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噪声相关条件下Unscented卡尔曼滤波器设计

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摘要: 针对传统Unscented卡尔曼滤波器(UKF)在噪声相关条件下非线性滤波失效的问题, 研究了一类带相关噪声的非线性离散系统UKF设计方法. 文中基于最小均方误差估计准则, 给出了系统噪声和量测噪声相关时UKF滤波递推公式, 并采用Unscented变换(UT)来计算系统状态的后验均值和协方差. 所设计的噪声相关条件下UKF有效克服了传统UKF必须假设系统噪声和量测噪声为互不相关高斯白噪声的局限性, 拓展了UKF的应用范围. 仿真实例验证了其可行性和有效性.

关键词: 非线性离散系统; 噪声相关条件下UKF; 最小均方误差估计; Unscented变换; 可行性; 有效性

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Design of unscented Kalman filter with correlative noises

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Abstract: An Unscented-Kalman Filter(UKF) for a class of nonlinear discrete-time systems with correlated noises is designed to deal with the problem of nonlinear filtering failure in conventional UKF when system noise is correlated with measurement noise. Recursive filtering equations of UKF with correlated noises are given based on the minimum mean-square-error estimation; and unscented transformation(UT) is applied to the calculation the posterior means and covariances of the nonlinear system states. The proposed UKF breaks through the limitations on the conventional UKF that the system noise and measurement noise must be uncorrelated Gauss white noises, thus extending the applications of the conventional UKF. A simulation example shows its feasibility and effectiveness.

Key words: nonlinear discrete-time systems; UKF with correlated noises; minimum mean-square-error estimation; unscented transformation; feasibility; effectiveness

1　引言(Introduction)

Unscented卡尔曼滤波器(unscented Kalman filter, UKF)是一种新兴的非线性滤波方法[1,2], 其核心思想是基于最小均方误差估计, 采用Unscented变换(unscented transformation, UT)对非线性高斯系统状态的后验均值及协方差进行近似. 与EKF相比, UKF无需计算非线性状态函数和量测函数的雅可比矩阵, 且不论系统非线性程度如何, UT变换理论上至少可以3阶泰勒精度逼近任何非线性高斯系统状态的后验均值和协方差, 具有实现简单, 滤波精度高, 收敛性好等优点, 因此UKF已得到了国内外学者的广泛关注[3].

Julier等[4]提出了UT变换中Sigma点采样策略的一般性选择依据, 且对UT变换的精度给出了详细证明; Merver等[5]设计了一种平方根UKF算法, 有效克服了因协方差失去正定而引起的滤波器计算发散, 提高了UKF的数值稳定性; Seongr等[6]将交互式多模型算法(IMM)引入到UKF设计中, 提出一种自适应融合滤波算法, 以此来解决UKF在系统模型不确定时鲁棒性差的问题. 需要特别强调的是, 已有关于UKF的文献, 都是基于系统噪声和量测噪声为互不相关高斯白噪声的假设, 来设计非线性系统UKF滤波器的. 然而, 受内外部环境变化的影响, 噪声互不相关的条件并不能完全得到满足, 而传统UKF在噪声相关时非线性滤波将会失效; 在现实世界中, 势必存在系统噪声和量测噪声相关的情况, 因此讨论噪声相关条件下UKF的设计问题极具理论价值和现实意义. 但是, 到目前为止, 针对此方面问题的解决方案, 国内外尚未见有相关文献报导.

为此, 本文基于最小均方误差估计准则, 详细推

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导了噪声相关条件下UKF递推公式, 分别给出了系统状态的最优一步预报估计和最优滤波估计, 并应用UT变换来计算非线性系统状态的后验均值和协方差. 仿真实例证明了所设计UKF滤波器的正确性和有效性.

2 问题描述(Problem formulation)

考虑如下所示的非线性离散系统:

$$\begin{cases} \boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{u}_k) + \boldsymbol{\Gamma}_k \boldsymbol{w}_k, \\ \boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k. \end{cases} \tag{1}$$

其中: $\boldsymbol{x}_k$与$\boldsymbol{z}_k$分别为系统n维状态向量和m维量测向量; $\boldsymbol{u}_k$是l维控制输入向量; $\boldsymbol{w}_k$和$\boldsymbol{v}_k$分别为p维系统噪声和m维量测噪声. $\boldsymbol{f}_k(\cdot)$和$\boldsymbol{h}_k(\cdot)$分别为系统非线性状态函数和测量函数; $\boldsymbol{\Gamma}_k$是$n \times p$维噪声输入矩阵.

假设 1 $\boldsymbol{w}_k$和$\boldsymbol{v}_k$是相关的高斯白噪声, 且具有如下统计特性:

$$\begin{cases} \mathrm{E}(\boldsymbol{w}_k) = \boldsymbol{q}_k, \ \mathrm{Cov}(\boldsymbol{w}_k, \boldsymbol{w}_j^{\mathrm{T}}) = \boldsymbol{Q}_k \delta_{kj}, \\ \mathrm{E}(\boldsymbol{v}_k) = \boldsymbol{r}_k, \ \mathrm{Cov}(\boldsymbol{v}_k, \boldsymbol{v}_j^{\mathrm{T}}) = \boldsymbol{R}_k \delta_{kj}, \\ \mathrm{Cov}(\boldsymbol{w}_k, \boldsymbol{v}_j^{\mathrm{T}}) = \boldsymbol{S}_k \delta_{kj}. \end{cases} \tag{2}$$

其中: $\boldsymbol{Q}_k$是非负定对称阵, $\boldsymbol{R}_k$是正定对称阵; δ_{kj}为kronecker$-\delta$函数.

假设 2 初始状态$\boldsymbol{x}_0$与$\boldsymbol{w}_k, \boldsymbol{v}_k$互不相关, 且服从高斯正态分布, 其均值和协方差阵为

$$\begin{cases} \hat{\boldsymbol{x}}_0 = \mathrm{E}(\boldsymbol{x}_0), \\ \boldsymbol{P}_0 = \mathrm{Cov}(\boldsymbol{x}_0) = \mathrm{E}[(\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0)(\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0)^{\mathrm{T}}]. \end{cases} \tag{3}$$

问题是基于最小均方误差估计准则, 利用量测值$\boldsymbol{Z}^{k+1} = \{\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots, \boldsymbol{z}_{k+1}\}$, 求噪声相关条件下的Unscented卡尔曼滤波器$\hat{\boldsymbol{x}}_{k+1}$.

3 噪声相关条件下Unscented卡尔曼滤波器(Unscented Kalman filter with correlative noises)

3.1 UT变换(Unscented transformation)

设n维随机向量$\boldsymbol{x} \sim N(\bar{\boldsymbol{x}}, \boldsymbol{P}_x)$, m维随机向量$\boldsymbol{z}$为$\boldsymbol{x}$的某一非线性函数:

$$\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}). \tag{4}$$

$\boldsymbol{x}$的统计特性为$(\bar{\boldsymbol{x}}, \boldsymbol{P}_{\boldsymbol{x}})$, 其通过非线性函数$\boldsymbol{f}(\cdot)$进行传播得到$\boldsymbol{z}$的统计特性$(\bar{\boldsymbol{z}}, \boldsymbol{P}_{\boldsymbol{z}})$. UT变换就是根据$(\bar{\boldsymbol{x}}, \boldsymbol{P}_{\boldsymbol{x}})$, 设计一系列的点$\boldsymbol{\xi}_i$, $i = 0, 1, \cdots, L$, 称其为Sigma点; 对设定的Sigma点计算其经过$\boldsymbol{f}(\cdot)$传播所得的结果γ_i, $i = 0, 1, \cdots, L$; 然后基于γ_i, $i = 0, 1, \cdots, L$计算$(\bar{\boldsymbol{z}}, \boldsymbol{P}_{\boldsymbol{z}})$.

在UT变换算法中, 最重要的是确定Sigma点的采样策略, 也就是确定使用Sigma点的个数、位置以及相应得权值. 目前已有的Sigma采样策略[3]有对称采样、单形采样、3阶矩偏度采样以及高斯分布4阶矩对称采样等. 其后, 为了保证输出变量$\boldsymbol{z}$协方差的半正定性, 提出了对上述基本采样策略进行比例修正的算法框架.

UT变换实现过程描述如下:

1) 根据所选择的采样策略, 利用 $\boldsymbol{x}$ 的统计特性$(\bar{\boldsymbol{x}}, \boldsymbol{P}_{\boldsymbol{x}})$来计算Sigma采样点及其权系数. 设对应于$\boldsymbol{\xi}_i$, $i = 0, 1, \cdots, L$的权值为W_i^{m}和W_i^{c}, 它们分别为求一阶和二阶统计特性时的权系数.

2) 计算Sigma点通过非线性函数$\boldsymbol{f}(\cdot)$的传播结果:

$$\gamma_i = \boldsymbol{f}(\boldsymbol{\xi}_i), \ i = 0, 1, \cdots, L. \tag{5}$$

从而得到随机向量$\boldsymbol{x}$经非线性函数$\boldsymbol{f}(\cdot)$传递之后的后验均值$\bar{\boldsymbol{z}}$、协方差$\boldsymbol{P}_{\boldsymbol{z}}$及互协方差$\boldsymbol{P}_{\boldsymbol{xz}}$:

$$\bar{\boldsymbol{z}} = \sum_{i=0}^{L} W_i^{\mathrm{m}} \gamma_i, \tag{6}$$

$$\boldsymbol{P}_{\boldsymbol{z}} = \sum_{i=0}^{L} W_i^{\mathrm{c}} (\gamma_i - \bar{\boldsymbol{z}})(\gamma_i - \bar{\boldsymbol{z}})^{\mathrm{T}}, \tag{7}$$

$$\boldsymbol{P}_{\boldsymbol{xz}} = \sum_{i=0}^{L} W_i^{\mathrm{c}} (\boldsymbol{\xi}_i - \bar{\boldsymbol{x}})(\gamma_i - \bar{\boldsymbol{z}})^{\mathrm{T}}. \tag{8}$$

上述UT变换中, 应用不同的采样策略区别仅在于第1) 步和后续计算的Sigma点个数L.

3.2 噪声相关条件下UKF(UKF with correlative noises)

从线性最小均方误差估计理论可知: 基于观测量$\boldsymbol{z}$的$\boldsymbol{x}$最小均方误差估计$\hat{\boldsymbol{x}}$唯一等于$\boldsymbol{x}$在$\boldsymbol{z}$上的正交投影, 记为$\hat{\boldsymbol{x}} = \hat{\mathrm{E}}(\boldsymbol{x}/\boldsymbol{z})$. 下面不加证明地给出关于正交投影的结论[7~9]:

引理 1 设$\boldsymbol{x}$和$\boldsymbol{z}$为具有二阶矩的随机向量, 则$\boldsymbol{x}$在$\boldsymbol{z}$上的正交投影$\hat{\boldsymbol{x}}$唯一地等于基于$\boldsymbol{z}$的线性最小均方误差估计, 即

$$\hat{\mathrm{E}}(\boldsymbol{x}/\boldsymbol{z}) = \mathrm{E}(\boldsymbol{x}) + \mathrm{Cov}(\boldsymbol{x}, \boldsymbol{z})[\mathrm{Var}(\boldsymbol{z})]^{-1}[\boldsymbol{z} - \mathrm{E}(\boldsymbol{z})]. \tag{9}$$

引理 2 设$\boldsymbol{x}, \boldsymbol{y}$和$\boldsymbol{z}$为具有二阶矩的随机向量, $\boldsymbol{A}$和$\boldsymbol{B}$为具有相应维数的非随机矩阵, 则有

$$\hat{\mathrm{E}}[(\boldsymbol{Ax} + \boldsymbol{By})/\boldsymbol{z}] = \boldsymbol{A}\hat{\mathrm{E}}(\boldsymbol{x}/\boldsymbol{z}) + \boldsymbol{B}\hat{\mathrm{E}}(\boldsymbol{y}/\boldsymbol{z}). \tag{10}$$

引理 3 设$\boldsymbol{x}, \boldsymbol{z}_a$和$\boldsymbol{z}_b$为具有二阶矩的随机向量, 且$\boldsymbol{z} = [\boldsymbol{z}_a \ \ \boldsymbol{z}_b]$, 则有

$$\begin{aligned} &\hat{\mathrm{E}}(\boldsymbol{x}/\boldsymbol{z}) = \hat{\mathrm{E}}(\boldsymbol{x}/\boldsymbol{z}_a) + \hat{\mathrm{E}}(\tilde{\boldsymbol{x}}/\tilde{\boldsymbol{z}}_b) = \\ &\hat{\mathrm{E}}(\boldsymbol{x}/\boldsymbol{z}_a) + \mathrm{E}(\tilde{\boldsymbol{x}}\tilde{\boldsymbol{z}}_b^{\mathrm{T}})[\mathrm{E}(\tilde{\boldsymbol{z}}_b\tilde{\boldsymbol{z}}_b^{\mathrm{T}})]^{-1}\tilde{\boldsymbol{z}}_b. \end{aligned} \tag{11}$$

上式中:

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\mathrm{E}}(\boldsymbol{x}/\boldsymbol{z}_a), \ \tilde{\boldsymbol{z}}_b = \boldsymbol{z}_b - \hat{\mathrm{E}}(\boldsymbol{z}_b/\boldsymbol{z}). \tag{12}$$

定理 1 基于最小均方误差估计准则和量测值

$\boldsymbol{Z}^k=\{z_1,z_2,\cdots,z_k\}$, 在假设1, 2下, 非线性离散系统(1)的最优一步状态预报递推公式为

$$\hat{\boldsymbol{x}}_{k+1/k}=\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}+\boldsymbol{\varGamma}_k\boldsymbol{q}_k+\boldsymbol{M}_k\varepsilon_k, \tag{13}$$

$$\varepsilon_k=z_k-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}-\boldsymbol{r}_k, \tag{14}$$

$$\boldsymbol{M}_k=[\mathrm{E}(\boldsymbol{\varLambda}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})+\boldsymbol{\varGamma}_k\boldsymbol{S}_k][\mathrm{E}(\boldsymbol{\varTheta}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})+\boldsymbol{R}_k]^{-1}, \tag{15}$$

$$\boldsymbol{P}_{k+1/k}=\mathrm{E}(\boldsymbol{\varLambda}_k\boldsymbol{\varLambda}_k^{\mathrm{T}})-\boldsymbol{M}_k[\mathrm{E}(\boldsymbol{\varLambda}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})+\boldsymbol{\varGamma}_k\boldsymbol{S}_k]^{\mathrm{T}}+\boldsymbol{\varGamma}_k\boldsymbol{Q}_k\boldsymbol{\varGamma}_k^{\mathrm{T}}. \tag{16}$$

其中:

$$\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}=\hat{\mathrm{E}}[\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)/\boldsymbol{Z}^{k-1}], \tag{17}$$

$$\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}=\hat{\mathrm{E}}[\boldsymbol{h}_k(\boldsymbol{x}_k)/\boldsymbol{Z}^{k-1}], \tag{18}$$

$$\boldsymbol{\varLambda}_k=\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)-\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}, \tag{19}$$

$$\boldsymbol{\varTheta}_k=\boldsymbol{h}_k(\boldsymbol{x}_k)-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}. \tag{20}$$

证 已知$\boldsymbol{v}_k$为高斯白噪声, 故$\boldsymbol{v}_k$与$k-1$时刻前量测值$\boldsymbol{Z}^{k-1}$互不相关, 于是有

$$\mathrm{Cov}(\boldsymbol{v}_k,\boldsymbol{Z}^{k-1})=0. \tag{21}$$

则根据引理1可知

$$\hat{\mathrm{E}}(\boldsymbol{v}_k/\boldsymbol{Z}^{k-1})=\mathrm{E}(\boldsymbol{v}_k)=\boldsymbol{r}_k. \tag{22}$$

联合式(22), 同时根据引理2可得k时刻输出预测为

$$\hat{z}_{k/k-1}=\hat{\mathrm{E}}(z_k/\boldsymbol{Z}^{k-1})=\hat{\mathrm{E}}[\boldsymbol{h}_k(\boldsymbol{x}_k)/\boldsymbol{Z}^{k-1}]+\hat{\mathrm{E}}(\boldsymbol{v}_k/\boldsymbol{Z}^{k-1})=\hat{\mathrm{E}}[\boldsymbol{h}_k(\boldsymbol{x}_k)/\boldsymbol{Z}^{k-1}]+\boldsymbol{r}_k. \tag{23}$$

其中$\hat{\mathrm{E}}[\boldsymbol{h}_k(\boldsymbol{x}_k)/\boldsymbol{Z}^{k-1}]$表示的含义是: k时刻状态一步预测值$\hat{\boldsymbol{x}}_{k/k-1}$经非线性量测函数$\boldsymbol{h}_k(\cdot)$传递之后的后验均值. 对于线性KF, $\hat{\mathrm{E}}[\boldsymbol{h}_k(\boldsymbol{x}_k)/\boldsymbol{Z}^{k-1}]$可通过线性量测函数传递精确已知; 而对于非线性UKF, $\hat{\mathrm{E}}[\boldsymbol{h}_k(\boldsymbol{x}_k)/\boldsymbol{Z}^{k-1}]$只能通过UT变换以三阶泰勒精度近似已知. 定义

$$\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}=\hat{\mathrm{E}}[\boldsymbol{h}_k(\boldsymbol{x}_k)/\boldsymbol{Z}^{k-1}], \tag{24}$$

则k时刻输出残差可表示为

$$\varepsilon_k=z_k-\hat{z}_{k/k-1}=z_k-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}-\boldsymbol{r}_k=\boldsymbol{h}_k(\boldsymbol{x}_k)-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}+\boldsymbol{v}_k-\boldsymbol{r}_k. \tag{25}$$

显然$\boldsymbol{Z}^k=\{\boldsymbol{Z}^{k-1},z_k\}$, 根据引理3可得$k+1$时刻状态最优一步预测值$\hat{\boldsymbol{x}}_{k+1/k}$:

$$\hat{\boldsymbol{x}}_{k+1/k}=\hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^k)=\hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k-1})+\hat{\mathrm{E}}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}/\tilde{z}_k)=\hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k-1})+\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{z}_k^{\mathrm{T}})[\mathrm{E}(\tilde{z}_k\tilde{z}_k^{\mathrm{T}})]^{-1}\tilde{z}_k. \tag{26}$$

其中:

$$\tilde{z}_k=z_k-\hat{E}[z_k/\boldsymbol{Z}^{k-1}]=z_k-\hat{z}_{k/k-1}=\varepsilon_k, \tag{27}$$

$$\tilde{\tilde{\boldsymbol{x}}}_{k+1}=\boldsymbol{x}_{k+1}-\hat{\mathrm{E}}[\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k-1}]. \tag{28}$$

已知$\boldsymbol{w}_k$为高斯白噪声, 故$\boldsymbol{w}_k$与$k-1$时刻前量测值$\boldsymbol{Z}^{k-1}$互不相关, 于是有

$$\mathrm{Cov}(\boldsymbol{w}_k,\boldsymbol{Z}^{k-1})=0, \tag{29}$$

则根据引理1可知

$$\hat{\mathrm{E}}(\boldsymbol{w}_k/\boldsymbol{Z}^{k-1})=\mathrm{E}(\boldsymbol{w}_k)=\boldsymbol{q}_k. \tag{30}$$

联合式(30), 根据引理2可知

$$\hat{\mathrm{E}}[\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k-1}]=\hat{\mathrm{E}}[\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)/\boldsymbol{Z}^{k-1}]+\boldsymbol{\varGamma}_k\hat{\mathrm{E}}(\boldsymbol{w}_k/\boldsymbol{Z}^{k-1})=\hat{\mathrm{E}}[\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)/\boldsymbol{Z}^{k-1}]+\boldsymbol{\varGamma}_k\boldsymbol{q}_k. \tag{31}$$

其中$\hat{\mathrm{E}}[\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)/\boldsymbol{Z}^{k-1}]$表示的含义是: k时刻状态一步预测值$\hat{\boldsymbol{x}}_{k/k-1}$经非线性状态函数$\boldsymbol{f}_k(\cdot)$传递之后的后验均值. 对于线性KF, $\hat{\mathrm{E}}[\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)/\boldsymbol{Z}^{k-1}]$可通过线性状态函数传递精确已知; 而对于非线性UKF, $\hat{\mathrm{E}}[\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)/\boldsymbol{Z}^{k-1}]$只能通过UT变换以三阶泰勒精度近似已知. 定义

$$\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}=\hat{\mathrm{E}}[\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)/\boldsymbol{Z}^{k-1}]. \tag{32}$$

将式(31)(32)代入到式(28)可得

$$\tilde{\tilde{\boldsymbol{x}}}_{k+1}=\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)-\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}+\boldsymbol{\varGamma}_k(\boldsymbol{w}_k-\boldsymbol{q}_k). \tag{33}$$

相应地, 定义

$$\boldsymbol{\varLambda}_k=\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)-\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}, \tag{34}$$

$$\boldsymbol{\varTheta}_k=\boldsymbol{h}_k(\boldsymbol{x}_k)-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}, \tag{35}$$

则式(25)(33)变成

$$\tilde{\tilde{\boldsymbol{x}}}_{k+1}=\boldsymbol{\varLambda}_k+\boldsymbol{\varGamma}_k(\boldsymbol{w}_k-\boldsymbol{q}_k), \tag{36}$$

$$\tilde{z}_k=\varepsilon_k=\boldsymbol{\varTheta}_k+\boldsymbol{v}_k-\boldsymbol{r}_k. \tag{37}$$

已知$\boldsymbol{w}_k,\boldsymbol{v}_k$为高斯白噪声, 且考虑到$\boldsymbol{w}_k,\boldsymbol{v}_k$都与$\boldsymbol{Z}^{k-1}$互不相关, 故$\boldsymbol{w}_k,\boldsymbol{v}_k$都与$\boldsymbol{\varLambda}_k,\boldsymbol{\varTheta}_k$互不相关, 于是可求得式(26)中的$\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{z}_k^{\mathrm{T}})$和$\mathrm{E}(\tilde{z}_k\tilde{z}_k^{\mathrm{T}})$:

$$\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{z}_k^{\mathrm{T}})=\mathrm{E}\{[\boldsymbol{\varLambda}_k+\boldsymbol{\varGamma}_k(\boldsymbol{w}_k-\boldsymbol{q}_k)][\boldsymbol{\varTheta}_k+\boldsymbol{v}_k-\boldsymbol{r}_k]^{\mathrm{T}}\}=\mathrm{E}(\boldsymbol{\varLambda}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})+\boldsymbol{\varGamma}_k\mathrm{E}[(\boldsymbol{w}_k-\boldsymbol{q}_k)(\boldsymbol{v}_k-\boldsymbol{r}_k)^{\mathrm{T}}]=\mathrm{E}(\boldsymbol{\varLambda}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})+\boldsymbol{\varGamma}_k\boldsymbol{S}_k, \tag{38}$$

$$\mathrm{E}(\tilde{z}_k\tilde{z}_k^{\mathrm{T}})=\mathrm{E}[(\boldsymbol{\varTheta}_k+\boldsymbol{v}_k-\boldsymbol{r}_k)(\boldsymbol{\varTheta}_k+\boldsymbol{v}_k-\boldsymbol{r}_k)^{\mathrm{T}}]=\mathrm{E}(\boldsymbol{\varTheta}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})+\mathrm{E}[(\boldsymbol{v}_k-\boldsymbol{r}_k)(\boldsymbol{v}_k-\boldsymbol{r}_k)^{\mathrm{T}}]=\mathrm{E}(\boldsymbol{\varTheta}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})+\boldsymbol{R}_k. \tag{39}$$

其中: $\mathrm{E}(\boldsymbol{\varTheta}_k\boldsymbol{\varTheta}_k^{\mathrm{T}})$表示$k$时刻状态一步预测值$\hat{\boldsymbol{x}}_{k/k-1}$经非线性量测函数$\boldsymbol{h}_k(\cdot)$传递之后的后验协方差;

$\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Theta}_k^{\mathrm{T}})$表示$k$时刻状态一步预测值$\hat{\boldsymbol{x}}_{k/k-1}$经非线性状态函数$\boldsymbol{f}_k(\cdot)$及量测函数$\boldsymbol{h}_k(\cdot)$传递之后的后验互协方差. 对于线性KF, 它们可通过线性状态函数传递精确已知; 而对于非线性UKF, 它们只能通过UT变换以三阶泰勒精度近似已知.

定义预报增益矩阵$\boldsymbol{M}_k$为

$$\boldsymbol{M}_k = \mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\boldsymbol{z}}_k^{\mathrm{T}})[\mathrm{E}(\tilde{\boldsymbol{z}}_k\tilde{\boldsymbol{z}}_k^{\mathrm{T}})]^{-1}. \tag{40}$$

显然

$$\boldsymbol{M}_k=[\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Theta}_k^{\mathrm{T}})+\boldsymbol{\Gamma}_k\boldsymbol{S}_k][\mathrm{E}(\boldsymbol{\Theta}_k\boldsymbol{\Theta}_k^{\mathrm{T}})+\boldsymbol{R}_k]^{-1}. \tag{41}$$

将式(27)(31)及(41)代入式(26)可得

$$\begin{aligned}\hat{\boldsymbol{x}}_{k+1/k} &= \hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k-1}) + \boldsymbol{M}_k\tilde{\boldsymbol{z}}_k = \\ &\hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k-1}) + \boldsymbol{M}_k\boldsymbol{\varepsilon}_k = \\ &\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}} + \boldsymbol{\Gamma}_k\boldsymbol{q}_k + \boldsymbol{M}_k\boldsymbol{\varepsilon}_k.\end{aligned} \tag{42}$$

下面来计算一步状态预测误差协方差$\boldsymbol{P}_{k+1/k}$. 联合式(28)及式(42)可知

$$\begin{aligned}\tilde{\boldsymbol{x}}_{k+1/k} &= \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1/k} = \\ &\boldsymbol{x}_{k+1} - \hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k-1}) - \boldsymbol{M}_k\tilde{\boldsymbol{z}}_k = \\ &\tilde{\tilde{\boldsymbol{x}}}_{k+1} - \boldsymbol{M}_k\tilde{\boldsymbol{z}}_k.\end{aligned} \tag{43}$$

相应地

$$\begin{aligned}\boldsymbol{P}_{k+1/k} &= \mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{\boldsymbol{x}}_{k+1/k}^{\mathrm{T}}) = \\ &\mathrm{E}[(\tilde{\tilde{\boldsymbol{x}}}_{k+1} - \boldsymbol{M}_k\tilde{\boldsymbol{z}}_k)(\tilde{\tilde{\boldsymbol{x}}}_{k+1} - \boldsymbol{M}_k\tilde{\boldsymbol{z}}_k)^{\mathrm{T}}] = \\ &\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\tilde{\boldsymbol{x}}}_{k+1}^{\mathrm{T}}) - \mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\boldsymbol{z}}_k^{\mathrm{T}})\boldsymbol{M}_k^{\mathrm{T}} - \\ &\boldsymbol{M}_k\mathrm{E}(\tilde{\boldsymbol{z}}_k\tilde{\tilde{\boldsymbol{x}}}_{k+1}^{\mathrm{T}}) + \boldsymbol{M}_k\mathrm{E}(\tilde{\boldsymbol{z}}_k\tilde{\boldsymbol{z}}_k^{\mathrm{T}})\boldsymbol{M}_k^{\mathrm{T}}.\end{aligned} \tag{44}$$

根据$\boldsymbol{M}_k$的表达式(40), 注意到

$$\begin{aligned}&\boldsymbol{M}_k\mathrm{E}(\tilde{\boldsymbol{z}}_k\tilde{\boldsymbol{z}}_k^{\mathrm{T}})\boldsymbol{M}_k^{\mathrm{T}} = \\ &\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\boldsymbol{z}}_k^{\mathrm{T}})[\mathrm{E}(\tilde{\boldsymbol{z}}_k\tilde{\boldsymbol{z}}_k^{\mathrm{T}})]^{-1}\mathrm{E}(\tilde{\boldsymbol{z}}_k\tilde{\boldsymbol{z}}_k^{\mathrm{T}})\boldsymbol{M}_k^{\mathrm{T}} = \\ &\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\boldsymbol{z}}_k^{\mathrm{T}})\boldsymbol{M}_k^{\mathrm{T}},\end{aligned} \tag{45}$$

则式(44)可以简化为

$$\begin{aligned}\boldsymbol{P}_{k+1/k} &= \mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\tilde{\boldsymbol{x}}}_{k+1}^{\mathrm{T}}) - \boldsymbol{M}_k\mathrm{E}(\tilde{\boldsymbol{z}}_k\tilde{\tilde{\boldsymbol{x}}}_{k+1}^{\mathrm{T}}) = \\ &\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\tilde{\boldsymbol{x}}}_{k+1}^{\mathrm{T}}) - \boldsymbol{M}_k\mathrm{E}[(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\boldsymbol{z}}_k^{\mathrm{T}})]^{\mathrm{T}}.\end{aligned} \tag{46}$$

由$\boldsymbol{w}_k$与$\boldsymbol{\Lambda}_k$的互不相关性, 根据式(36)可计算出$\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\tilde{\boldsymbol{x}}}_{k+1}^{\mathrm{T}})$:

$$\begin{aligned}&\mathrm{E}(\tilde{\tilde{\boldsymbol{x}}}_{k+1}\tilde{\tilde{\boldsymbol{x}}}_{k+1}^{\mathrm{T}}) = \\ &\mathrm{E}\{[\boldsymbol{\Lambda}_k + \boldsymbol{\Gamma}_k(\boldsymbol{w}_k - \boldsymbol{q}_k)][\boldsymbol{\Lambda}_k + \boldsymbol{\Gamma}_k(\boldsymbol{w}_k - \boldsymbol{q}_k)]^{\mathrm{T}}\} = \\ &\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Lambda}_k^{\mathrm{T}}) + \boldsymbol{\Gamma}_k\mathrm{E}[(\boldsymbol{w}_k - \boldsymbol{q}_k)(\boldsymbol{w}_k - \boldsymbol{q}_k)^{\mathrm{T}}]\boldsymbol{\Gamma}_k^{\mathrm{T}} = \\ &\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Lambda}_k^{\mathrm{T}}) + \boldsymbol{\Gamma}_k\boldsymbol{Q}_k\boldsymbol{\Gamma}_k^{\mathrm{T}}.\end{aligned} \tag{47}$$

其中$\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Lambda}_k^{\mathrm{T}})$表示$k$时刻状态一步预测值$\hat{\boldsymbol{x}}_{k/k-1}$经非线性状态函数$\boldsymbol{f}_k(\cdot)$传递之后的后验协方差. 对于线性KF, 它可通过线性状态函数传递精确已知; 而对于非线性UKF, 它只能通过UT变换以三阶泰勒精度近似已知.

将式(38)(47)代入到式(46), 可知

$$\begin{aligned}&\boldsymbol{P}_{k+1/k} = \\ &\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Lambda}_k^{\mathrm{T}})-\boldsymbol{M}_k[\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Theta}_k^{T})+\boldsymbol{\Gamma}_k\boldsymbol{S}_k]^{\mathrm{T}}+\boldsymbol{\Gamma}_k\boldsymbol{Q}_k\boldsymbol{\Gamma}_k^{\mathrm{T}}.\end{aligned} \tag{48}$$

证毕.

定理 2 基于最小均方误差估计准则和量测值$\boldsymbol{Z}^{k+1} = \{\boldsymbol{z}_1,\boldsymbol{z}_2,\cdots,\boldsymbol{z}_{k+1}\}$, 在假设1, 2下, 非线性离散系统(1)的最优状态滤波递推公式为

$$\hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{x}}_{k+1/k} + \boldsymbol{K}_{k+1}\boldsymbol{\varepsilon}_{k+1}, \tag{49}$$

$$\boldsymbol{\varepsilon}_{k+1}=\boldsymbol{z}_{k+1}-\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}}-\boldsymbol{r}_{k+1}, \tag{50}$$

$$\begin{aligned}&\boldsymbol{K}_{k+1} = \\ &\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})[\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})+\boldsymbol{R}_{k+1}]^{-1},\end{aligned} \tag{51}$$

$$\boldsymbol{P}_{k+1}=\boldsymbol{P}_{k+1/k}-\boldsymbol{K}_{k+1}[\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})+\boldsymbol{R}_{k+1}]\boldsymbol{K}_{k+1}^{\mathrm{T}}. \tag{52}$$

其中:

$$\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}}=\hat{\mathrm{E}}[\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})/\boldsymbol{Z}^{k}], \tag{53}$$

$$\boldsymbol{\Theta}_{k+1}=\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})-\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}}. \tag{54}$$

证 显然$\boldsymbol{Z}^{k+1} = \{\boldsymbol{Z}^k,\boldsymbol{z}_{k+1}\}$, 根据引理3可得$k+1$时刻状态最优滤波估计值$\hat{\boldsymbol{x}}_{k+1}$:

$$\begin{aligned}&\hat{\boldsymbol{x}}_{k+1} = \hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k+1}) = \\ &\hat{\mathrm{E}}(\boldsymbol{x}_{k+1}/\boldsymbol{Z}^{k}) + \hat{E}(\tilde{\boldsymbol{x}}_{k+1/k}/\tilde{\boldsymbol{z}}_{k+1}) = \\ &\hat{\boldsymbol{x}}_{k+1/k} + \mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{\boldsymbol{z}}_{k+1}^{\mathrm{T}})[\mathrm{E}(\tilde{\boldsymbol{z}}_{k+1}\tilde{\boldsymbol{z}}_{k+1}^{\mathrm{T}})]^{-1}\tilde{\boldsymbol{z}}_{k+1}.\end{aligned} \tag{55}$$

其中:

$$\tilde{\boldsymbol{x}}_{k+1/k} = \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1/k}, \tag{56}$$

$$\begin{aligned}&\tilde{\boldsymbol{z}}_{k+1} = \boldsymbol{z}_{k+1} - \hat{\boldsymbol{z}}_{k+1/k} = \boldsymbol{\varepsilon}_{k+1} = \\ &\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1}) - \boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}} + \\ &\boldsymbol{v}_{k+1} - \boldsymbol{r}_{k+1} = \boldsymbol{\Theta}_{k+1} + \boldsymbol{v}_{k+1} - \boldsymbol{r}_{k+1}.\end{aligned} \tag{57}$$

上式(57)中$\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}}$表示的含义参见定理1中的说明.

考虑到$\boldsymbol{v}_{k+1}$与$\boldsymbol{w}_k$, $\boldsymbol{Z}^k$互不相关, 故$\boldsymbol{v}_{k+1}$与$\tilde{\boldsymbol{x}}_{k+1/k}$, $\boldsymbol{\Theta}_{k+1}$互不相关, 于是有

$$\begin{aligned}&\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{\boldsymbol{z}}_{k+1}^{\mathrm{T}}) = \\ &\mathrm{E}[\tilde{\boldsymbol{x}}_{k+1/k}(\boldsymbol{\Theta}_{k+1} + \boldsymbol{v}_{k+1} - \boldsymbol{r}_{k+1})^{\mathrm{T}}] = \\ &\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}}),\end{aligned} \tag{58}$$

$$\begin{aligned}&\mathrm{E}(\tilde{\boldsymbol{z}}_{k+1}\tilde{\boldsymbol{z}}_{k+1}^{\mathrm{T}}) = \\ &\mathrm{E}[(\boldsymbol{\Theta}_{k+1}+\boldsymbol{v}_{k+1}-\boldsymbol{r}_{k+1})(\boldsymbol{\Theta}_{k+1}+\boldsymbol{v}_{k+1}-\boldsymbol{r}_{k+1})^{\mathrm{T}}]=\end{aligned}$$

$$\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})+\mathrm{E}[(\boldsymbol{v}_{k+1}-\boldsymbol{r}_{k+1})(\boldsymbol{v}_{k+1}-\boldsymbol{r}_{k+1})^{\mathrm{T}}]=\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})+\boldsymbol{R}_{k+1}. \quad (59)$$

其中$\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})$和$\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})$的含义参见定理1中的说明.

定义滤波增益矩阵$\boldsymbol{K}_{k+1}$为

$$\boldsymbol{K}_{k+1}=\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{z}_{k+1}^{\mathrm{T}})[\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})]^{-1}. \quad (60)$$

显然

$$\boldsymbol{K}_{k+1}=\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})[\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})+\boldsymbol{R}_{k+1}]^{-1}, \quad (61)$$

则式(55)变成

$$\hat{\boldsymbol{x}}_{k+1}=\hat{\boldsymbol{x}}_{k+1/k}+\boldsymbol{K}_{k+1}\tilde{z}_{k+1}=\hat{\boldsymbol{x}}_{k+1/k}+\boldsymbol{K}_{k+1}\boldsymbol{\varepsilon}_{k+1}. \quad (62)$$

下面来计算最优状态估计协方差$\boldsymbol{P}_{k+1}$. 由式(62)可知

$$\begin{aligned}\boldsymbol{P}_{k+1}&=\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1}\tilde{\boldsymbol{x}}_{k+1}^{\mathrm{T}})=\\&\mathrm{E}[(\tilde{\boldsymbol{x}}_{k+1/k}-\boldsymbol{K}_{k+1}\tilde{z}_{k+1})(\tilde{\boldsymbol{x}}_{k+1/k}-\boldsymbol{K}_{k+1}\tilde{z}_{k+1})^{\mathrm{T}}]=\\&\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{\boldsymbol{x}}_{k+1/k}^{\mathrm{T}})-\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{z}_{k+1}^{\mathrm{T}})\boldsymbol{K}_{k+1}^{\mathrm{T}}-\\&\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{\boldsymbol{x}}_{k+1/k}^{\mathrm{T}})+\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})\boldsymbol{K}_{k+1}^{\mathrm{T}}. \quad (63)\end{aligned}$$

根据$\boldsymbol{K}_{k+1}$的表达式(60), 注意到

$$\begin{aligned}&\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})\boldsymbol{K}_{k+1}^{\mathrm{T}}=\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})\times\\&\{\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{z}_{k+1}^{\mathrm{T}})[\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})]^{-1}\}^{\mathrm{T}}=\\&\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})[\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})]^{-1}\mathrm{E}(\tilde{z}_{k+1}\tilde{\boldsymbol{x}}_{k+1/k}^{\mathrm{T}})=\\&\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{\boldsymbol{x}}_{k+1/k}^{\mathrm{T}}), \quad (64)\\&\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})\boldsymbol{K}_{k+1}^{\mathrm{T}}=\\&\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{z}_{k+1}^{\mathrm{T}})[\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})]^{-1}\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})\boldsymbol{K}_{k+1}^{\mathrm{T}}=\\&\mathrm{E}(\tilde{\mathrm{x}}_{k+1/k}\tilde{\mathrm{z}}_{k+1}^{\mathrm{T}})\boldsymbol{K}_{k+1}^{\mathrm{T}}. \quad (65)\end{aligned}$$

联合式(59), 则式(63)简化为

$$\begin{aligned}\boldsymbol{P}_{k+1}&=\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\tilde{\boldsymbol{x}}_{k+1/k}^{\mathrm{T}})-\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{\boldsymbol{x}}_{k+1/k}^{\mathrm{T}})=\\&\boldsymbol{P}_{k+1/k}-\boldsymbol{K}_{k+1}\mathrm{E}(\tilde{z}_{k+1}\tilde{z}_{k+1}^{\mathrm{T}})\boldsymbol{K}_{k+1}^{\mathrm{T}}=\\&\boldsymbol{P}_{k+1/k}-\boldsymbol{K}_{k+1}[\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})+\boldsymbol{R}_{k+1}]\boldsymbol{K}_{k+1}^{\mathrm{T}}. \quad (66)\end{aligned}$$

证毕.

推论 1 基于定理1, 2和UT变换, 噪声相关条件下非线性离散系统(1)的UKF滤波递推计算公式如下:

① 选择UT变换Sigma点采样策略[3].

② 状态预测.

由定理1可知, 计算$k+1$时刻状态预测$\hat{\boldsymbol{x}}_{k+1/k}$的关键是如何计算$k$时刻状态预测$\hat{\boldsymbol{x}}_{k/k-1}$经非线性状态函数$\boldsymbol{f}_k(\cdot)$及量测函数$\boldsymbol{h}_k(\cdot)$传递之后的后验均值和协方差, 为此可以采用UT变换来实现.

按照第①步所选择的Sigma点采样策略, 由$\hat{\boldsymbol{x}}_{k/k-1}$和$\boldsymbol{P}_{k/k-1}$来计算Sigma点$\boldsymbol{\xi}_{i,k/k-1}$, 其通过非线性状态函数$\boldsymbol{f}_k(\cdot)$及量测函数$\boldsymbol{h}_k(\cdot)$传播为$\boldsymbol{\gamma}_{i,k/k-1}$及$\boldsymbol{\chi}_{i,k/k-1}$, 由$\boldsymbol{\gamma}_{i,k/k-1}$及$\boldsymbol{\chi}_{i,k/k-1}$来计算后验均值$\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}$和$\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}$、后验自协方差$\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Lambda}_k^{\mathrm{T}})$和$\mathrm{E}(\boldsymbol{\Theta}_k\boldsymbol{\Theta}_k^{\mathrm{T}})$及后验互协方差$\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Theta}_k^{\mathrm{T}})$:

$$\boldsymbol{\gamma}_{i,k/k-1}=\boldsymbol{f}_k(\boldsymbol{\xi}_{i,k/k-1},\boldsymbol{u}_k),\ i=0,1,\cdots,L, \quad (67)$$

$$\boldsymbol{\chi}_{i,k/k-1}=\boldsymbol{h}_k(\boldsymbol{\xi}_{i,k/k-1}),\ i=0,1,\cdots,L, \quad (68)$$

$$\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}=\sum_{i=0}^{L}W_i^{\mathrm{m}}\boldsymbol{\gamma}_{i,k/k-1}=\sum_{i=0}^{L}W_i^{\mathrm{m}}\boldsymbol{f}_k(\boldsymbol{\xi}_{i,k/k-1},\boldsymbol{u}_k), \quad (69)$$

$$\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}}=\sum_{i=0}^{L}W_i^{\mathrm{m}}\boldsymbol{\chi}_{i,k/k-1}=\sum_{i=0}^{L}W_i^{\mathrm{m}}\boldsymbol{h}_k(\boldsymbol{\xi}_{i,k/k-1}), \quad (70)$$

$$\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Lambda}_k^{\mathrm{T}})=\sum_{i=0}^{L}W_i^{\mathrm{c}}(\boldsymbol{\gamma}_{i,k/k-1}-\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}})\times(\boldsymbol{\gamma}_{i.k/k-1}-\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}})^{\mathrm{T}}, \quad (71)$$

$$\mathrm{E}(\boldsymbol{\Theta}_k\boldsymbol{\Theta}_k^{\mathrm{T}})=\sum_{i=0}^{L}W_i^{\mathrm{c}}(\boldsymbol{\chi}_{i,k/k-1}-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}})\times(\boldsymbol{\chi}_{i,k/k-1}-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}})^{\mathrm{T}}, \quad (72)$$

$$\mathrm{E}(\boldsymbol{\Lambda}_k\boldsymbol{\Theta}_k^{\mathrm{T}})=\sum_{i=0}^{L}W_i^{\mathrm{c}}(\boldsymbol{\gamma}_{i,k/k-1}-\boldsymbol{f}_k(\boldsymbol{x}_k,\boldsymbol{u}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}})\times(\boldsymbol{\chi}_{i,k/k-1}-\boldsymbol{h}_k(\boldsymbol{x}_k)|_{\boldsymbol{x}_k\leftarrow\hat{\boldsymbol{x}}_{k/k-1}})^{\mathrm{T}}. \quad (73)$$

将式(69)~(73)代入式(13)~(16), 即可计算出$k+1$时刻最优一步状态预测$\hat{\boldsymbol{x}}_{k+1/k}$及误差协方差$\boldsymbol{P}_{k+1/k}$.

③ 状态估计.

由定理2可知, 计算$k+1$时刻状态估计$\hat{\boldsymbol{x}}_{k+1}$的关键是如何计算$k+1$时刻状态预测$\hat{\boldsymbol{x}}_{k+1/k}$经非线性量测函数传递之后的后验均值和协方差, 为此可以采用UT变换来实现.

已知由第②步已经计算得到$\hat{\boldsymbol{x}}_{k+1/k}$及$\boldsymbol{P}_{k+1/k}$, 按照第①步所选择的Sigma点采样策略, 由$\hat{\boldsymbol{x}}_{k+1/k}$和$\boldsymbol{P}_{k+1/k}$来计算Sigma点$\boldsymbol{\xi}_{i,k+1/k}$, 其通过非线性量测函数$\boldsymbol{h}_{k+1}(\cdot)$传播为$\boldsymbol{\chi}_{i,k+1/k}$, 由$\boldsymbol{\chi}_{i,k+1/k}$来计算后验均值$\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}}$、后验自协方差$\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})$及后验互协方差$\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})$:

$$\boldsymbol{\chi}_{i,k+1/k}=\boldsymbol{h}_{k+1}(\boldsymbol{\xi}_{i,k+1/k}),\ i=0,1,\cdots,L,\tag{74}$$

$$\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}}=\sum_{i=0}^{L}W_i^{\mathrm{m}}\boldsymbol{\chi}_{i,k+1/k}=\sum_{i=0}^{L}W_i^{\mathrm{m}}\boldsymbol{h}_{k+1}(\boldsymbol{\xi}_{i,k+1/k}),\tag{75}$$

$$\mathrm{E}(\boldsymbol{\Theta}_{k+1}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})=\sum_{i=0}^{L}W_i^{\mathrm{c}}(\boldsymbol{\chi}_{i,k+1/k}-\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}})\times(\boldsymbol{\chi}_{i,k+1/k}-\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}})^{\mathrm{T}},\tag{76}$$

$$\mathrm{E}(\tilde{\boldsymbol{x}}_{k+1/k}\boldsymbol{\Theta}_{k+1}^{\mathrm{T}})=\sum_{i=0}^{L}W_i^{\mathrm{c}}(\boldsymbol{\xi}_{i,k+1/k}-\hat{\boldsymbol{x}}_{k+1/k})\times(\boldsymbol{\chi}_{i,k+1/k}-\boldsymbol{h}_{k+1}(\boldsymbol{x}_{k+1})|_{\boldsymbol{x}_{k+1}\leftarrow\hat{\boldsymbol{x}}_{k+1/k}})^{\mathrm{T}}.\tag{77}$$

将式(75)~(77)代入式(49)~(52), 即可计算出$k+1$时刻最优滤波状态估计$\hat{\boldsymbol{x}}_{k+1}$及误差协方差$\boldsymbol{P}_{k+1}$.

从上述理论分析中不难发现, 噪声相关条件下UKF具有两个计算回路: 状态预测回路和状态估计回路. 其中状态预测回路是独立计算的, 可以离线进行, 而状态估计回路依赖于状态预测计算回路.

4 仿真实例(Simulation examples)

考虑如下所示的强非线性高斯系统模型来验证所提出的噪声相关条件下UKF的有效性:

$$\boldsymbol{x}_{k+1}=\begin{bmatrix}x_{1,k+1}\\x_{2,k+1}\\x_{3,k+1}\end{bmatrix}=\begin{bmatrix}3\cos x_{2,k}\\2x_{1,k}^2\mathrm{e}^{-0.05x_{2,k}}\\x_{2,k}/(1+x_{3,k}^2)\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}w_k,\tag{78}$$

$$z_k=x_{1,k}-\mathrm{e}^{-5x_{2,k}x_{3,k}}+v_k.\tag{79}$$

其中w_k和v_k均为高斯白噪声, 且它们的统计特性如下所示:

$$q_k=0.1,\ Q_k=0.001,\ r_k=0.2,\ R_k=0.004.\tag{80}$$

设非线性系统(78)(79)的理论初始值为

$$\boldsymbol{x}_0=[-0.5\ \ 1\ \ 1]^{\mathrm{T}}.\tag{81}$$

同时取状态估计的初始值为:

$$\hat{\boldsymbol{x}}_0=[-0.5\ \ 1\ \ 1]^{\mathrm{T}},\ \boldsymbol{P}_0=\boldsymbol{I}.\tag{82}$$

且$\hat{\boldsymbol{x}}_0$与w_k,v_k是互不相关的.

例 1　当$\boldsymbol{S}_k=0$时.

分别采用EKF、传统UKF及本文所设计的UKF对式(78)(79)所示的非线性系统状态进行估计. 特别地, 针对噪声相关条件下UKF递推公式, 取

$$\hat{\boldsymbol{x}}_{0/-1}=\hat{\boldsymbol{x}}_0=0,\ \boldsymbol{P}_{0/-1}=\boldsymbol{P}_0=\boldsymbol{I},\tag{83}$$

则状态估计及误差曲线如图1,2所示.

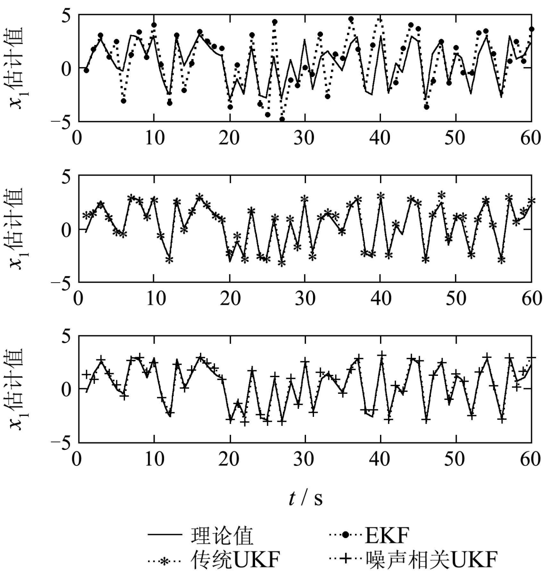

图 1　状态x_1估计值

Fig. 1　Estimation of the state x_1

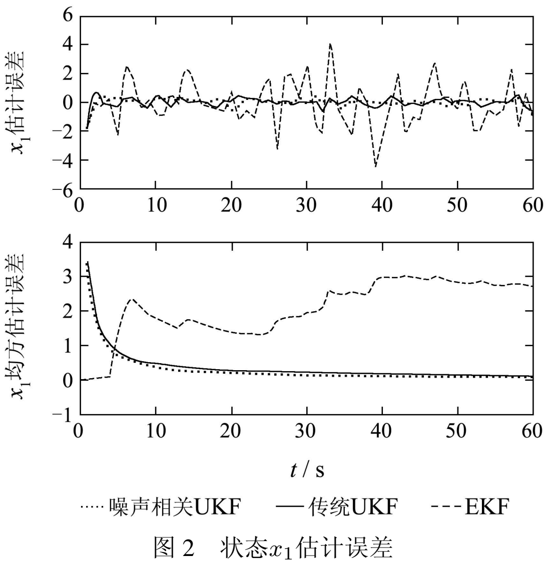

图 2　状态x_1估计误差

Fig. 2　Estimation error of the state x_1

从仿真图中不难看出, 当噪声互协方差$\boldsymbol{S}_k$为零时, 噪声相关条件下UKF状态x_1估计误差与传统UKF相近, 估计精度相同, 说明本文所提出的噪声相关条件下UKF递推公式在系统噪声和量测噪声互不相关时自动退化为传统UKF; 且它们的精度都高于EKF, 这是因为UT变换对非线性状态后验均值和协方差的逼近精度高于雅可比矩阵的计算.

例 2　当$\boldsymbol{S}_k=0.25$时.

噪声互协方差$\boldsymbol{S}_k$非零时, 噪声相关条件下UKF及传统UKF的状态估计及误差曲线如图3,4所示.

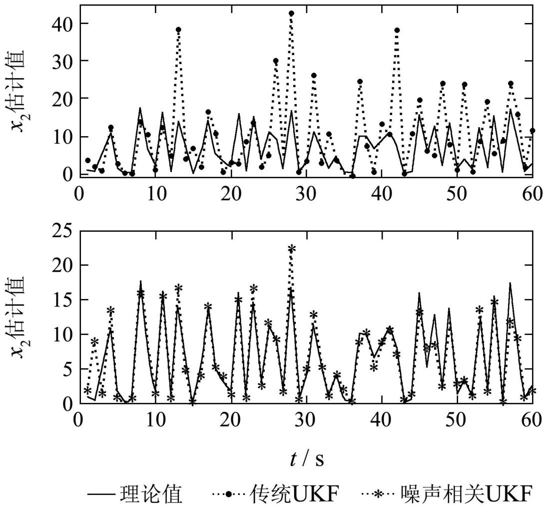

图 3　状态x_2估计值

Fig. 3　Estimation of the state x_2

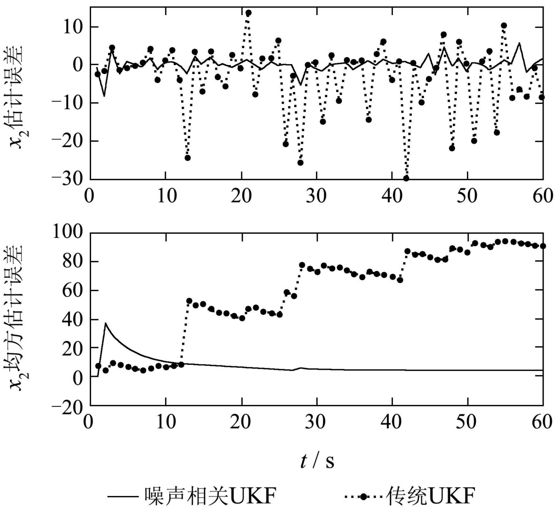

图 4　状态x_2估计误差

Fig. 4　Estimation error of the state x_2

不难发现, 传统UKF对状态x_2的估计效果不佳, 且状态均方估计误差随时间迅速积累, 其仿真曲线发散, 这充分说明传统UKF在系统噪声和量测噪声相关时非线性滤波失效; 而本文所设计的噪声相关UKF能实现对状态x_2的有效跟踪, 其均方误差仿真曲线不仅收敛, 而且状态估计精度高, 证明了噪声相关条件下UKF在解决带相关噪声的非线性系统滤波问题时的有效性和可行性.

5　结论(Conclusion)

传统UKF采用UT变换能以三阶泰勒精度逼近任何非线性高斯系统状态的后验均值和协方差, 其滤波精度高于EKF, 然而UKF要求系统噪声和量测噪声必须为互不相关高斯白噪声的局限性限制了其应用范围. 针对传统UKF在噪声相关条件下非线性滤波失效的问题, 本文基于最小均方误差估计准则, 推导了噪声相关条件下UKF滤波递推公式, 并采用UT变换来计算预测回路和估计回路中的状态后验均值和协方差. 仿真结果表明: 所设计的噪声相关条件下UKF不仅有效克服了传统UKF必须假设噪声互不相关的局限性, 拓展了UKF的应用范围; 而且在噪声互协方差为零时, 噪声相关条件下UKF自动退化为传统UKF, 其在实际应用中比传统UKF具有更强的适应能力.

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