-1}Y_k + X'_0,\ Y_0 = X'_0$,

then we can get

$$Y_k = (\sum_{i=0}^{k} A^{-i}) X'_0 \subset \sum_{i=1}^{k} A^{-i} X'_0 \subset X'_k.$$

The fact that $\{Y_k\}$ is a subset of $\{X'_k\}$ implies that $\{Y_k\}$ can be used in place of $\{X'_k\}$. As the shape of the polytopes is not changed, it is very easy to be computed.

Then we will use the results above for the system where the states constraints exist. As the system is controllable, must be a robust control invariant set. That means for every point $x \in E$, there exists a $u \in \Omega$, such that

$$x' = Ax + Gu + w \in E,$$

so that

$$x = A^{-1}x' - A^{-1}Bu - A^{-1}w \in (A^{-1}E - A^{-1}B\Omega).$$

Hence $E \subset (A^{-1}E - A^{-1}B\Omega)$.

Theorem 3 If $X'_k = ((\sum_{i=0}^{k} A^{-i})(-A^{-1}B\Omega) \cap E) \succ M$, then $X'_{k+1} = ((\sum_{i=0}^{k+1} A^{-i}(-A^{-1}B\Omega) \cap E) \succ M$.

Proof

$$X'_{K+1} = (A^{-1}X'_k - A^{-1}B\Omega) \cap E \succ M = \{(A^{-1}(\sum_{i=0}^{k} A^{-i})(-A^{-1}B\Omega \cap E \succ M) - A^{-1}B\Omega \succ M\} \cap (E \succ M) \supset (\sum_{i=0}^{k+1} A^{-i})(-A^{-1}B\Omega \succ M) \cap ((A^{-1}E - A^{-1}B\Omega) \succ M) \cap (E \succ M) = ((\sum_{i=0}^{k+1} A^{-i})(-A^{-1}B\Omega) \cap E) \succ M.$$

As in this case,

$$X_0 = -A^{-1}B\Omega \cap E,$$

$$X'_0 = (-A^{-1}B\Omega \cap E) \succ M,$$

then take

$$Y_0 = X_0,\ Y'_0 = X'_0,$$

$$Y_{k+1} = (\sum_{i=0}^{k+1} A^{-i})(-A^{-1}B\Omega) \cap E,$$

$$Y'_{k+1} = ((\sum_{i=0}^{k+1} A^{-i})(-A^{-1}B\Omega) \cap E) \succ M.$$

As $Y'_k \subset X'_k$, we take $\{Y'_k\}$ instead of $\{X'_k\}$ as the level sets. Then the computation will be much more easier than before. It can be conducted on line. But as the spaces of the level sets are more greatly lessened than before, it is not the time-optimal solution for the system. For some initial point, the time to terminal set is increased. To solve this problem a new method for searching the time-optimal solution is proposed in 3.3.

3.3 Control inputs

The serial number of the smallest level set that the initial states lies is the same as the number of the smallest level set with the initial states, which is the same as the number of the control steps. That means if the initial state lies in k-level, then it can be steered to the terminal set in k steps. If the initial state lies in k-level, it can be steered to and only to $(k-1)$-level. So the states can be steered to the terminal set in minimum time with polytope techniques as there is no space loosing.

The state spaces, control input spaces and level sets are all lessened in the algorithm above, k-level set contains the points, not all of which can be steered to the terminal set in k steps. If the initial state lies in k-level, it may be steered not only to $(k-1)$-level. With suitable control input, the state may be steer to $(k-1)$-level, $(k-2)$-level and so on. If the state is steered to the smallest level set and it can be steered to with suitable control input, the control steps will be the least. The aim of the optimizing algorithm for the control input is to search the smallest level set which the state can be steered to with constraints satisfied control input.

The whole algorithm is as follows.

Algorithm 2

Data Given A, B, E, Ω, M and N.

Step 1 Compute terminal set as

$$X_0 = -A^{-1}B\Omega \cap E,$$

$$X'_0 = X_0 \succ M.$$

Step 2 Let $k = 0$.

Step 3 Compute level sets as

$$Y_{k+1} = (\sum_{i=0}^{k+1} A^{-i})(-A^{-1}B\Omega) \cap E,$$

$$Y'_{k+1} = ((\sum_{i=0}^{k+1} A^{-i})(-A^{-1}B\Omega) \cap E) > M.$$

Step 4 Let $k = k + 1$.

Step 5 If all vertices of E are inside Y_k, go to Step 6; else go to Step 3.

Step 6 Let $k = 0, x[k] = x_0$.

Step 7 Compute which ellipsoid Y_k the current state $x[k]$ lies inside.

Step 8 If $Y'_k = X'_0, u[k] = -B^{-1}Ax[k]$, stop. Else, go to Step 9.

Step 9 Compute set $O = B\Omega + Ax[k]$.

Step 10 Compute set $G = O \cap E$.

Step 11 Let $j = 2$.

Step 12 If $G \cap G'_{k-j} = \phi$, go to Step 14.

Step 13 Let $j = j + 1$, go to Step 12.

Step 14 Compute set $S = G \cap Y'_{k'-j+1}, R = B^{-1}(S - Ax[k])$.

Step 15 Compute the center value of R as the single-value control $u[k]$.

Step 16 Get x_0, go to Step 6.

To determine whether the intersection of two polytope exits or not, we use the method of collision detection in robot path planning as the method in S. J. Ren et al[4].

4 Examples

For such a system

$$x[k+1] = \begin{bmatrix} 0.99 & 0.09 \\ -0.17 & 0.73 \end{bmatrix} x[k] + \begin{bmatrix} 0.02 & -0.20 \\ 0.34 & 0.02 \end{bmatrix} u[k] + w[k]$$

with following constraints:

$$x \in E = [-1,1] \times [-1,1] \in \mathbb{R}^2,$$

$$u \in \Omega = [-1,1] \times [-1,1] \in \mathbb{R}^2,$$

$$w \in M = [-0.1,0.1] \times [-0.1,0.1] \in \mathbb{R}^2.$$

Initial state is $[-0.99, -0.98]$.

Figure 1 shows the control results of polytope techniques, Fig.2 ellipsoidal techniques and Fig.3 the new algorithm.

We can see that the new algorithm is much simpler. Although the level sets are more than those in polytope techniques and ellipsoidal techniques, the time for the state to be steered to the terminal set does not increase so much as in ellipsoidal techniques.

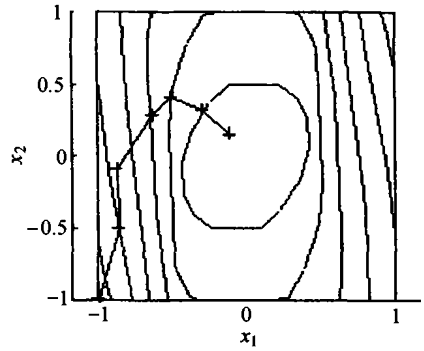

Fig. 1 Polytope techniques

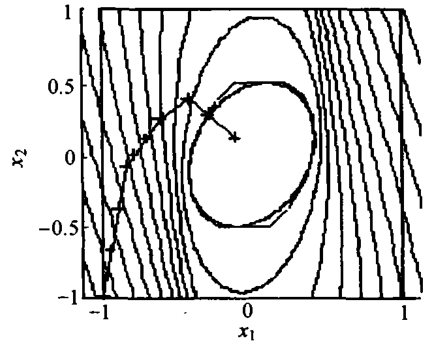

Fig. 2 Ellipsoidal techniques

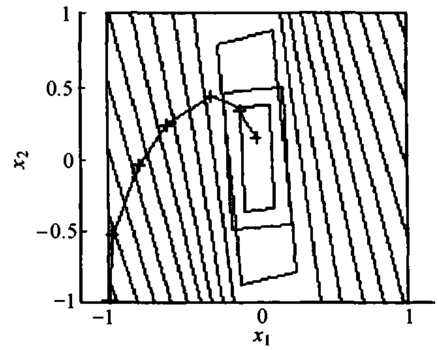

Fig. 3 New techniques

5 Conclusion

A simple method is proposed to solve the control problem for linear system with state and control constraints. By choosing the suitable terminal set and simplifying the computation for the level sets, the computation is much simpler. The new algorithm for optimizing the control inputs is proposed to solve the problem of time increase. The terminal set and level sets in the new algorithm can be computed on-line as it is easy to compute. So the parameters of the system can be adjusted on-line. The algorithm can be more robust and more useful than before.

As the level sets and terminal set are computed according to the control constraints, there are some limit for its application as it can not solve the control problem as single-input multi-output system.

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