第 36 卷第 11 期
2019 年 11 月

控 制 理 论 与 应 用
Control Theory & Applications

Vol. 36 No. 11
Nov. 2019

稳定降阶控制器设计的新方法

段志生[1], 郝宇清[2†]

(1. 北京大学 工学院 力学与工程科学系 湍流与复杂系统国家重点实验室, 北京 100871;
2. 北京航空航天大学 航空科学与工程学院 动力学与控制系, 北京 100191)

摘要: 本文首先提出了稳定降阶控制器的一个特殊结构, 然后给出了两种控制器设计方法: 一种是基于新的松弛变量的特殊构造, 另一种是基于分离李雅普诺夫矩阵和控制器矩阵的两步法. 新的方法可以处理具有不同阶数和不同数目不稳定极点的几个系统的同时镇定问题, 而这个问题是很难用一般的鲁棒控制方法处理的. 本文提出的新方法还可以拓展到处理H_∞、严格正实以及分散控制问题. 同时, 本文讨论了控制器的范数约束问题. 在该方法中, 甚至每一个控制器参数都可以被施加绝对值约束. 本文还讨论了相关的NP难问题, 给出了相应的控制器设计算法. 最后, 给出了几个例子验证了所提出的设计方法的有效性, 并且对这两种设计方法进行了对比.

关键词: 稳定控制器; 同时镇定; H_∞控制; 严格正实; 范数约束; NP难问题

引用格式: 段志生, 郝宇清. 稳定降阶控制器设计的新方法. 控制理论与应用, 2019, 36(11): 1850 – 1860

DOI: 10.7641/CTA.2019.90566

New methods for stable and reduced-order controller design

DUAN Zhi-sheng[1], HAO Yu-qing[2†]

(1. State Key Laboratory for Turbulence and Complex Systems, Department of Mechanics and Engineering Science, College of Engineering, Peking University, Beijing 100871, China;
2. Department of Dynamics and Control, Beihang University, Beijing 100191, China)

Abstract: In this paper, a special structure is first proposed for stable and reduced-order controllers. Then, two controller design methods are presented, where one is based on a special construction on the new introduced slack matrix variables, the other is based on the separation of Lyapunov matrix and control matrix from the two stage algorithms. The new methods can deal with the simultaneous stabilization problems for several plants with different orders and different numbers of unstable poles which can not be solved by general robust control methods. The new methods are also generalized to solve H_∞, strict positive real (SPR) and decentralized control problems. Finally, norm constraint problems of controllers are discussed, even an absolute value constraint can be added to each controller parameter in the proposed methods. Related NP hard problems have been discussed in this paper, and related controller design algorithms are proposed. Two new methods are compared with each other in several examples, and the effectiveness of the new methods are illustrated.

Key words: stable controllers; simultaneous stabilization; H_∞ control; strict positive real (SPR); norm constraint; NP-hard problems

Citation: DUAN Zhisheng, HAO Yuqing. New methods for stable and reduced-order controller design. *Control Theory & Applications*, 2019, 36(11): 1850 – 1860

1 Introduction

Strong stabilization, i.e., stabilizing plants by stable controllers[1], has lasted for a long time and has been studied by some authors, see [2–3] and references therein. Strong stabilization is a very challenging problem, which is not only important in theory, but also very valuable in engineering applications. Stable controllers are strongly preferred by engineers in practical applications, since unstable controllers may lead to problems with actuator and sensor failure, sensitivity to plant uncertainties, nonlinearities and implementation[4–5]. On the other hand, simultaneous stabilization of n plants by a general controller is equivalent to simultaneous stabilization of $n-1$ plants by a stable controller[1]. One given plant is strongly stabilizable if and only if it satisfies the parity interlacing property[1]. Some procedures are available to design stable stabilizing controllers involving with interpolation constraints, but may result in very high order controllers[1,6]. Some interpolation techniques or numerical methods have also been developed for multiple input systems about sensitivity reduction problems, time delay problems and mixed sen-

Received 15 July 2019; revised 14 November 2019.
†Corresponding author. E-mail: haoyq@buaa.edu.cn.
Recommended by Associate Editor: YU Li.
Supported by the National Natural Science Foundation of China (U1713223, 61673026, 11802006).

sitivity reduction problems[2,7–11]. Strong stabilization of a class of multi-input-multi-output (MIMO) systems with restrictions on the zeros in the right-hand complex plane was considered in [2]. A sensitivity reduction problem by stable stabilizing controller for a linear time-invariant MIMO distributed parameter system was investigated in [9]. Stable H_∞ controller design for MIMO systems with multiple input/output time delays was studied in [10]. Stable controller design was considered for two-link underactuated planar robots in [3, 12]. A method to design stable controllers for mixed sensitivity reduction for systems that have input/output delays and infinitely many unstable zeros was proposed in [11].

Because of the essential difficulty for stable controller design, so far the existence of a stable stabilizing controller for more than two plants is still an open problem, which is rationally undecidable or NP hard[13], i.e., the controller cannot be designed by polynomial time algorithms. Also, some linear control problems including static output feedback design, decentralized control and simultaneous stabilization with low-order controllers are NP hard[14–17]. A nonsmooth H_∞ synthesis method without involving Lyapunov matrix was presented in [18] which can deal with some NP hard design problems. There were some research results about simultaneous stabilization for different systems, such as distributed networked systems, time-varying systems and time-delay linear systems. The simultaneous H_∞ stabilization for distributed networked multimode control systems with multiple packet dropouts was investigated in [15]. A controller design method was provided in [16] to simultaneously stabilize a collection of time-varying linear systems within the framework of nest algebras. The design of simultaneous static output feedback controllers for a finite collection of time-delay linear systems was considered in [17]. There were also some approaches to deal with the simultaneous stabilization problems. A necessary and sufficient condition for a static output feedback controller was found by using the inverse LQ approach[19]. A new approach using geometric terms[20] was proposed to deal with the problem of simultaneously strong stabilization. Theories and developments in the simultaneous stabilization of linear systems were discussed in [21]. This paper aims to propose new techniques to simplify bilinear matrix inequalities (BMIs) related to stable controller design and some related NP hard problems. Low-order controller design has also been studied by some authors and effective numerical algorithms were proposed, see [22–26] and references therein. A decentralized dynamic feedback H_∞ control design algorithm was presented in [27] allowing the design of low-order controllers.

Different from the above mentioned stable controller design method, this paper presents simpler and more generalized stable controller design methods by predetermining the controller structure and combining with the iterative algorithms, which are not limited to single input or multiple input problems and controller orders. To the best knowledge of authors, this is the first time to determine simple and effective structure for stable controllers. Based on the understanding of controller robustness, a controller structure that has no conservativeness is proposed by this paper, and this controller structure plays an important role in stable controller design. By introducing the slack matrices, the matrix variables are relaxed from the general bilinear terms, and inequalities are easier to be solved by the iterative LMI method. For the problems of simultaneous stabilization of two plants, the proposed theorem has no conservativeness, because two plants can have different Lyapunov matrices. And, this paper proposes new approaches to simplify bilinear matrix inequalities related to stable controller design and some related NP hard problems

The rest of this paper is organized as follows. In Section 2, a stable controller structure and two design methods are presented. The methods are extended to strong stabilization of two plants, and can be used to deal with the simultaneous stabilization of more plants. In Section 3, the methods are generalized to H_∞ and SPR control problems. In Section 4, controller constraint problems are discussed. The large gain problem in the general state feedback and full-order dynamic controller designs can be effective avoided. In Section 5, four examples are given to show that the effectiveness of the proposed methods. Section 6 concludes the paper.

Throughout this paper, B and C are supposed to be matrices of full column rank and full row rank, $B^{\perp}$ satisfies that $B^{\perp\mathrm{T}}B = 0$ and $[B,\ B^{\perp}]$ is nonsingular. The superscript T means transpose for real matrices. $R(\cdot)$ denotes the column space of the corresponding matrix. The notation $\mathrm{sym}\{A\} = A + A^{\mathrm{T}}$ is used. A star$(*)$ indicates symmetric terms in matrix inequalities.

2 Stable and reduced-order controllers

Consider the following linear system

$$\begin{cases}\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u},\\ \boldsymbol{y} = C\boldsymbol{x},\end{cases} \tag{1}$$

and the dynamic output feedback controller

$$\begin{cases}\dot{\boldsymbol{x}}_{\boldsymbol{k}} = A_k\boldsymbol{x}_{\boldsymbol{k}} + B_k\boldsymbol{y},\\ \boldsymbol{u} = C_k\boldsymbol{x}_{\boldsymbol{k}} + D_k\boldsymbol{y}.\end{cases} \tag{2}$$

Then, the closed-loop system matrix is

$$A_{\mathrm{cl}} = \begin{bmatrix} A + BD_kC & BC_k \\ B_kC & A_k \end{bmatrix} = \hat{A} + \hat{B}\hat{K}\hat{C}, \tag{3}$$

where $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B & 0 \\ 0 & I_k \end{bmatrix}$, $\hat{K} = \begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix}$,

$\hat{C} = \begin{bmatrix} C & 0 \\ 0 & I_k \end{bmatrix}$. Actually, the dynamic output feedback control problem can be viewed as a special case of static output feedback control problems.

If A_k is stable and the order of controller (2) is less than the order of plant (1), then controller (2) is called a stable and reduced-order controller. If $D_k = 0$, it is also called a strictly proper controller.

Throughout this paper, the following assumption is required.

Assumption 1 A_k in controller (2) has no Jordan blocks for complex eigenvalues and has at most Jordan blocks of order 2 for real eigenvalues.

Note that Assumption 1 is not so conservative based on the understanding of controller robustness. If A_k has higher-order Jordan blocks, such blocks will disappear after a small perturbation of controller parameters.

Based on Assumption 1, suppose A_k has the following structure,

$$A_k = \mathrm{diag}\{K_i\},\ i = 1, \cdots, l, \tag{4}$$

where $K_i = k_i < 0$ or $K_i = \begin{bmatrix} 0 & k_{i3} \\ k_{i1} & k_{i2} \end{bmatrix}$, $k_{i1} < 0,\ k_{i2} < 0,\ k_{i3} > 0$. Obviously, A_k is stable and K_i in (4) can have a couple of conjugate complex eigenvalues or two real eigenvalues. Of course, K_i can also be supposed as $K_i = \begin{bmatrix} 0 & 1 \\ k_{i1} & k_{i2} \end{bmatrix}$. For convenience of assuming linear matrix inequality (LMI) variables, this paper considers the form of K_i as in (4). When the order of A_k is odd, at least one of K_i is a real scalar.

Motivated by the method in [28], together with the above controller structure one can get the following theorem for stable controller design.

Theorem 1 A stable controller as in (2) and (4) stabilizes system (1) if and only if there are a symmetric matrix $P > 0$, and any matrices G_{11}, G_{12}, G_{21}, G_{22}, F_{11}, F_{12}, F_{21} and F_{22} such that

$$\begin{bmatrix} -\mathrm{sym}\{G\} & G\hat{A} + P - F^{\mathrm{T}} + \varPhi \\ * & \mathrm{sym}\{F\hat{A} + \varPsi\} \end{bmatrix} < 0, \tag{5}$$

where

$$\varPhi = \hat{B}G_{11}\hat{B}^{\mathrm{T}}\hat{B}\hat{K}\hat{C} + \hat{B}^{\perp}G_{21}\hat{B}^{\mathrm{T}}\hat{B}\hat{K}\hat{C},$$

$$G = [\hat{B}\ \ \hat{B}^{\perp}] \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} [\hat{B}\ \ \hat{B}^{\perp}]^{\mathrm{T}},$$

$$\varPsi = \hat{B}F_{11}\hat{B}^{\mathrm{T}}\hat{B}\hat{K}\hat{C} + \hat{B}^{\perp}F_{21}\hat{B}^{\mathrm{T}}\hat{B}\hat{K}\hat{C},$$

and

$$F = [\hat{B}\ \ \hat{B}^{\perp}] \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} [\hat{B}\ \ \hat{B}^{\perp}]^{\mathrm{T}}.$$

Proof Actually, the inequality (5) is just

$$\begin{bmatrix} -(G + G^{\mathrm{T}}) & GA_{\mathrm{cl}} + P - F^{\mathrm{T}} \\ * & FA_{\mathrm{cl}} + A_{\mathrm{cl}}^{\mathrm{T}}F^{\mathrm{T}} \end{bmatrix} < 0,$$

which is just equivalent to the Lyapunov inequality $PA_{\mathrm{cl}} + A_{\mathrm{cl}}^{\mathrm{T}}P < 0$ by the general parameter dependent Lyapunov function method[29–31]. Obviously by introducing the matrix $B^{\perp}$, the matrix variables G_{12}, G_{22} and F_{12}, G_{22} are relaxed from the general bilinear terms $G\hat{B}\hat{K}\hat{C}$ and $F\hat{B}\hat{K}\hat{C}$, which makes the inequality (5) easier to solve by the iterative LMI method.

Let

$$\hat{A}^b = [\hat{B}\ \ \hat{B}^{\perp}]^{\mathrm{T}}\hat{A}[\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}} = \begin{bmatrix} \hat{A}^b_{11} & \hat{A}^b_{12} \\ \hat{A}^b_{21} & \hat{A}^b_{22} \end{bmatrix}. \tag{6}$$

One can get the following corollary easily.

Corollary 1 If $G_{21} = 0$ and $F_{21} = 0$, then Theorem 1 implies

$$\begin{bmatrix} -(G_{22} + G_{22}^{\mathrm{T}}) & G_{22}\hat{A}^b_{22} + P_{22} - F_{22}^{\mathrm{T}} \\ * & F_{22}\hat{A}^b_{22} + \hat{A}^{b\mathrm{T}}_{22}F_{22}^{\mathrm{T}} \end{bmatrix} < 0, \tag{7}$$

which is free from the control matrix $\hat{K}$, where P_{22} is the block in $P^* = [\hat{B}\ \ \hat{B}^{\perp}]^{-1}P[\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}}$ corresponding to G_{22}. (7) implies $\hat{A}^b_{22} = \hat{B}^{\perp\mathrm{T}}\hat{A}\hat{B}^{\perp} = B^{\perp\mathrm{T}}AB^{\perp}$ is stable. QED.

Proof If $G_{21} = 0$ and $F_{21} = 0$, multiplying $\mathrm{diag}([\hat{B}\ \ \hat{B}^{\perp}]^{-1}, [\hat{B}\ \ \hat{B}^{\perp}]^{-1})$ and $\mathrm{diag}([\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}}, [\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}})$ on the left- and right-hand sides of (9) gives

$$\begin{bmatrix} -\mathrm{sym}\{G^*\} & G^*\hat{A}^b + P^* - F^{*\mathrm{T}} + \varPhi^* \\ * & \mathrm{sym}\{F^*\hat{A}^b + \varPsi^*\} \end{bmatrix} < 0, \tag{8}$$

where $G^* = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}$, $F^* = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}$, $\varPhi^* = [I\ \ 0]^{\mathrm{T}}G_{11}\hat{B}^{\mathrm{T}}\hat{B}\hat{K}\hat{C}^*$ and $\varPsi^* = [I\ \ 0]^{\mathrm{T}}F_{11}\hat{B}^{\mathrm{T}}\hat{B}\hat{K}\hat{C}^*$, $\hat{C}^* = \hat{C}[\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}}$. By this inequality, one can get (7) easily. QED.

One can also establish the following result for controller design by introducing $\hat{C}^{\mathrm{T}\perp}$.

Corollary 2 A stable controller as given in (2) and (4) stabilizes system (1) if and only if there are a symmetric matrix $Q > 0$, and any matrices H_{11}, H_{12}, H_{21}, H_{22}, J_{11}, J_{12}, J_{21} and J_{22} such that

$$\begin{bmatrix} -\mathrm{sym}\{H\} & \hat{A}^{\mathrm{T}}H^{\mathrm{T}} + Q - J + \varPhi^{\mathrm{T}} \\ * & \mathrm{sym}\{\hat{A}J + \varPsi\} \end{bmatrix} < 0, \tag{9}$$

where

$$\varPhi = \hat{B}\hat{K}\hat{C}\hat{C}^{\mathrm{T}}[H_{11}\ \ H_{12}][\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}],$$

$$H = [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}] \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{\mathrm{T}},$$

$$\varPsi = \hat{B}\hat{K}\hat{C}\hat{C}^{\mathrm{T}}[J_{11}\ \ J_{12}][\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}],$$

and

$$J = [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}] \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{\mathrm{T}}.$$

Actually, Corollary 2 can be viewed as a dual case of Theorem 1. Similar to corollary 1, if $H_{12} = 0$ and $J_{12} = 0$, one knows that $\hat{C}^{\mathrm{T}\perp\mathrm{T}}\hat{A}\hat{C}^{\mathrm{T}\perp} = C^{\mathrm{T}\perp\mathrm{T}}AC^{\mathrm{T}\perp}$ is stable. Based on Theorem 1, one can establish the following algorithm for stable controller design.

Algorithm 1

Step 1 Take initial matrices G_{11}, G_{21}, F_{11} and F_{21}, then design $\hat{K}$, G_{12}, G_{22}, F_{12}, F_{22} and $P > 0$ by solving the following LMI with a small scalar $\epsilon_1 \geqslant 0$,

$$\begin{bmatrix} -(G+G^{\mathrm{T}}) & GA+P-F^{\mathrm{T}}+\varPhi \\ * & -\epsilon_1 P+\mathrm{sym}\{FA+\varPsi\} \end{bmatrix} < 0, \quad (10)$$

where G, F, $\varPhi$ and $\varPsi$ are as given in Theorem 1. If $A_{\mathrm{cl}} = \hat{A} + \hat{B}\hat{K}\hat{C}$ is stable, then stop the algorithm. Otherwise, turn to Step 2.

Step 2 With the designed $\hat{K}$ in Step 1, solve the following LMI for the matrix variables G, F and $P > 0$,

$$\begin{bmatrix} -(G+G^{\mathrm{T}}) & GA_{\mathrm{cl}}+P-F^{\mathrm{T}} \\ * & -\epsilon_2 P+FA_{\mathrm{cl}}+A_{\mathrm{cl}}^{\mathrm{T}}F^{\mathrm{T}} \end{bmatrix} < 0,$$

where $0 \leqslant \epsilon_2 \leqslant \epsilon_1$.

Step 3 With the matrices G and F obtained in Step 2, let $\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = [\hat{B}\ \ \hat{B}^{\perp}]^{-1}G\,[\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}}$ and $\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = [\hat{B}\ \hat{B}^{\perp}]^{-1}F[\hat{B}\ \hat{B}^{\perp}]^{-\mathrm{T}}$. Substitute the matrices G_{11}, G_{21}, F_{11} and F_{21} in Step 1. Solve the inequality in Step 1 for another small ϵ_1. Repeat the above two steps until that A_{cl} is stable.

Remark 1 The inequality (10) in Algorithm 1 is equivalent to $PA_{\mathrm{cl}}+A_{\mathrm{cl}}^{\mathrm{T}}P < \epsilon_1 P$, which implies the real part of each eigenvalue of A_{cl} is less than $0.5\epsilon_1$. Because of a special choice of G_{11} and G_{21}, and F_{11} and F_{21}, sometimes solving inequality (10) can give a stabilizing solution even if $\epsilon_1 > 0$, see the following Example 1. One can also design controllers to assign eigenvalues of closed-loop system by combining Theorem 1 with the general LMI method[31–32]. Of course, one can also give a corresponding dual algorithm based on Corollary 2. Algorithm 1 can also be used to design decentralized controllers.

On the other hand, one can also introduce the method based on the separation of Lyapunov matrix and control matrix from the two stage algorithm given by [33–34].

Lemma 1 A stable controller as given in (2) and (4) stabilizes system (1) if and only if there are a symmetric matrix $P > 0$, a diagonal matrix $X > 0$, a state feedback matrix K_0 and a structured matrix $\hat{K}$ as given in (3) and (4) such that

$$\begin{bmatrix} P\hat{A}+\hat{A}^{\mathrm{T}}P & P\hat{B} \\ \hat{B}^{\mathrm{T}}P & 0 \end{bmatrix} - \varOmega - \varOmega^{\mathrm{T}} < 0, \quad (11)$$

where $\varOmega = \begin{bmatrix} K_0^{\mathrm{T}} \\ -I \end{bmatrix} X[\hat{K}\hat{C}\ \ -I]$.

Based on Lemma 1, one can also establish the following algorithm.

Algorithm 2

Step 1 Take an initial state feedback matrix K_0 such that $\hat{A} + \hat{B}K_0$ is stable, then design $\hat{K}^*$, diagonal matrix $X > 0$ and $P > 0$ by solving the following LMI with a small scalar $\epsilon_1 \geqslant 0$,

$$\begin{bmatrix} P\hat{A}+\hat{A}^{\mathrm{T}}P-\epsilon_1 P & P\hat{B} \\ * & 0 \end{bmatrix} - \varOmega - \varOmega^{\mathrm{T}} < 0, \quad (12)$$

where $\varOmega = \begin{bmatrix} K_0^{\mathrm{T}} \\ -I \end{bmatrix} [\hat{K}^*\hat{C}\ \ -X]$ and $\hat{K}^*$ has the same structure as $\hat{K}$. Let $\hat{K} = X^{-1}\hat{K}^*$. If $A_{\mathrm{cl}} = \hat{A}+\hat{B}\hat{K}\hat{C}$ is stable, then stop the algorithm. Otherwise, turn to Step 2.

Step 2 With the designed $\hat{K}^*$ and X in Step 1, solve the inequality (12) again for K_0, $P > 0$ and a small scalar $\epsilon_2 \geqslant 0$ with $\epsilon_2 \leqslant \epsilon_1$. Repeat the above two steps until that A_{cl} is stable.

Remark 2 The inequality (12) in Algorithm 2 implies $PA_{\mathrm{cl}} + A_{\mathrm{cl}}^{\mathrm{T}}P < \epsilon_1 P$, which means the real part of each eigenvalue of A_{cl} is smaller than $0.5\epsilon_1$. Also, the inequality (12) in Algorithm 2 implies $P[\hat{A} + \hat{B}K_0] + [\hat{A} + \hat{B}K_0]^{\mathrm{T}}P < \epsilon_1 P$. K_0 can be viewed as a state feedback matrix and the initial K_0 in Step 1 of Algorithm 2 can be taken as stabilizing solution with some pole assignment requirement. Therefore, Algorithm 2 mainly from [33] searches alternatively between a state feedback solution K_0 and a structure constrained controller $\hat{K}$.

Remark 3 Although dynamic output feedback stabilization can be viewed as a special case of static output feedback stabilization, dynamic controllers are much harder to design because the order of closed-loop systems is higher. Both Algorithms 1 and 2 can be used to design dynamic controllers with any fixed order and decentralized controllers. The higher the order of controllers, the harder the choice of the initial matrices in Algorithm 1. Because of the introduction of slack matrices G and F, Algorithm 1 has good robustness against the choice of initial matrices, and Algorithm 2 also has robustness against the choice of initial state feedback matrix.

The simultaneous stabilization of several given plants is a very challenging problem because of the NP hardness in design algorithms[14,35]. The method in the above section can be extended to design common stable controllers for several plants. For simplicity, this section mainly considers the simultaneously strong stabilization of two plants, which is equivalent to simultaneous stabilization of three plants[1].

Consider the following two plants:

$$\begin{cases} \dot{\boldsymbol{x}}^{i} = A^{i}\boldsymbol{x}^{i} + B^{i}\boldsymbol{u}^{i}, \\ \boldsymbol{y}^{i} = C^{i}\boldsymbol{x}^{i},\ i = 1,\ 2, \end{cases} \quad (13)$$

possibly with different orders, but with the same number of inputs and the same number of outputs, and the common dynamic output feedback controller

$$\begin{cases}\dot{\boldsymbol{x}}_k = A_k\boldsymbol{x}_k + B_k\boldsymbol{y}^i,\\ \boldsymbol{u}^i = C_k\boldsymbol{x}_k + D_k\boldsymbol{y}^i.\end{cases} \tag{14}$$

Then, the two closed-loop matrices are

$$A^i_{\rm cl} = \begin{bmatrix} A^i + B^iD_kC^i & B^iC_k\\ * & A_k\end{bmatrix} = \hat{A}^i + \hat{B}^i\hat{K}\hat{C}^i,\ i = 1,\ 2, \tag{15}$$

where $\hat{A}^i = \begin{bmatrix} A^i & 0\\ 0 & 0\end{bmatrix}$, $\hat{B}^i = \begin{bmatrix} B^i & 0\\ 0 & I_k\end{bmatrix}$, $\hat{C}^i = \begin{bmatrix} C^i & 0\\ 0 & I_k\end{bmatrix}$, and $\hat{K}$ is as given in (3). The structure of A_k is as given in (4).

Similar to Theorem 1, the following result can be presented for simultaneous stabilization problem.

Theorem 2 A stable controller as given in (4) and (14) simultaneously stabilizes two plants in (13) if and only if there are symmetric matrices $P^i > 0$, and any matrices $G^i_{11},\ G^i_{12},\ G^i_{21},\ G^i_{22},\ F^i_{11},\ F^i_{12},\ F^i_{21}$ and F^i_{22} such that

$$\begin{bmatrix} -{\rm sym}\{G^i\} & G^i\hat{A}^i + P^i - F^{i\rm T} + \varPhi^i\\ * & {\rm sym}\{F^i\hat{A}^i + \varPsi^i\}\end{bmatrix} < 0,\ i=1,\ 2, \tag{16}$$

where

$$\varPhi^i = \hat{B}^iG^i_{11}\hat{B}^{i\rm T}\hat{B}^i\hat{K}\hat{C}^i + \hat{B}^{i\perp}G^i_{21}\hat{B}^{i\rm T}\hat{B}^i\hat{K}\hat{C}^i,$$

$$G^i = [\hat{B}^i\ \ \hat{B}^{i\perp}]\begin{bmatrix} G^i_{11} & G^i_{12}\\ G^i_{21} & G^i_{22}\end{bmatrix}[\hat{B}^i\ \ \hat{B}^{i\perp}]^{\rm T},$$

$$\varPsi^i = \hat{B}^iF^i_{11}\hat{B}^{i\rm T}\hat{B}^i\hat{K}\hat{C}^i + \hat{B}^{i\perp}F^i_{21}\hat{B}^{i\rm T}\hat{B}^i\hat{K}\hat{C}^i,$$

and

$$F^i = [\hat{B}^i\ \ \hat{B}^{i\perp}]\begin{bmatrix} F^i_{11} & F^i_{12}\\ F^i_{21} & F^i_{22}\end{bmatrix}[\hat{B}^i\ \ \hat{B}^{i\perp}]^{\rm T}.$$

Similar to Corollary 1, if $G^i_{21} = 0,\ i = 1, 2$ and $F^i_{21} = 0,\ i = 1, 2$, then Theorem 2 implies $B^{i\perp\rm T}A^iB^{i\perp}$ are stable. Also, similar to Corollary 2, one can establish a corresponding result of Theorem 2 by using the information of $C^{i\perp}$.

Remark 4 Two plants in (13) can have different orders and different numbers of unstable eigenvalues, which is different from robust control problems subject to stable dynamic perturbations[25] and parametric uncertainties[30–31]. Theorem 2 has no any conservativeness because two plants can have different Lyapunov matrices and different G and F. Theorem 3 can also be generalized to simultaneous stabilization of more plants.

Remark 5 It is well-known that simultaneous stabilization of three plants or simultaneously strong stabilization of two plants is NP hard, i.e. one cannot get polynomial time algorithms[13]. Based on Theorem 2, one can establish the algorithms like Algorithm 1 and Algorithm 2 for the above simultaneous stabilization problem.

3 $\rm H_\infty$ and SPR control problems

In this section, the methods in the above sections are extended to $\rm H_\infty$ and SPR control problems. Consider a general system

$$\begin{cases}\dot{\boldsymbol{x}} = A\boldsymbol{x} + B_1\boldsymbol{w} + B\boldsymbol{u},\\ \boldsymbol{z} = C_1\boldsymbol{x} + D_{11}\boldsymbol{w} + D_{12}\boldsymbol{u},\\ \boldsymbol{y} = C\boldsymbol{x} + D_{21}\boldsymbol{w},\end{cases} \tag{17}$$

where $\boldsymbol{w}$ is the external disturbance and $\boldsymbol{z}$ is the regulated output. The dynamic output feedback controller is as given in (2). Then, the closed-loop transfer function is given by

$$G_{\rm cl}(s) = C_{\rm cl}(sI - A_{\rm cl})^{-1}B_{\rm cl} + D_{\rm cl}, \tag{18}$$

where $A_{\rm cl} = \hat{A} + \hat{B}\hat{K}\hat{C}$ and $\hat{A},\ \hat{B},\ \hat{K}$ and $\hat{C}$ are as given in (2), $B_{\rm cl} = \hat{B}_1 + \hat{B}\hat{K}\hat{D}_{21}, C_{\rm cl} = \hat{C}_1 + \hat{D}_{12}\hat{K}\hat{C}$ and $D_{\rm cl} = D_{11} + \hat{D}_{12}\hat{K}\hat{D}_{21}, \hat{B}_1 = \begin{bmatrix} B_1\\ 0\end{bmatrix}$, $\hat{D}_{21} = \begin{bmatrix} D_{21}\\ 0\end{bmatrix}$, $\hat{C}_1 = \begin{bmatrix} C_1 & 0\end{bmatrix}$, $\hat{D}_{12} = \begin{bmatrix} D_{12} & 0\end{bmatrix}$.

Combining Theorem 1 with the general bounded real lemma, one can get the following result.

Theorem 3 A stable controller as given in (2) and (4) stabilizes system (17) and makes $\|G_{\rm cl}(s)\|_\infty < \gamma I$ if and only if any one of the following two conditions holds:

i) there are a symmetric matrix $P > 0$, and any matrices $G_1,\ G_2,\ F_1$ and F_2 such that

$$\begin{bmatrix} -{\rm sym}\{G\} & M^1_{12} & M^1_{13} & 0\\ * & {\rm sym}\{F\hat{A} + \varPsi\hat{C}\} & M^1_{23} & C^{\rm T}_{\rm cl}\\ * & * & -\gamma I & D^{\rm T}_{\rm cl}\\ * & * & * & -\gamma I\end{bmatrix} < 0, \tag{19}$$

where

$$M^1_{12} = G\hat{A} + P - F^{\rm T} + \varPhi\hat{C},$$

$$M^1_{13} = G\hat{B}_1 + \varPhi\hat{D}_{21},$$

$$M^1_{23} = F\hat{B}_1 + \varPsi\hat{D}_{21},$$

$$\varPhi = [\hat{B}\ \ \hat{B}^\perp]G_1\hat{B}^{\rm T}\hat{B}\hat{K},$$

$$G = [\hat{B}\ \ \hat{B}^\perp][G_1\ \ G_2][\hat{B}\ \ \hat{B}^\perp]^{\rm T},$$

$$\varPsi = [\hat{B}\ \ \hat{B}^\perp]F_1\hat{B}^{\rm T}\hat{B}\hat{K},$$

and

$$F = [\hat{B}\ \ \hat{B}^\perp][F_1\ \ F_2][\hat{B}\ \ \hat{B}^\perp]^{\rm T}.$$

ii) there are a symmetric matrix $Q > 0$, and any matrices $H_1,\ H_2,\ J_1$ and J_2 such that

$$\begin{bmatrix} -{\rm sym}\{H\} & M^2_{12} & M^2_{13} & 0\\ * & {\rm sym}\{\hat{A}J + \hat{B}\varPsi\} & M^2_{23} & B_{\rm cl}\\ * & * & -\gamma I & D_{\rm cl}\\ * & * & * & -\gamma I\end{bmatrix} < 0, \tag{20}$$

where

$$M_{12}^2 = H^{\mathrm{T}}\hat{A}^{\mathrm{T}} + P - J + \Phi^{\mathrm{T}}\hat{B}^{\mathrm{T}},$$
$$M_{13}^2 = H^{\mathrm{T}}\hat{C}_1^{\mathrm{T}} + \Phi^{\mathrm{T}}\hat{D}_{12}^{\mathrm{T}},$$
$$M_{23}^2 = J^{\mathrm{T}}\hat{C}_1^{\mathrm{T}} + \Psi^{\mathrm{T}}\hat{D}_{12}^{\mathrm{T}},$$
$$\Phi = \hat{K}\hat{C}\hat{C}^{\mathrm{T}}H_1[\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{\mathrm{T}},$$
$$H = [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}][H_1^{\mathrm{T}}\ \ H_2^{\mathrm{T}}]^{\mathrm{T}}[\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{\mathrm{T}},$$
$$\Psi = \hat{K}\hat{C}\hat{C}^{\mathrm{T}}J_1[\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{\mathrm{T}},$$

and

$$J = [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}][J_1\ \ J_2][\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{\mathrm{T}}.$$

Proof Actually, the LMI (19) is just

$$\begin{bmatrix} -\mathrm{sym}\{G\} & GA_{\mathrm{cl}} + P - F^{\mathrm{T}} & GB_{\mathrm{cl}} & 0 \\ * & \mathrm{sym}\{FA_{\mathrm{cl}}\} & FB_{\mathrm{cl}} & C_{\mathrm{cl}}^{\mathrm{T}} \\ * & * & -\gamma I & D_{\mathrm{cl}}^{\mathrm{T}} \\ * & * & * & -\gamma I \end{bmatrix} < 0, \tag{21}$$

which is equivalent to the bounded real lemma inequality[29]

$$\begin{bmatrix} PA_{\mathrm{cl}} + A_{\mathrm{cl}}^{\mathrm{T}}P & PB_{\mathrm{cl}} & C_{\mathrm{cl}}^{\mathrm{T}} \\ B_{\mathrm{cl}}^{\mathrm{T}}P & -\gamma I & D_{\mathrm{cl}}^{\mathrm{T}} \\ C_{\mathrm{cl}} & D_{\mathrm{cl}} & -\gamma I \end{bmatrix} < 0 \tag{22}$$

for the H_∞ norm condition $\|G_{\mathrm{cl}}(s)\|_\infty < \gamma I$. Theorem 3–ii) can be proved similarly. QED.

Similar to Corollary 2, Theorem 3–ii) can be viewed as a dual case of Theorem 3–i). Compared with Theorems 1 and 2, in Theorem 3, G_1 can be viewed as $[G_{11}^{\mathrm{T}}\ \ G_{21}^{\mathrm{T}}]^{\mathrm{T}}$ and F_1 can be viewed as $[F_{11}^{\mathrm{T}}\ \ F_{21}^{\mathrm{T}}]^{\mathrm{T}}$. For convenience of statement, they are written as a compact form. In this theorem, matrices P, G_2 and F_2, and Q, H_2 and J_2 are relaxed from the control matrix $\hat{K}$. If G_1 and F_1, and H_1 and J_1 are fixed, then (19) and (20) are LMIs.

The following algorithm gives an alternately iterative design method based on Theorem 3–i) and –ii) for stable controllers.

Algorithm 3

Step 1 Design a stable stabilizing controller first by Algorithm 1 and solve the inequality (21) to get γ and matrices G and F. Let $[G_1\ \ G_2] = [\hat{B}\ \ \hat{B}^{\perp}]^{-1}\cdot G[\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}}$ and $[F_1\ \ F_2] = [\hat{B}\ \ \hat{B}^{\perp}]^{-1}F[\hat{B}\ \ \hat{B}^{\perp}]^{-\mathrm{T}}$.

Step 2 With the obtained G_1 and F_1, solve (19) for a new controller and γ. With this controller, solve (21) again to get a smaller γ, and matrices G and F.

Step 3 If γ obtained in Step 2 is small enough, stop the algorithm. Otherwise, turn to Step 1 with the new matrices G_1 and F_1 getting from G and F. Repeat Step 2 again to minimize γ. If Steps 1 and 2 get matured, then turn to Step 4.

Step 4 With the controller and γ obtained in Step 3, solve the following inequality:

$$\begin{bmatrix} -\mathrm{sym}\{H\} & H^{\mathrm{T}}A_{\mathrm{cl}}^{\mathrm{T}} + Q - J & H^{\mathrm{T}}C_{\mathrm{cl}}^{\mathrm{T}} & 0 \\ * & \mathrm{sym}\{A_{\mathrm{cl}}J\} & J^{\mathrm{T}}C_{\mathrm{cl}}^{\mathrm{T}} & B_{\mathrm{cl}} \\ * & * & -\gamma I & D_{\mathrm{cl}} \\ * & * & * & -\gamma I \end{bmatrix} < 0, \tag{23}$$

to get matrices H and J. Let

$$[H_1^{\mathrm{T}}\ \ H_2^{\mathrm{T}}]^{\mathrm{T}} = [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{-1}H[\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{-\mathrm{T}}$$

and

$$[J_1\ \ J_2] = [\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{-1}J[\hat{C}^{\mathrm{T}}\ \ \hat{C}^{\mathrm{T}\perp}]^{-\mathrm{T}}.$$

Step 5 With the obtained H_1 and J_1 in Step 4, solve (20) for a new controller and γ. With this controller, solve (23) again to get a smaller γ, and matrices H and J.

Step 6 Turn to Step 4 with the new matrices H_1 and J_1 getting from H and J. Repeat Step 4 again to minimize γ. To get a desirable controller and γ by alternately iterative searching between Steps 1–3 and Steps 4–6.

Of course, one can also establish an algorithm for H_∞ control based on the separation of Lyapunov matrix and control matrix as in Lemma 1 and Algorithm 2. Actually, the bounded real lemma condition (22) is equivalent to

$$\begin{bmatrix} M_{11}^3 & P\hat{B}_1 + \hat{C}_1^{\mathrm{T}}D_{11} & P\hat{B} + \hat{C}_1^{\mathrm{T}}\hat{D}_{12} \\ * & -\gamma^2 I + D_{11}^{\mathrm{T}}D_{11} & D_{11}^{\mathrm{T}}\hat{D}_{12} \\ * & * & \hat{D}_{12}^{\mathrm{T}}\hat{D}_{12} \end{bmatrix} - 2\Theta^{\mathrm{T}}X\Theta < 0, \tag{24}$$

where $M_{11}^3 = \mathrm{sym}\{P\hat{A}\} + \hat{C}_1^{\mathrm{T}}\hat{C}_1$, X is a diagonal positive definite matrix, and $\Theta = [\hat{K}\hat{C}\ \ \hat{K}\hat{D}_{21}\ \ -I]$. Noticing that Lyapunov matrix P and control matrix $\hat{K}$ are separated from each other in the above inequality, one can establish the following algorithm.

Algorithm 4

Step 1 Design a stabilizing controller $\hat{K}$ first by Algorithm 2 and solve the inequality

$$\begin{bmatrix} M_{11}^4 & P\hat{B}_1 + \hat{C}_1^{\mathrm{T}}D_{11} & P\hat{B} + \hat{C}_1^{\mathrm{T}}\hat{D}_{12} \\ * & -\gamma^2 I + D_{11}^{\mathrm{T}}D_{11} & D_{11}^{\mathrm{T}}\hat{D}_{12} \\ * & * & \hat{D}_{12}^{\mathrm{T}}\hat{D}_{12} \end{bmatrix} - \Omega^{\mathrm{T}}\Theta - \Theta^{\mathrm{T}}\Omega < 0, \tag{25}$$

where $M_{11}^4 = \mathrm{sym}\{P\hat{A}\} + \hat{C}_1^{\mathrm{T}}\hat{C}_1$, Θ is as given in (24) and $\Omega = [K_{01}\ \ K_{02}\ \ -X]$, to get γ, matrices $P > 0$, any matrices K_{01}, K_{02}, and a diagonal matrix $X > 0$.

Step 2 With the obtained matrices P, K_{01}, K_{02} and X, solve (23) again to get a smaller γ. Then, repeat Steps 1 and 2 to minimize γ. If γ is small enough, stop the algorithm. Otherwise, turn to Step 3.

Step 3 With the controller and γ obtained in Step

2, solve the following inequality

$$\begin{bmatrix} M_{11}^5 & Q\hat{C}_1^{\mathrm{T}}+\hat{B}_1D_{11}^{\mathrm{T}} & Q\hat{C}^{\mathrm{T}}+\hat{B}_1\hat{D}_{21}^{\mathrm{T}} \\ * & -\gamma^2I+D_{11}D_{11}^{\mathrm{T}} & D_{11}\hat{D}_{21}^{\mathrm{T}} \\ * & * & \hat{D}_{21}\hat{D}_{21}^{\mathrm{T}} \end{bmatrix} -$$
$$\varGamma\varOmega-\varOmega^{\mathrm{T}}\varGamma^{\mathrm{T}}<0, \qquad (26)$$

where $M_{11}^5=\hat{A}Q+Q\hat{A}^{\mathrm{T}}+\hat{B}_1\hat{B}_1^{\mathrm{T}},\ \varGamma=[\hat{K}^{\mathrm{T}}\hat{B}^{\mathrm{T}}\ \hat{K}^{\mathrm{T}}\hat{D}_{12}^{\mathrm{T}}\ \ -I]^{\mathrm{T}}$ and $\varOmega$ is as given in (25), to get γ and matrices $Q>0$, any matrices $K_{01},\ K_{02}$, and a diagonal matrix $X>0$.

Step 4 With the obtained $K_{01},\ K_{02}$ and X in Step 3, solve (26) again for a new controller $\hat{K}$ and γ. Repeat Steps 3 and 4 to minimize γ.

Remark 6 Similar to Theorem 3–i) and –ii), (26) can be viewed as a dual inequality of (25). Since the searching directions are different, an alternately iterative algorithm between Steps 1–3 and Steps 4–6 of Algorithm 3, or between Steps 1–2 and Steps 3–4 of Algorithm 4, may sometimes further reduce the design conservativeness.

The above method can also be extended to stable controller design for SPR control problems.

Theorem 4 A stable controller $\hat{K}$ as in (2) and (4) stabilizes system (17) and makes that $G_{\mathrm{cl}}(s)$ (if it is square) is SPR, if and only if there are a symmetric matrix $P>0$, and any matrices $G_1,\ G_2,\ F_1$ and F_2 such that

$$\begin{bmatrix} -\mathrm{sym}\{G\} & M_{12}^6 & G\hat{B}_1+\varPhi\hat{D}_{21} \\ * & M_{22}^6 & F\hat{B}_1+\varPsi\hat{C}-C_{\mathrm{cl}}^{\mathrm{T}} \\ * & * & -D_{\mathrm{cl}}-D_{\mathrm{cl}}^{\mathrm{T}} \end{bmatrix}<0, \quad (27)$$

where

$$M_{12}^6=G\hat{A}+P-F^{\mathrm{T}}+\varPhi\hat{C},$$
$$M_{22}^6=\mathrm{sym}\{F\hat{A}+\varPsi\hat{C}\},$$
$$\varPhi=[\hat{B}\ \ \hat{B}^{\perp}]G_1\hat{B}^{\mathrm{T}}\hat{B}\hat{K},$$
$$G=[\hat{B}\ \ \hat{B}^{\perp}][G_1\ \ G_2][\hat{B}\ \ \hat{B}^{\perp}]^{\mathrm{T}},$$
$$\varPsi=[\hat{B}\ \ \hat{B}^{\perp}]F_1\hat{B}^{\mathrm{T}}\hat{B}\hat{K},$$

and

$$F=[\hat{B}\ \ \hat{B}^{\perp}][F_1\ \ F_2][\hat{B}\ \ \hat{B}^{\perp}]^{\mathrm{T}}.$$

Proof Actually, the inequality (27) is just

$$\begin{bmatrix} -(G+G^{\mathrm{T}}) & GA_{\mathrm{cl}}+P-F^{\mathrm{T}} & GB_{\mathrm{cl}} \\ * & FA_{\mathrm{cl}}+A_{\mathrm{cl}}^{\mathrm{T}}F^{\mathrm{T}} & FB_{\mathrm{cl}}-C_{\mathrm{cl}}^{\mathrm{T}} \\ * & * & -D_{\mathrm{cl}}-D_{\mathrm{cl}}^{\mathrm{T}} \end{bmatrix}<0,$$

which is just equivalent to the bounded real lemma inequality[27]

$$\begin{bmatrix} PA_{\mathrm{cl}}+A_{\mathrm{cl}}^{\mathrm{T}}P & PB_{\mathrm{cl}}-C_{\mathrm{cl}}^{\mathrm{T}} \\ B_{\mathrm{cl}}^{\mathrm{T}}P-C_{\mathrm{cl}} & D_{\mathrm{cl}}-D_{\mathrm{cl}}^{\mathrm{T}} \end{bmatrix}<0.$$

QED.

Similar to Theorem 3-ii), a dual case can also be provided to (27).

Remark 7 SPR characteristic is closely related to absolute stability of Lur'e systems[38–40]. One can also design stable and reduced-order controllers for Lur'e systems by the method of this paper.

4 Controller constraint problems

Generally, one needs to solve bilinear matrix inequalities (BMIs) to design controllers in most of control problems. So far, only state feedback controller and full-order dynamic output feedback controller design problems can be transformed into LMIs. However, in such LMIs, effective constraints cannot be exerted on controllers and controllers with large parameters always appear. The algorithms given in the above sections can not only solve BMIs, but also can add norm constraints on the designed variables to get controllers with reasonable parameters. As discussed in Example 3, in Algorithms 1 and 3, one can add the following constrains:

$$\begin{bmatrix} \alpha_{\mathrm{G}}I & G \\ G^{\mathrm{T}} & I \end{bmatrix}>0, \qquad (28)$$

$$\begin{bmatrix} \alpha_{\mathrm{F}}I & F \\ F^{\mathrm{T}} & I \end{bmatrix}>0, \qquad (29)$$

and

$$\begin{bmatrix} \alpha_{\mathrm{K}}I & \hat{K} \\ \hat{K}^{\mathrm{T}} & I \end{bmatrix}>0 \qquad (30)$$

with some scalars $\alpha_{\mathrm{G}},\ \alpha_{\mathrm{F}}$ and α_{K}.

Similar constraints can be added in Algorithms 2 and 4.

Similar controller constraint problems can be considered on dynamic controllers.

Remark 8 In the general LMI method for the pole assignment problem, it is not easy to exert constraints on the state feedback controllers because the control matrix is solved from $K=YP^{-1}$. Similarly, in full-order controller design problems, controller parameters are eliminated by projection lemma[41] which results in that one cannot effectively exert controller constraints. For the new methods in this paper, one can add controller norm constraints easily as discussed above, even the bound constraint can be added for each controller parameters. As discussed in [14], state feedback controller design with a bound constraint for each parameters is also NP hard. Controller constraint problems can be solved by the iterative method in this paper.

5 Examples

Example 1 Consider system (1) with matrix data

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -3 & -2 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$C = [-2\ \ 1\ \ 2\ \ 1]$.

Now, A has two unstable eigenvalues $0.2881 + 0.6475\text{i}$ and $0.2881 - 0.6475\text{i}$. By taking the initial matrix $G_{11} = F_{11} = I_3,\ G_{21} = F_{21} = \text{diag}\{1,\ 1,\ 0\}$ and $\epsilon_1 = 1.1$, the first step of Algorithm 1 gives a second-order stable controller stabilizing the system,

$$A_k = \begin{bmatrix} 0 & 1.7351 \\ -1.6747 & -3.7258 \end{bmatrix},\ B_k = \begin{bmatrix} -0.0809 \\ -0.1492 \end{bmatrix},$$
$$C_k = [0.0862\ \ -0.4056],\ D_k = -0.9526.$$

On the other hand, one can also use Algorithm 2 to design stable controllers for this example. First, take an initial state feedback controller K_0 such that the real part of each eigenvalue of $\hat{A} + \hat{B}K_0$ is less than -0.25. Such an initial K_0 can be solved by the general LMI method as

$$K_0 = \begin{bmatrix} -1.4295 & -8.7890 & -5.8360 & -1.0760 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.75 \end{bmatrix}.$$

Then, iteratively search a stable controller by Algorithm 2. Unfortunately, this algorithm cannot get a stabilizing solution. The main reason lies in that $\hat{A}$ and $\hat{B}$ are both diagonal blocked, so K_0 given by the general LMI algorithm in MATLAB is also diagonal blocked. This results in that the searched controller $\hat{K}$ falls into a "diagonal blocked trap". To avoid such a "diagonal blocked trap", one can revise K_0 by adding nonzero elements in the skew-diagonal blocks of K_0 as

$$K_0 = \begin{bmatrix} -1.4295 & -8.789 & -5.836 & -1.076 & 0.1 & 0.2 \\ 1 & 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0.500 & 0 & 0 & 0 & -0.75 \end{bmatrix}.$$

With this new K_0, by taking $\epsilon_1 = 0.45$, the first step of Algorithm 2 gives a stable controller

$$A_k = \begin{bmatrix} 0 & 0.7878 \\ -0.6020 & -1.1587 \end{bmatrix},\ B_k = \begin{bmatrix} -0.0955 \\ -0.1230 \end{bmatrix},$$
$$C_k = [0.2409\ \ -0.3396],\ D_k = -0.7071.$$

Remark 9 Stable controller design has been studied by some authors[2–3, 8]. Compared with the existing methods for stable controller design, the new methods in this paper are much simpler and more generalized. Here, Algorithm 1 is based on a special structure of the new introduced slacked matrices by using the information of $\hat{B}$ and $\hat{B}^{\perp}$ Algorithm 2 is based on the technique of Lyapunov matrix P and control matrix $\hat{K}$ from [33]. Both Algorithms 1 and 2 heavily rely on the new stable controller structure (4). This controller structure is very effective, see the subsequent examples. Actually, Algorithms 1 and 2 can be viewed as two parallel algorithms, one pays attention to the initial choice of the introduce slack matrices which relax Lyapunov matrix free from control matrix, the other pays attention to the initial choice of control matrix. Example 1 shows that sometimes the initial G_{11} and F_{11} can be chosen as identity matrices, and G_{21} and F_{21} can be chosen as diagonal matrices with many zero elements. Of course, one can also first design a state feedback matrix as in Algorithm 2, then determine the initial G_{11}, G_{21}, and F_{11} and F_{21} by solving the second step of Algorithm 1. However, Example 1 shows that the initial state feedback matrix can easily fall into a 'diagonal blocked trap' which can be revised as in Example 1 by adding some nonzero elements, or can be avoided by the method of assigning complex eigenvalues.

Example 2 Consider two systems (13) with matrix data $A^1,\ B^1$ and C^1 are equal to $A,\ B$ and C as given in Example 1, and

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.8 & -3 & -2 \end{bmatrix},\ B^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C^2 = [1\ \ 1\ \ 2].$$

Now, A^1 has two unstable eigenvalues as in Example 1 and A^2 has one unstable eigenvalue 0.228. And, the orders of A^1 and A^2 are different. Let the order of controller be 2. By taking the initial matrices $G_{11}^1 = G_{11}^2 = F_{11}^1 = F_{11}^2 = I_3,\ G_{21}^1 = F_{21}^1 = \text{diag}\{1,\ 1,\ 0\}, G_{21}^2 = F_{21}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ and $\epsilon_1^1 = \epsilon_1^2 = 1.1$, the first step like algorithm 1 gives a second-order stable controller stabilizing two systems,

$$A_k = \begin{bmatrix} 0 & 1.4543 \\ -1.2332 & -3.1366 \end{bmatrix},\ B_k = \begin{bmatrix} -0.1344 \\ -0.1296 \end{bmatrix},$$
$$C_k = [-0.0207\ \ -0.7324],\ D_k = -0.9114.$$

On the other hand, one can also use algorithm like Algorithm 2 to design second-order stable controllers for this example. First, take initial state feedback controllers K_0^i such that the real part of each eigenvalue of $\hat{A}^i + \hat{B}^i K_0^i$ is less than -0.25. Similar to Example 1, K_0^i also fall into a "diagonal blocked trap", where they are given by

$$K_0^1 = \begin{bmatrix} -1.6033 & -9.2651 & -6.2799 & -1.2897 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.75 \end{bmatrix}$$

and

$$K_0^2 = \begin{bmatrix} -3.0272 & -1.9042 & -0.2796 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & -0.75 \end{bmatrix}.$$

In this case, algorithm like Algorithm 2 cannot give a stabilizing solution. By adding nonzero elements in the skew-diagonal blocks of K_0^i, revise them as

$$K_0^1 = \begin{bmatrix} -1.6033 & -9.2651 & -6.2799 & -1.2897 & 0.1 & 0.2 \\ 1 & 0 & 0 & 0 & -0.75 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & -0.75 \end{bmatrix}$$

and

$$K_0^2=\begin{bmatrix}-3.0272 & -1.9042 & -0.2796 & 0.1 & 0.2\\ 1 & 0 & 0 & -0.75 & 0\\ 0 & 0.5 & 0 & 0 & -0.75\end{bmatrix}.$$

With these new K_0^i, by taking $\epsilon_1^1=0.6$ and $\epsilon_1^2=0.1$, the first step of algorithm like Algorithm 2 gives a stable controller

$$A_k=\begin{bmatrix}0 & 0.5143\\ -0.40697 & -1.1831\end{bmatrix},\ B_k=\begin{bmatrix}0.0377\\ 0.0010\end{bmatrix},$$
$$C_k=[-0.0361\ \ 0.4113],\ D_k=-0.9319.$$

Remark 10 The NP hardness of simultaneous stabilization problems have been discussed by some authors[13–14,35]. A rank-one LMI method was given in [35] for simultaneous stabilization problem involving with Hurwitz criterion. By predetermining the stable controller structure as in (4), two new iterative LMI methods can be given like Algorithms 1 and 2. The above Examples 1 and 2 and the subsequent examples show that the new methods in this paper are very effective.

Example 3 Consider a two-cart-one-spring system (Benchmark problem) given in [42] with matrix data

$$A=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k/m_1 & k/m_1 & -c/m_1 & 0\\ k/m_2 & -k/m_2 & 0 & -c/m_2\end{bmatrix},$$
$$B=\begin{bmatrix}0\\ 0\\ 1/m_1\\ 0\end{bmatrix},\ B_1=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1/m_1 & 0 & 0\\ 0 & 1/m_2 & 0\end{bmatrix},$$

$C=C_1=(0,1,0,0)$ and $D_{11}=0,\ D_{12}=0,\ D_{21}=(0,0,1)$, where $m_1=m_2,\ k=2$ and $c=0.2$.

Here, design a second-order stable controller to stabilize the system and satisfy a prescribed H_∞ performance index. First, taking the initial matrix $G_{11}=F_{11}=I_3,\ G_{21}=F_{21}=\mathrm{diag}\{1,-1,0\}$ and $\epsilon_1=0.5$, the first step of Algorithm 1 gives a second-order stable controller stabilizing the system,

$$A_k=\begin{bmatrix}0 & 1.0198\\ -0.9958 & -2.6089\end{bmatrix},\ B_k=\begin{bmatrix}0.1180\\ 0.4875\end{bmatrix},$$
$$C_k=[0.1121\ \ -0.0402],\ D_k=-0.9480,$$

which achieves the H_∞ performance index of the closed-loop system $\gamma=6.0$.

In order to avoid computing with large parameters, one can further add constraints

$$\begin{bmatrix}10000I_6 & G\\ G^{\mathrm{T}} & I_6\end{bmatrix}>0$$

and

$$\begin{bmatrix}10000I_6 & F\\ F^{\mathrm{T}} & I_6\end{bmatrix}>0$$

in Algorithm 3. By the first three steps of Algorithm 3, one can get a new controller,

$$A_k=\begin{bmatrix}0 & 1.8004\\ -1.0092 & -1.8493\end{bmatrix},\ B_k=\begin{bmatrix}0.0202\\ 0.3954\end{bmatrix},$$
$$C_k=[0.0166\ \ -3.2712],\ D_k=-0.5989,$$

which achieves the H_∞ performance index of the closed-loop system $\gamma=2.77$. Further, by Steps 4–6 of Algorithm 3, one can get another controller with $\gamma=2.73$. But such an improvement is comparatively minor.

On the other hand, one can also use Algorithm 4 to design stable controllers for this example. First, take an initial state feedback controller K_0 such that $\hat{A}+\hat{B}K_0$ is stable. Unfortunately, a diagonal blocked K_0 appears as in Examples 1 and 2. By making some similar revisions as in Example 1, one can get an initial K_0 as

$$K_0=\begin{bmatrix}-6.9721 & 1.6741 & -3.9675 & -1.2586 & 0.1 & 0.2\\ 1 & 0 & 0 & 0 & -0.5 & 0\\ 0 & -1 & 0 & 0 & 0 & -0.5\end{bmatrix}.$$

With this K_0, by taking $\epsilon_1=0.5$, the first step of Algorithm 2 gives a stable controller. In order to avoid computing with large parameters, one can further add constraints

$$\begin{bmatrix}10000I_3 & K_{01}\\ K_{01}^{\mathrm{T}} & I_6\end{bmatrix}>0$$

and

$$\begin{bmatrix}10000I_3 & K_{02}\\ K_{02}^{\mathrm{T}} & I_3\end{bmatrix}>0$$

in Algorithm 4. However, with such an initial stable controller, one can not get a controller with $\gamma<3.2$.

Remark 11 Example 3 shows that Algorithm 3 may provide a better H_∞ controller compared with Algorithm 4. Since the introduction of slack matrices G and F, Algorithm 3 has good robustness against the choice of initial matrices G_1 and F_1. Comparatively, Algorithm 4 has less matrix variables to solve. In addition, a nonsmooth H_∞ control approach was given in [18], which can be used to simultaneous stabilization, decentralized control and fixed-order controller design problems without involving Lyapunov function. The methods in this paper are based on the traditions LMI methods. The new techniques presented here effectively simply the simultaneous stabilization and stable reduced-order H_∞ control problems. The new methods are very simple and can be used to deal with other control problems such as decentralized control and static output feedback control problems[27–28,36–37].

Example 4 Consider the system matrices given in Example 3. Here, consider the pole assignment problems by state feedback controllers. First, solve the following inequality:

$$AP+PA^{\mathrm{T}}+BY+Y^{\mathrm{T}}B^{\mathrm{T}}+\epsilon_0P<0 \qquad (31)$$

with $P > 0$ and $\epsilon_0 = 2.1$. One can get a controller $K = YP^{-1} = (-123.7857 \ \ -13.4501 \ \ -17.5409 \ \ -113.4392)$. Clearly, the parameters in this K are very large. In this case, the real part of each eigenvalue of $A + BK$ is less than -1.39.

On the other hand, add a new constraint (30) in the first step of Algorithm 1 with $\alpha_{\mathrm{K}} = 1000$ and two new constraints (28) and (29) in the second step of Algorithm 1 with $\alpha_{\mathrm{G}} = \alpha_{\mathrm{F}} = 10000$. Substitute $\hat{A}$, $\hat{B}$, $\hat{C}$ by A, B and I, and taking initial matrices $G_{11} = F_{11} = 1$ and $G_{21} = F_{21} = (1 \ \ 0 \ \ 0)^{\mathrm{T}}$. Then, the first step of Algorithm 1 gives a state feedback stabilizing solution with $\epsilon_1 = 0.5$. And one can get some new matrices G and F by the second step with $\epsilon_2 = 0.1$. Through several iterative steps by decreasing ϵ_1 and ϵ_2, and finally by taking $\epsilon_1 = \epsilon_2 = -1.8$, one can get a new controller $K = (-20.7978 \ \ 11.3897 \ \ -8.2325 \ \ -7.2084)$. As one can see, the controller parameters are much smaller than the ones from directly solving LMI (31). And in this case, the real part of each eigenvalue of $A + BK$ is less than -1.4.

6 Conclusions

A new stable controller structure has been presented, which is not so conservative based on the understanding of controller robustness. Combining with a new structure on the introduce slack matrix variables and the separation technique from the two stage algorithms, two new methods have been proposed for strong stabilization problems. Compared with the existing methods, the new methods are simple and easy to be generalized to simultaneous stabilization of several plants, H_∞ control and SPR control problems and decentralized controller design. Stable controllers with any fixed-order can be effectively designed by the new iteratively, even the first step of the proposed algorithms can give stable solutions without any iterative steps for several examples. Controller parameter bound constraints can be easily exerted in the new algorithms. Some related NP hard problems have been discussed and can be partly solved by the mew methods.

References:

[1] VIDYASAGAR M. *Control System Synthesis: A factorization Approach*. Cambridge, MA: MIT Press, 1985.

[2] AN G H O. Strong stabilization of a class of MIMO systems. *IEEE Transactions on Automatic Control*, 2011, 56(6): 1445 – 1452.

[3] XIN X, LIU Y. Reduced-order stable controllers for two-link underactuated plana robots. *Automatica*, 2013, 49(7): 2176 – 2183.

[4] IAN R P. Robust H_∞ control of an uncertain system via a strict bounded real output feedback controller. *Optimal Control Applications and Methods*, 2009, 30(3): 247 – 266.

[5] ZHU Y, YANG F, LI C, et al. Strong γ_{c}–$\gamma_{\mathrm{c}1}$ H_∞ stabilization for networked control systems under denial of service attacks. *Journal of the Franklin Institute*, 2019, 356(5): 2723 – 2741.

[6] PETER D, PARK H B. An algorithm with interpolation in units in H_∞ with applications to feedback stabilization. *Automatica*, 1989, 25(3): 427 – 430.

[7] CHOU Y S, LEU J L, CHU Y C. Stable controller design for MIMO systems: An LMI method. *IET Control Theory & Applications*, 2007, 1(3): 817 – 829.

[8] DU B, LAM J, SHU Z. Strong stabilization by output feedbak controllr for linear systems with delayed input. *IET Control Theory & Applications*, 2011, 6(10): 1329 – 1340.

[9] WAKAIKI M, YAMAMOTO Y, OZBAY H. Sensitivity reduction by strongly stabilizing controllers for MIMO distributed parameter systems. *IEEE Transactions on Automatic Control*, 2011, 57(8): 2089 – 2094.

[10] ÜNAL H U, İFTAR A. Stable H_∞ controller design for systems with multiple input/output time-delays. *Automatica*, 2012, 48(3): 563 – 568.

[11] WAKAIKI M, YAMAMOTO Y. Stable controller design for mixed sensitivity reduction of infinite-dimensional systems. *Systems & Control Letters*, 2014, 72: 80 – 85.

[12] LIU Y, XIN X. Stable stabilizing controllers for remotely driven acrobot: Existence and design methods. *SICE Journal of Control, Measurement, and System Integration*, 2015, 8(6): 396 – 403.

[13] BLONDEL V, GEVERS M. Simultaneous stabilizability question of three linear systems is rationally undecidable. *Mathematics of Control, Signals and Systems*, 1993, 6(2): 135 – 145.

[14] BLONDEL V, TSITSIKLIS J N. NP-hardness of some linear control design problems. *SIAM Journal on Control and Optimization*, 1997, 35(6): 2118 – 2127.

[15] ZHU Y, YANG F, HAN Q L. Simultaneous H_∞ stabilisation for distributed networked multimode control systems with multiple packet dropouts. *IET Control Theory & Applications*, 2016, 10(6): 625 – 636.

[16] WANG H, YU T, XIAO J. Simultaneously stabilising controllers for time-varying linear systems within the framework of nest algebras. *International Journal of Control*, 2016, 89(8): 1632 – 1640.

[17] WU J, CHANG Y , CHEN S. Design of simultaneous static output feedback low-gain H_∞ controllers for a collection of time-delay system. *Optimal Control Applications and Methods*, 2017, 38(6): 973 – 981.

[18] APKARIAN P, NOLL D. Nonsmooth H_∞ synthesis. *IEEE Transactions on Automatic Control*, 2006, 51(1): 71 – 86.

[19] CHAO Y, SUN Y. Simultaneous stabilization: inverse LQ approach. *Control Theory & Applications*, 1999, 16(6): 807 – 810.

[20] CHENG Chuwang, SUN Youxian, GUO Zhaohui. Simultaneously strong stabilization of two plants – A New npproach ssing geometric terms. *Control Theory & Applications*, 1997, 14(5): 738 – 741.
(程储旺, 孙优贤, 郭朝辉. 两个对象的同时强镇定——一种基于几何知识的新方法. 控制理论与应用, 1997, 14(5): 738 – 741.)

[21] GUAN Qiang, HE Guannan, WANG Long, et al. Simultaneous stabilization of linear systems. *Control Theory & Applications*, 2011, 28(1): 1 – 12.
(关强, 何冠男, 王龙, 等. 线性系统的同时镇定问题. 控制理论与应用, 2011, 28(1): 1 – 12.)

[22] KHATIBI H, KARIMI A, LONGCHAMP R. Fixed-order controller design for polytopic systems using LMIs. *IEEE Transactions on Automatic Control*, 2008, 53(1): 428 – 434.

[23] GRIGORIADIS K M, SKELTON R E. Low-order control design for LMI problems using alternating projection methods. *Automatica*, 1996, 32(8): 1117 – 1125.

[24] YANG F, GANI M, HENRION D. Fixed-order robust H_∞ controller design with regionnal pole assignment. *IEEE Transactions on Automatic Control*, 2007, 52(10): 1959 – 1963.

[25] ZHOU K, DOYLE J C, GLOVER K. *Robust and Optimal Control*. New Jersey: Prentice hall, 1996.

[26] GUO Lei, XIN Xin, FENG Chunbo. A linear–matrix–inequality based design method of reduced-order controllers for singular H_∞ control problems. *Control Theory & Applications*, 1996, 13(6): 709 – 716.
(郭雷, 忻欣, 冯纯伯. 线性系统的同时镇定问题. 控制理论与应用, 1996, 13(6): 709 – 716.)

[27] ZHAI G, IKEDA M, FUJISAKI Y. Decentralized H_∞ controller Design: A matrix inequality design using a homotopy method. *IEEE Transactions on Automatic Control*, 2001, 37(4): 565 – 572.

[28] DUAN Z, JIANG Z, HUANG L. A new decentralised controller design method for a class of strongly interconnected systems. *International Journal of Control*, 2017, 90(2): 201 – 217.

[29] DUAN Z, ZHANG J, ZHANG C, et al. Robust H_2 and H_∞ filtering for uncertain linear systems. *Automatica*, 2006, 42(11): 1919 – 1926.

[30] DE OLIVEIRA M C, BERNUSSOU J, GEROMEL J C. A new discrete-time robust stability condition. *Systems & Control Letters*, 1999, 37(4): 261 – 265.

[31] PEAUCELLE D, ARZELIER D, BACHELIER O, et al. A new robust D–stability condition for real convex polytopic uncertainty. *Systems & Control Letters*, 2000, 40(1): 21 – 30.

[32] GAHINET P, NEMIROVSKI A, LAUB A J, et al. *LMI Control Toolbox for Use with MATLAB*. Natick, MA: The MathWorks, Inc, 1995.

[33] ARZELIER D, GRYAZINA E N, PEAUCELLE D, et al. Mixed LMI/Randomized methods for static output feedback control design. *Proceedings of the 2010 American Control Conference*. Baltimore: IEEE, 2010: 4683 – 4688.

[34] LI X, GAO H. A heuristic approach to static output feedback controller synthesis with restricted frequency-domain specifications. *IEEE Transactions on Automatic Control*, 2014, 39(4): 1008 – 1014.

[35] HENRION D, TARBOURIECH S, ŠEBEK M. Rank-one LMI approach to simultaneous stabilization of linear systems. *Systems & Control Letters*, 1999, 38(2): 79 – 89.

[36] SILJAK D D. *Decentralized Control of Complex Systems*. New York: Academic, 1991.

[37] HAO Y, DUAN Z. Static output-feedback controller synthesis with restricted frequency domain specifications for time-delay systems. *IET Control Theory & Applications*, 2015, 9(10): 1608 – 1614.

[38] PARK P. A revisited Popov criterion for nonlinear Lur'e systems with sector-restrictions. *International Journal of Control*, 1997, 68(3): 461 – 469.

[39] SMIRNOVA V B. *Frequency-Domain Methods for Nonlinear Analysis: Theory and Applications*. Singapore: World Scientific, 1996.

[40] WANG J, DUAN Z, YANG Y. *Analysis and Control of Nonlinear Systems with Stationary Sets: Time-Domain and Frequency-Domain Methods*. Singapore: World Scientific, 2009.

[41] IWASAKI T, SKELTON R E. All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas. *Automatica*, 1994, 30(8): 1307 – 1317.

[42] WIE B, BERNSTEIN D S. Benchmark problems for robust control design. *Journal of Guidance, Control, and Dynamics*, 1992, 15(5): 1057 – 1059.

作者简介:

段志生 教授, 第7届“关肇直奖”(2001年)获奖论文作者, 国家杰出青年基金获得者, 获得2015年国家自然科学二等奖, 2013年任长江学者特聘教授, 目前研究方向为鲁棒控制、关联系统稳定性、复杂动态网络分析与控制等, E-mail: duanzs@pku.edu.cn;

郝宇清 讲师, 目前研究方向为复杂网络分析与控制、机械臂控制、鲁棒控制等, E-mail: haoyq@buaa.edu.cn.