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Exponential stability for stochastic interval delayed Hopfield neural networks

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Abstract: A type of stochastic interval delayed Hopfield neural network had been studied. By using Itô formula and Lyapunov function, some new delay-dependent and delay-independent sufficient conditions of its global exponential stability had been given. All the results obtained were generalizations of some recent results reported in the literature for stochastic neural networks with constant delays or their certain cases with variable-delays.

Key words: stochastic interval delayed Hopfield neural network; Brownian motion; Itô formula; Lyapunov function; robust stability

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带有时滞的随机区间 Hopfield 神经网络的指数稳定性

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摘要：讨论了带有可变时滞的随机区间 Hopfield 神经网络的指数稳定性，利用 Itô 公式和 Lyapunov 函数，得到了几个关于其指数稳定时滞无关和时滞相关的充分性条件，推广了现有文献中关于定常时滞随机神经网络及其确定形式的许多结果.

关键词：随机区间时滞 Hopfield 神经网络；布朗运动；Itô 公式；Lyapunov 函数；鲁棒稳定性

1 Introduction

The theoretical research on neural network has made great progress since it was born. In many networks, such as in electronic neural networks, time delay can not be avoided. In fact, the stochastic perturbations can not be avoided either[1~13]. On the other hand, the system is unavoidable uncertainty, which is due to the existence of modeling errors, can also destroy the stability of the neural networks. So it is very important to discuss the stability and robustness of network against such error and fluctuation[4,7,8]. To overcome this difficulty, we will propose the stochastic interval delayed Hopfield neural networks (SIDHNN), and derive some robust stability criteria for the networks.

Consider an SIDHNN state equation as follows:

$$\begin{cases} du(t) = [-A_I u(t) + W_I^{\tau} f(u^{\tau}(t))]dt + \\ \qquad \sigma(t, u(t), u^{\tau}(t))dw(t),\ t \geqslant 0, \\ u(s) = \xi(s),\ -\bar{\tau} \leqslant s \leqslant 0. \end{cases} \tag{1}$$

Where $A_I := \{A = \mathrm{diag}\,(a_{ii})_{n\times n} \in \mathbb{R}^{n\times n}: \underline{A} \leqslant A \leqslant \bar{A}$, i.e. $\underline{a}_{ii} \leqslant a_{ii} \leqslant \bar{a}_{ii}, i = 1,2,\cdots,n\}$, $W_I^{\tau} := \{W^{\tau} = (w_{ij}^{\tau})_{n\times n} \in \mathbb{R}^{n\times n}: \underline{W}^{\tau} \leqslant W^{\tau} \leqslant \overline{W}^{\tau}$, i.e. $\underline{w}_{ij}^{\tau} \leqslant w_{ij}^{\tau} \leqslant \bar{w}_{ij}^{\tau}, i,j = 1,2,\cdots,n\}$ denote weight matrix, $w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^{\mathrm{T}} (m \leqslant n)$ is an m-dimensional Brownian motion defined on a complete probability space (Ω, F, P) with a natural filtration $\{F_t\}_{t\geqslant 0}$ (i.e. $F_t = \sigma\{w(s): 0 \leqslant s \leqslant t\}$), and $\xi \in C([-2\bar{\tau}, 0]; \mathbb{R}^n)$, $\xi \in L^2_{F_0}([-2\bar{\tau}, 0]; \mathbb{R}^n)$ is F_0-me-asurable, $\int_{-\bar{\tau}}^{0} \mathrm{E}\,|\xi(s)|^2 ds < \infty$. $u(t) = (u_1(t), u_2(t), \cdots, u_n(t))$, $u^{\tau}(t) = (u_1(t-\tau_1(t)), u_2(t-\tau_2(t)), \cdots, u_n(t-\tau_n(t)))$ (where $-\bar{\tau} \leqslant -\tau_i(t) \leqslant 0, i = 1,2,\cdots,n$), $\sigma: t \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, that is $\sigma(t, u, u^{\tau'}) = (\sigma_{ij}(t, u, u^{\tau'}))_{n\times m}$ (where $\tau_i'(t)$ be similar to $\tau_i(t)$). Define $f(u(t)) = (f_1(u_1(t)), f_2(u_2(t)), f_n(u_n(t)))$, similar to $f(u^{\tau}(t))$. Assume, in the whole paper, tho-

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se $\sigma(t,\cdot,\cdot)$ and $f(\cdot)$ (nonlinear sigmoidal activation function) satisfy

H1) There exist constants $L_j, 0 < L_j < \infty, j = 1,2,\cdots,n$ such that the incremental ratio for $f_j:\mathbb{R}\to\mathbb{R}$ satisfies

$$0 \leqslant f_j(x_j) - f_j(y_j) \leqslant L_j(x_j - y_j), \forall x_j, y_j \in \mathbb{R}.$$

H2) There exist constants $b_{ik}, c_{ik}, i = 1,2,\cdots,n$, $k = 1,2,\cdots,m$ satisfy

$$\sigma_{ik}^2(t, x_i(t), x_i(t-\tau_i'(t))) \leqslant b_{ik}x_i^2(t) + c_{ik}x_i^2(t-\tau_i'(t)).$$

It is known[6] that Eq.(1) has a unique global solution on $t \geqslant 0$, which is denoted by $u(t;\xi)$. Moreover, assume also that $\sigma(t,0,0) \equiv 0$ for the the sake of stability. Thus system (1) admits an equation solution $u(t;0) \equiv 0$.

Let $w_{ij}^{\tau^*} = \max\limits_{1\leqslant j\leqslant n}\{|\underline{w}_{ij}^{\tau}|, |\bar{w}_{ij}^{\tau}|\}, i = 1,2,\cdots,n$.

Similarly we can define $w_{ji}^{\tau^*}$.

2 Delay-dependent sufficient criteria

Theorem 1 Let $W = (\max\{\underline{w}_{ij}^{\tau}, \bar{w}_{ij}^{\tau}\})_{n\times n}$. Assume that

$$\delta = \frac{n\max\limits_{1\leqslant i\leqslant n}\{\underline{a}_i^{-2}\}\|W\|^2}{2(\min\limits_{1\leqslant i\leqslant n}\{\underline{a}_iL_i^{-1}\} - \|W\|)} > 0, \qquad (2)$$

$$\varepsilon = \max_{1\leqslant i\leqslant n}\left\{\left(\bar{a}_i\bar{\tau} + \sqrt{n\bar{\tau}^2L_i^2\sum_{j=1}^n w_{ji}^{\tau^{*2}}} + \sqrt{m\bar{\tau}\sum_{k=1}^m(b_{ik}+c_{ik})}\right)\right\},$$

$$\alpha = \min_{1\leqslant i\leqslant n}\left\{1 - \left(\frac{1}{\underline{a}_i} + \delta L_i\right)^2 - \sum_{k=1}^m\left(\frac{1}{\underline{a}_i} + \delta L_i\right)b_{ik}\right\},$$

$$\beta = \max_{1\leqslant i\leqslant n}\left\{3\varepsilon nL_i^2\sum_{j=1}^n w_{ji}^{\tau^{*2}} + \sum_{k=1}^m\left(\frac{1}{\underline{a}_i} + \delta L_i\right)c_{ik}\right\}.$$

If $\alpha > \beta$, then for $\xi \in C([-2\bar{\tau},0];\mathbb{R}^n)$ the zero-solution of system (1) is global 2nd moment exponential stable and almost surely exponential stable.

Proof Define the Lyapunov function as follows

$$V(t,u) = \sum_{i=1}^n(\lambda_iu_i^2 + 2\delta\int_0^{u_i}f_i(s)\mathrm{d}s),$$

$$LV(t,u,u^\tau) =$$

$$2\sum_{i=1}^n(\lambda_iu_i(t) + \delta f_i(u_i(t)))\times$$

$$(-a_iu_i(t) + \sum_{j=1}^n w_{ij}^\tau f_j(u_j(t-\tau_j(t)))) +$$

$$\sum_{i=1}^n\sum_{k=1}^m(\lambda_i + \delta f_i')\sigma_{ik}^2(t,u_i(t),u_i(t-\tau_i'(t))) \leqslant$$

$$-\sum_{i=1}^n\left(1 - \left(\frac{1}{\underline{a}_i} + \delta L_i\right)^2 - \left(\frac{1}{\underline{a}_i} + \delta L_i\right)\sum_{k=1}^m b_{ik}\right)\times$$

$$\mathrm{E}\,u_i^2(t) + \sum_{i=1}^n\left(3\varepsilon nL_i^2\sum_{j=1}^n w_{ji}^{\tau^{*2}} + \left(\frac{1}{\underline{a}_i} + \delta L_i\right)\sum_{k=1}^m c_{ik}\right)\times$$

$$\sup_{-2\bar{\tau}\leqslant\vartheta\leqslant 0}\mathrm{E}\,u_i^2(t+\vartheta) \leqslant$$

$$-\alpha\sum_{i=1}^n\mathrm{E}\,u_i^2(t) + \beta\sum_{i=1}^n\sup_{-2\bar{\tau}\leqslant\vartheta\leqslant 0}\mathrm{E}\,u_i^2(t+\vartheta).$$

By the Razuminkhin-type theory in [14] and condition $\alpha > \beta$, we can conclude that for every $\xi \in C([-2\bar{\tau},0];\mathbb{R}^n)$, the zero-solution of system (1) is globally 2nd moment exponential stable and almost surely exponential stable.

Corollary 1 Let $w^{\tau^*} = \max\limits_{1\leqslant j\leqslant n}\{w_{ji}^{\tau^*}\}$, assume that

$$\delta = \frac{n\max\limits_{1\leqslant i\leqslant n}\{\underline{a}_i^{-2}\}\|W\|^2}{2(\min\limits_{1\leqslant i\leqslant n}\{\underline{a}_iL_i^{-1}\} - \|W\|)} > 0,$$

$$\varepsilon = \max_{1\leqslant i\leqslant n}\left\{\left(\bar{a}_i\bar{\tau} + n\bar{\tau}w^{\tau^*}L_i + \sqrt{m\bar{\tau}\sum_{k=1}^m(b_{ik}+c_{ik})}\right)^2\right\},$$

$$\alpha = \min_{1\leqslant i\leqslant n}\left\{1 - \left(\frac{1}{\underline{a}_i} + \delta L_i\right)^2 - \left(\frac{1}{\underline{a}_i} + \delta L_i\right)\sum_{k=1}^m b_{ik}\right\},$$

$$\beta = \max_{1\leqslant i\leqslant n}\left\{3\varepsilon n^2L_i^2w^{\tau^{*2}} + \left(\frac{1}{\underline{a}_i} + \delta L_i\right)\sum_{k=1}^m c_{ik}\right\}.$$

If $\alpha < \beta$, for $\xi \in C([-2\bar{\tau},0];\mathbb{R}^n)$ the zero-solution of system (1) is globally 2nd moment exponential stable and almost surely exponential stable

Example 1 Let

$$r_1 = r_2 = 0.5,\ \bar{\tau} = 0.01,\ L_1 = L_2 = 0.5,$$

$$\underline{A} = \begin{pmatrix}1.3 & 0\\ 0 & 1.3\end{pmatrix},\ \bar{A} = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix},$$

$$\underline{W}^\tau = \begin{pmatrix}0.1 & -0.1\\ 0.05 & -0.15\end{pmatrix},\ \bar{W}^\tau = \begin{pmatrix}0.33 & 0.14\\ 0.1 & 0.15\end{pmatrix},$$

$$B = \begin{pmatrix}0.01 & 0.02\\ 0.05 & 0.001\end{pmatrix},\ C = \begin{pmatrix}0.03 & 0.04\\ 0.05 & 0.2\end{pmatrix}.$$

By Corollary 1, it is easy to find that $\alpha = 0.3342, \beta = 0.2471$. Obviously, $\alpha > \beta$, so this system is globally exponentially robust stable when $\xi \in C([-0.02,0],\mathbb{R}^2)$.

Remark 1 As far as the existing papers are concerned, [15 ~ 17] have discussed constant time delay, and those systems are special case of our systems. [17] has studied stochastic Hopfield neural networks with two variable delays, and the delays are single respectively in the certain part and uncertain part. Though this paper has discussed the same problem with multi-delays, when $\sigma(t,u,u^{\tau}) \equiv 0$, we have generalized those certainty systems such as [18,19].

3 Delay-independent sufficient criteria

Theorem 2 Assume those $\lambda_1,\lambda_2,\cdots,\lambda_n$ are non-negative numbers and, $r \in [0,1]$, let

$$\alpha > \beta + \gamma, \tag{3}$$

where $\alpha = \min_{1\leqslant i\leqslant n} \{2\underline{a}_i - \sum_{j=1}^{n}(w_{ij}^{\tau *}L_j^{2r} + b_{ij})\}(b_{ij} = 0, j = k+1,k+2,\cdots,n)$, $\gamma = \max_{1\leqslant i\leqslant n}\{\sum_{k=1}^{m} c_{ik}\}$, $\beta = \max_{1\leqslant i\leqslant n}\left\{\frac{L_i^{2(1-r)}}{\lambda_i}\sum_{j=1}^{m}\lambda_j w_{ji}^{\tau *}\right\}$. Then for every $\xi \in C([-\bar{\tau},0];\mathbb{R}^n)$, zero-solution of system (1) is globally 2nd moment exponential stable and almost surely exponential stable.

The proof is omitted.

Corollary 2 Define $w^* = \max_{1\leqslant i\leqslant n}\{w_{ij}^{\tau *}\}$ and

$$\alpha = \min_{1\leqslant i\leqslant n}\{2\underline{a}_i - w^*\sum_{j=1}^{n}(L_j^{2r} + b_{ij})\},$$

$$\beta = \max_{1\leqslant i\leqslant n}\left\{\frac{L_i^{2(1-r)}}{\lambda_i}w^*\sum_{j=1}^{m}\lambda_j\right\},$$

$$\gamma = \max_{1\leqslant i\leqslant n}\{\sum_{k=1}^{m}c_{ik}\}.$$

If (3) is satisfied, Theorem 1 is also true.

References:

[1] BALDI P, ATIYA A F. How delays affect neural dynamics and learning[J]. *IEEE Trans on Neural Networks*, 1994,38(5):612 - 621.

[2] LIAO X, MAO X. Exponential stability and instability of stochastic neural networks [J]. *Stochastic Analysis and Application*, 1996, 14 (2):165 - 168.

[3] LIAO X, MAO X. Stability of stochastic neural networks [J]. *Neural, Parallel Science Computing*, 1996,4(2):205 - 224.

[4] LIAO X. *Mathematic Theory and Application of Stability* [M]. Wuhan: Huazhong Normal University Press, 2001.

[5] CAO Jinde. Global exponential stability of Hopfield neural networks [J]. *Int J of Systems Science*, 2001,32(2):233 - 236.

[6] MAO X. *Stochastic Differential Equations and Application* [M]. London: Horwood Publishing, 1997

[7] LIAO X, YU J. Robust stability for interval Hopfield neural networks with time delays [J]. *IEEE Trans on Neural Networks*, 1998, 9 (1):1042 - 1046.

[8] LIAO X. Novel robust stability criteria for interval-delayed Hopfield neural networks [J]. *IEEE Trans on Circuits and Systems*, 2001,48 (11):1355 - 1359.

[9] BLYTHE S. Stability of stochastic delay neural networks [J]. *J of the Franklin Institute*, 2001,2(1):1 - 15.

[10] MAO X, SELFRIDGE C. Stability of stochastic systems with time delays [J]. *Systems & Control Letters*, 2001,42(2):279 - 290.

[11] SHEN Y. The stability of nonlinear stochastic delay systems with its applications [J]. *Control Theory & Applications*, 2000,17(1):19 - 22.

[12] SHEN Y. Global convergence of discrete-time neural networks [J]. *Systems Engineering and Electronics*, 1999,21(11):50 - 51.

[13] SHEN Y, LIAO X. Exponential stability of stochastic differential delay equations [J]. *Actc Mathematica Scientia*, 1999,19(2): 211 - 218.

[14] MAO X. Razumikhin-type theorems on exponential stability of stochastic functional differential equations [J]. *Stochastic Processes and their Applications*, 1996,65(2):233 - 250.

[15] SHEN Y. Robust stability of a family of nonlinear stochastic delay systems [J]. *Acta Automatica Sinica*, 1999,25(4):537 - 542.

[16] YE H. Robust stability of nonlinear time-delay systems with applications to neural networks [J]. *IEEE Trans on Circuits and Systems-I*, 1996,43(7):532 - 543.

[17] SHEN Y. Exponential stability of Hopfield-type stochastic neural networks with variable delays [J]. *Acta Mathematica Scientia*, 2000,20(3):400 - 404.

[18] CEHN J. On sufficient conditions for stabilty in dependent of delay [J]. *IEEE Trans on Autimatic Control*, 1995,40(8):1675 - 1680.

[19] CAO Jinde. Exponential stability and periodic solutions of delayed cellular neural neural networks [J]. *Science in China (Series E)*, 2000,43(3):328 - 336.

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