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不确定Euler-Bernoulli梁方程的边界输出跟踪

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摘要: 本文讨论边界带有控制输入非同位的内部不确定和外部扰动的Euler-Bernoulli梁方程输出跟踪问题. 为处理边界干扰, 文章首先设计了一个新的总扰动估计器, 在线估计未知扰动. 其次基于估计出来的总扰动, 设计一个伺服系统跟踪参考信号. 最后根据自抗扰方法获得控制输出跟踪的反馈控制. 闭环系统被证明是适定和有界的, 且受控系统的输出指数跟踪参考信号.

关键词: Euler-Bernoulli梁方程; 输出跟踪; 干扰抑制; 边界控制

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Output tracking for an Euler-Bernoulli beam equation with boundary uncertainty

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Abstract: This paper discusses output tracking problem on a boundary controlled Euler-Bernoulli beam equation suffered non-collocated unknown internal nonlinear uncertainty and external disturbance. To deal with boundary disturbance, we firstly design a novel disturbance estimator to estimate successfully the total disturbance. A servomechanism based on the estimated total disturbance is designed to track the reference signal. Finally, we obtain an output tracking controller by using active disturbance rejection method. It is shown that the closed-loop system is well-posed and bounded. Moreover, the performance output can track the reference signal exponentially.

Key words: Euler-Bernoulli beam equation; output tracking; disturbance rejection; boundary control

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1 引言

输出调节或伺服机制设计是控制理论的终极目标. 输出调节是设计一个输出反馈控制以使闭环系统的性能输出渐近跟踪参考信号而不受外部干扰和初始状态的影响, 同时使得闭环系统的所有子系统保持有界. 上世纪70年代, E.J. Davison[1], B.A. Francis与W. M. Wonham[2]对有限维系统发展了输出调节的内模原理. 在过去近半个世纪中, 大量的工作致力于将经典的有限维系统的输出调节的内模原理[1–6]推广到非线性系统[7]以及无限维系统[8–15]. 一方面, 虽然输出调节的内模原理允许参考信号和扰动由无穷维外系统生成, 但内模原理的干扰信号是几乎已知的. 而且, 绝大多数无穷维系统的输出跟踪问题考虑的控制和观测算子都是有界的. 另一方面, 在无穷维系统中, 性能输出跟踪问题远未得到充分解决. 2016年, 文献[16]率先将无扰动的有限维线性系统[17]的性能输出跟踪问题推广至具有未知的一般调和扰动的一维波动方程, 其设计了自适应跟踪控制器使得参考信号和输出之间的误差渐近收敛于零. 文献[19–20]针对具有未知一般有界扰动的一维波动方程设计了指数跟踪控制器得到文献[16]的两个改进的结果. 最近, 文献[16]中关于波动方程指数稳定性假设文献[21–22]被去除, 分别设计了伺服控制器使得控制输出调节到零和控制输出跟踪上调和信号. 不过文献[21–22]的结果都被最近基于观测器的偏微分系统的内模原理[18]所覆盖, 且结果要深刻的多. 关于一般有界干扰的输出调节要困难的多. 文献[26–27]分别对一维热方程和高维热方程在一般有界干扰下的一般参考信号的性能

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输出跟踪问题有很大的进展, 但控制和输出的信号必须是同位的.

最近, 文献[28]中研究了Euler-Bernoulli梁方程的输出跟踪问题, 且边界带有外部扰动, 其控制方式是通过边界弯矩控制. 另外一方面, 边界剪切力控制是Euler-Bernoulli梁方程更容易实现的控制方式, 如文献[29]中关于稳定性的讨论. 受以上文献启发, 本文考虑的是如下Euler-Bernoulli梁方程同时带有外部扰动和内部扰动通过剪切力控制的性能输出跟踪问题:

$$\begin{cases} y_{tt}(x,t)+y_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ y(0,t)=y_x(0,t)=0,\ t\geqslant 0,\\ y_{xx}(1,t)=-qy_{xt}(1,t)+f(y,y_t)+d(t),\\ y_{xxx}(1,t)=u(t),\\ y_m(t)=\{y_x(1,t),y_t(1,t)\},\\ e(t)=y_{\mathrm{ref}}-y_{\mathrm{out}}=y_{\mathrm{ref}}(t)-y(1,t), \end{cases} \tag{1}$$

其中: $y_x(x,t)$表示$y(x,t)$关于变量x的导数, $y_t(x,t)$表示$y(x,t)$关于变量t的导数, $u(t)$表示控制输入, $y_{\mathrm{out}}(t)$表示待调节的输出, $y_{\mathrm{ref}}(t)$表示参考信号, $d(t)$表示未知的干扰, q表示某个正的常数, 即$q>0$. 令$H_E^2(0,1)=\{\phi\in H^2(0,1):\phi(0)=\phi'(0)=0\}$. 系统(1)的状态空间取为Hilbert空间$\mathcal{H}=H_E^2(0,1)\times L^2(0,1)$, 其内积定义为对任意的$(\phi_i,\psi_i)^{\mathrm{T}}\in\mathcal{H}$, $i=1,2$,

$$\langle(\phi_1,\psi_1)^{\mathrm{T}},(\phi_2,\psi_2)^{\mathrm{T}}\rangle_{\mathcal{H}}=\int_0^1[\phi_1''(x)\bar{\phi}_2''(x)+\psi_1(x)\bar{\psi}_2(x)]\mathrm{d}x.$$

$f:H_E^2(0,1)\times L^2(0,1)\to\mathbb{R}$是一个未知的非线性函数, $f(y,y_t)$表示系统内部不确定性. 给定参考信号$y_{\mathrm{ref}}(t)$, $e(t)$表示跟踪误差, 本文的目的是针对不确定系统(1)设计输出反馈控制器$u(t)$使得: 1) $\lim\limits_{t\to\infty}e(t)=0$; 2) 有界的干扰信号$d(t)$得到抑制; 3) 闭环系统的所有环节都是有界的; 4) 当$d(t)=y_{\mathrm{ref}}(t)=0$时, 闭环系统指数内稳.

本文结构如下, 第2部分设计总扰动估计器来估计未知的外部干扰和内部不确定, 第3部分设计伺服系统用于跟踪参考信号并给出反馈控制器设计, 第4部分证明闭环系统的适定性与稳定性, 最后给出全文的总结.

2 总扰动估计器设计

系统(1)的总扰动包括系统内部非线性的不确定性和外部的干扰. 本节的主要目的是基于观测输出$y_m(t)=\{y_x(1,t),y_t(1,t)\}$, 设计如下总扰动估计器系统用于估计总扰动“$f(y,y_t)+d(t)$”:

$$\begin{cases} v_{tt}(x,t)+v_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ v(0,t)=v_x(0,t)=0,\ t\geqslant 0,\\ v_x(1,t)=y_x(1,t),\\ v_{xxx}(1,t)=u(t)+c_0[v_t(1,t)-y_t(1,t)], \end{cases} \tag{2}$$

其中c_0是正的调节参数. 显然系统(2)完全由测量$y_x(1,t)$, $y_t(1,t)$和控制输入u所确定, 因此系统(2)完全可知. 下面要证明

$$v_{xx}(1,t)+qv_{xt}(1,t)\approx f(y,y_t)+d(t), \tag{3}$$

即未知的总扰动$f(y,y_t)+d(t)$可以用估计器系统已知的量$v_{xx}(1,t)+qv_{xt}(1,t)$来近似. 令

$$p(x,t)=v(x,t)-y(x,t), \tag{4}$$

则$p(x,t)$满足

$$\begin{cases} p_{tt}(x,t)+p_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ p(0,t)=p_x(0,t)=0,\ t\geqslant 0,\\ p_x(1,t)=0,\\ p_{xxx}(1,t)=c_0p_t(1,t). \end{cases} \tag{5}$$

系统(5)可以写成如下发展方程:

$$\frac{\mathrm{d}}{\mathrm{d}t}(p(\cdot,t),p_t(\cdot,t))^{\mathrm{T}}=A(p(\cdot,t),p_t(\cdot,t))^{\mathrm{T}}, \tag{6}$$

其中算子$A:\mathcal{D}(A)(\subset\mathcal{H})\to\mathcal{H}$按照如下方式定义:

$$\begin{cases} A(\phi,\psi)^{\mathrm{T}}=(\psi,-\phi'''')^{\mathrm{T}},\\ \mathcal{D}(A)=\{(\phi,\psi)^{\mathrm{T}}\in(H^4(0,1)\times H_E^2(0,1))\cap\mathcal{H}:\\ \qquad \phi'''(1)=c_0\psi(1),\ \phi'(1)=0\}. \end{cases} \tag{7}$$

下面要证明算子A生成状态空间$\mathcal{H}$上指数稳定的C_0–半群. 如此, 则对任意初值$(p(\cdot,0),p_t(\cdot,0))\in\mathcal{D}(A)$, 系统(5)的解满足$|p_{xx}(1,t)|\leqslant M\mathrm{e}^{-\mu t}$, 其中$M$, $\mu>0$. 从而

$$\begin{aligned} 0\approx p_{xx}(1,t)&=v_{xx}(1,t)-y_{xx}(1,t)=\\ &v_{xx}(1,t)+qy_{xt}(1,t)-f(y,y_t)-d(t)=\\ &v_{xx}(1,t)+qv_{xt}(1,t)-f(y,y_t)-d(t). \end{aligned}$$

这意味着式(3)有严格的依据. 下面证明A生成指数稳定的C_0–半群. 为此, 需要一些准备工作.

引理 1 设A如(7)所定义. 则A^{-1}是$\mathcal{H}$上的紧算子, 因此$\sigma(A)=\sigma_p(A)$, 即$\sigma(A)$只包含A的特征根.

证 根据A的定义及边界条件容易计算得到

$$A^{-1}(\phi,\psi)^{\mathrm{T}}=$$

$$(-\frac{1}{6}\int_0^x(x-\xi)^3\psi(\xi)\mathrm{d}\xi-\frac{3x^2-2x^3}{12}(\int_0^1\psi(\xi)\mathrm{d}\xi+$$

$c_0\phi(1)) + \dfrac{x^2}{4}\int_0^1(1-\xi)^2\psi(\xi)\mathrm{d}\xi, \phi(x))^{\mathrm{T}}$.

根据Sobolev嵌入定理, A^{-1}是$\mathcal{H}$上的紧算子.

引理 2 设A如(7)所定义. 设$\lambda = i\tau^2 \neq 0$, 则$\lambda \in \sigma_p(A)$当且仅当τ满足如下特征方程:

$$\tau(\cos\tau\sinh\tau + \sin\tau\cosh\tau) + ic_0(1-\cosh\tau\cos\tau) = 0,$$

并且λ对应的特征函数是$(\phi, \lambda\phi)^{\mathrm{T}}$, 其中

$$\begin{aligned}\phi(x) = {}&\tau[\cosh\tau(1-x) - \cos\tau(1-x) - \\ &\cosh\tau\cos\tau x + \cos\tau\cosh\tau x + \\ &\sinh\tau\sin\tau x + \sin\tau\sinh\tau x] + \\ &ic_0[-\sinh\tau(1-x) - \sin\tau(1-x) + \\ &\sinh\tau\cos\tau x + \sin\tau\cosh\tau x - \\ &\cosh\tau\sin\tau x - \cos\tau\sinh\tau x].\end{aligned}$$

证 不难验证$i\tau^2 \in \sigma(A)$当且仅当存在$\phi(x) \neq 0$满足:

$$\begin{cases}\phi^{(4)}(x) - \tau^4\phi(x) = 0, \\ \phi(0) = \phi'(0) = \phi'(1) = 0, \\ \phi'''(1) = i\tau^2 c_0\phi(1),\end{cases} \tag{8}$$

使得$(\phi(x), i\tau^2\phi(x))^{\mathrm{T}}$为对应于$i\tau^2$的特征函数. 从而剩下只需要求解方程(8). 首先, 方程

$$\begin{cases}\phi^{(4)}(x) - \tau^4\phi(x) = 0, \\ \phi(0) = \phi'(0) = 0,\end{cases} \tag{9}$$

具有如下形式解:

$$\phi(x) = c_1(\cosh\tau x - \cos\tau x) + c_2(\sinh\tau x - \sin\tau x),$$

其中c_1, c_2为待确定的常数. 从而

$$\begin{aligned}&\phi'''(x) = \\ &c_1(\tau^3\sinh\tau x - \tau^3\sin\tau x) + \\ &c_2(\tau^3\cosh\tau x + \tau^3\cos\tau x),\end{aligned}$$

代入边界条件$\phi'''(1) = i\tau^2 c_0\phi(1)$得到

$$\begin{aligned}&c_1[-\tau(\sinh\tau - \sin\tau) + ic_0(\cosh\tau - \cos\tau)] = \\ &c_2[\tau(\cosh\tau + \cos\tau) - ic_0(\sinh\tau - \sin\tau)],\end{aligned}$$

可以取

$$\begin{aligned}&c_1 = \tau(\cosh\tau + \cos\tau) - ic_0(\sinh\tau - \sin\tau), \\ &c_2 = -\tau(\sinh\tau - \sin\tau) + ic_0(\cosh\tau - \cos\tau),\end{aligned}$$

从而有

$$\begin{aligned}&\phi(x) = \\ &[\tau(\cosh\tau + \cos\tau) - ic_0(\sinh\tau - \sin\tau)]\cdot \\ &(\cosh\tau x - \cos\tau x) + [-\tau(\sinh\tau - \sin\tau) + \\ &ic_0(\cosh\tau - \cos\tau)]\cdot(\sinh\tau x - \sin\tau x) = \\ &\tau[\cosh\tau(1-x) - \cos\tau(1-x) - \cosh\tau\cos\tau x + \\ &\cos\tau\cosh\tau x + \sinh\tau\sin\tau x + \sin\tau\sinh\tau x]\cdot \\ &ic_0[-\sinh\tau(1-x) - \sin\tau(1-x) + \sinh\tau\cos\tau x + \\ &\sin\tau\cosh\tau x - \cosh\tau\sin\tau x - \cos\tau\sinh\tau x].\end{aligned} \tag{10}$$

由边界条件$\phi'(1) = 0$可以得到τ满足如下特征方程:

$$\begin{aligned}&\tau(\cos\tau\sinh\tau + \sin\tau\cosh\tau) + \\ &ic_0(1-\cosh\tau\cos\tau) = 0.\end{aligned} \tag{11}$$

证毕.

引理 3 设A如式(7)所定义. 存在A的一族特征值$\{\lambda_n, \bar{\lambda}_n\}$, $\lambda_n = i\tau_n^2$具有如下渐近表示:

$$\lambda_n = i(n-\frac{1}{4})^2\pi^2 - c_0 + \mathcal{O}(\frac{1}{n}). \tag{12}$$

此外, 设$(\phi_n, \lambda_n\phi_n)^{\mathrm{T}}$为对应于$\lambda_n = i\tau_n^2$的特征向量, 则有

$$F_n(x) = \frac{2\tau_n^{-3}}{\mathrm{e}^{\tau_n}}\begin{pmatrix}\phi_n''(x) \\ \lambda_n\phi_n(x)\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{-(n-\frac{1}{4})\pi x} - \sin(n-\frac{1}{4})\pi x + \cos(n-\frac{1}{4})\pi x \\ i\mathrm{e}^{-(n-\frac{1}{4})\pi x} + i\sin(n-\frac{1}{4})\pi x - i\cos(n-\frac{1}{4})\pi x\end{pmatrix} + \mathcal{O}(\frac{1}{n}),$$

且有$\lim\limits_{n\to\infty}\|F_n\|^2_{L_2\times L_2} = 2$.

证 由特征方程(11)两边同时除以e^{τ}, 当$\mathrm{Im}\tau$有界以及$\mathrm{Re}\tau \to \infty$时有

$$\sin\tau + \cos\tau = \frac{ic_0}{\tau}\cos\tau + \mathcal{O}(\mathrm{e}^{-\mathrm{Re}\tau}), \tag{13}$$

进一步写成

$$\sin\tau + \cos\tau = \mathcal{O}(\frac{1}{|\tau|}). \tag{14}$$

等式(14)可以等价地写成

$$\sin(2\tau) = -1 + \mathcal{O}(\frac{1}{|\tau|^2}). \tag{15}$$

于是可以渐近地取

$$\tau_n = (n-\frac{1}{4})\pi + \mathcal{O}(\frac{1}{n}).$$

将τ_n代入方程(13)得到

$$\begin{aligned}&\text{左边} = \\ &\sin[(n-\frac{1}{4})\pi + \mathcal{O}(\frac{1}{n})] + \cos[(n-\frac{1}{4})\pi + \mathcal{O}(\frac{1}{n})] = \\ &\sin(n-\frac{1}{4})\pi + [\cos(n-\frac{1}{4})\pi]\mathcal{O}(\frac{1}{n}) - \\ &[\sin(n-\frac{1}{4})\pi]\mathcal{O}(\frac{1}{n^2}) + \cos(n-\frac{1}{4})\pi -\end{aligned}$$

$$[\sin(n-\frac{1}{4})\pi]\mathcal{O}(\frac{1}{n})-[\cos(n-\frac{1}{4})\pi]\mathcal{O}(\frac{1}{n^2})+o(\mathcal{O}(\frac{1}{n^2}))=$$
$$[\cos(n-\frac{1}{4})\pi-\sin(n-\frac{1}{4})\pi]\mathcal{O}(\frac{1}{n})+o(\mathcal{O}(\frac{1}{n^2})),\tag{16}$$

右边 $=$

$$\frac{ic_0}{(n-\frac{1}{4})\pi+\mathcal{O}(\frac{1}{n})}[\cos(n-\frac{1}{4})\pi-[\sin(n-\frac{1}{4})\pi]\mathcal{O}(\frac{1}{n})+o(\mathcal{O}(\frac{1}{n^2}))]+\mathcal{O}(\mathrm{e}^{-\mathrm{Re}\tau})=$$
$$\frac{ic_0}{(n-\frac{1}{4})\pi}[1-\frac{\mathcal{O}(\frac{1}{n})}{(n-\frac{1}{4})\pi}+o(\mathcal{O}(\frac{1}{n^2}))]\cdot[\cos(n-\frac{1}{4})\pi-[\sin(n-\frac{1}{4})\pi]\mathcal{O}(\frac{1}{n})+o(\mathcal{O}(\frac{1}{n}))]+\mathcal{O}(\mathrm{e}^{-\mathrm{Re}\tau}).\tag{17}$$

注意到$\sin(n-\frac{1}{4})\pi$与$\cos(n-\frac{1}{4})\pi$互为相反数且取值于$\{\pm\frac{\sqrt{2}}{2}\}$, 综合式(16)与式(17)可得

$$\mathcal{O}(\frac{1}{n})=\frac{ic_0}{2(n-\frac{1}{4})\pi}+\mathcal{O}(\frac{1}{n^2}).$$

从而有

$$\lambda_n=i\tau_n^2=i(n-\frac{1}{4})^2\pi^2-c_0+\mathcal{O}(\frac{1}{n}).$$

这就是式(12).

由式(10)可计算得

$$\begin{aligned}\frac{1}{\tau^2}\phi''(x)=\tau[&\cosh\tau(1-x)+\cos\tau(1-x)+\\&\cosh\tau\cos\tau x+\cos\tau\cosh\tau x-\\&\sinh\tau\sin\tau x+\sin\tau\sinh\tau x]\cdot\\&ic_0[-\sinh\tau(1-x)+\sin\tau(1-x)-\\&\sinh\tau\cos\tau x+\sin\tau\cosh\tau x+\\&\cosh\tau\sin\tau x-\cos\tau\sinh\tau x].\end{aligned}\tag{18}$$

等式(18)两边同时除以$\frac{\tau\mathrm{e}^{\tau}}{2}$, 当$\mathrm{Im}\tau$有界以及$\mathrm{Re}\tau\to\infty$时有

$$\frac{2\tau^{-3}}{\mathrm{e}^{\tau}}\phi''(x)=\mathrm{e}^{-\tau x}+\cos\tau x-\sin\tau x+(\sin\tau+\cos\tau)\mathrm{e}^{-\tau(1-x)}+\mathcal{O}(\mathrm{e}^{-\mathrm{Re}\tau}).\tag{19}$$

将$\tau=\tau_n=(n-\frac{1}{4})\pi+\mathcal{O}(\frac{1}{n}), \phi(x)=\phi_n(x)$代入等式(19), 注意到当$y>0$以及$0\leqslant x\leqslant 1$时有

$$\mathrm{e}^{-\tau_n y}=\mathrm{e}^{-(n-\frac{1}{4})\pi y}+\mathcal{O}(\frac{1}{n}),$$
$$\sin\tau_n x=\sin(n-\frac{1}{4})\pi x+\mathcal{O}(\frac{1}{n}),$$
$$\cos\tau_n x=\cos(n-\frac{1}{4})\pi x+\mathcal{O}(\frac{1}{n}).$$

从而有

$$\begin{aligned}&\frac{2\tau_n^{-3}}{\mathrm{e}^{\tau_n}}\phi_n''(x)=\\&\mathrm{e}^{-(n-\frac{1}{4})\pi x}+\cos(n-\frac{1}{4})\pi x-\sin(n-\frac{1}{4})\pi x+\\&[\sin(n-\frac{1}{4})\pi+\cos(n-\frac{1}{4})\pi]\cdot\\&\mathrm{e}^{-(n-\frac{1}{4})\pi(1-x)}+\mathcal{O}(\frac{1}{n})=\\&\mathrm{e}^{-(n-\frac{1}{4})\pi x}-[\sin(n-\frac{1}{4})\pi x-\\&\cos(n-\frac{1}{4})\pi x]+\mathcal{O}(\frac{1}{n}).\end{aligned}\tag{20}$$

同理可以得到

$$\frac{2\lambda_n\tau_n^{-3}}{\mathrm{e}^{\tau_n}}\phi_n(x)=i\mathrm{e}^{-(n-\frac{1}{4})\pi x}+i[\sin(n-\frac{1}{4})\pi x-\cos(n-\frac{1}{4})\pi x]+\mathcal{O}(\frac{1}{n}).\tag{21}$$

直接计算不难验证$\lim\limits_{n\to\infty}\|F_n(x)\|^2_{L_2\times L_2}=2$.

证毕.

下面将证明A的广义特征向量构成状态空间$\mathcal{H}$的一个Riesz基. 为此, 回顾一个有用的定理[23].

定理 1 设$\mathcal{A}$是一个Hilbert空间H上的一个稠定紧算子. 设$\{\phi_n\}_1^\infty$为H的一个Riesz基. 如果存在自然数$N\geqslant 0$及A的一列广义特征向量$\{\psi_n\}_{N+1}^\infty$使得

$$\sum_{n=N+1}^{\infty}\|\phi_n(x)-\psi_n(x)\|^2<\infty,$$

则

1) 存在自然数$M>N$及$\mathcal{A}$的广义特征向量$\{\psi_{n_0}\}_1^M$使得$\{\psi_{n_0}\}_1^M\cup\{\psi_n\}_{N+1}^\infty$构成$H$的一个Riesz基.

2) 设H的一个Riesz基$\{\psi_{n_0}\}_1^M\cup\{\psi_n\}_{N+1}^\infty$所对应的$A$的特征值记为$\{\lambda_n\}_1^\infty$, 则$\sigma(\mathcal{A})=\{\lambda_n\}_1^\infty$.

定理 2 设A如式(7)所定义. 则A的广义特征向量构成状态空间$\mathcal{H}$的一个Riesz基.

证 定义辅助系统

$$\begin{cases}\tilde p_{tt}(x,t)+\tilde p_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\\tilde p(0,t)=\tilde p_x(0,t)=0,\ t\geqslant 0,\\\tilde p_x(1,t)=0,\\\tilde p_{xxx}(1,t)=0.\end{cases}\tag{22}$$

系统(22)可以写成如下发展方程:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{p}(\cdot,t),\tilde{p}_t(\cdot,t))^{\mathrm{T}}=\tilde{A}(\tilde{p}(\cdot,t),\tilde{p}_t(\cdot,t))^{\mathrm{T}},\quad(23)$$

其中算子$\tilde{A}:\mathcal{D}(\tilde{A})(\subset\mathcal{H})\to\mathcal{H}$按照如下方式定义:

$$\begin{cases}\tilde{A}(\tilde{\phi},\tilde{\psi})^{\mathrm{T}}=(\tilde{\psi},-\tilde{\phi}'''')^{\mathrm{T}},\\ \mathcal{D}(\tilde{A})=\{(\tilde{\phi},\tilde{\psi})^{\mathrm{T}}\in(H^4(0,1)\times H_E^2(0,1))\cap\mathcal{H}:\\ \qquad\tilde{\phi}'''(1)=0,\ \tilde{\phi}'(1)=0\},\end{cases}\quad(24)$$

则$\tilde{A}$具有如下简单但非常重要的性质: 1) $\tilde{A}$是反自伴的; 2) $\tilde{A}^{-1}$是一个紧算子; 3) $\tilde{A}$的所有特征值落在虚轴上; 4) 存在一列$\tilde{A}$的广义特征值构成状态空间$\mathcal{H}$的一个Riesz基.

类似于引理2的讨论, 设$\tilde{\lambda}=i\tilde{\tau}^2\neq0$, 则$\tilde{\lambda}\in\sigma(\tilde{A})$当且仅当存在唯一特征向量$(\tilde{\phi},\tilde{\lambda}\tilde{\phi})$, 其中

$$\begin{aligned}\tilde{\phi}(x)=\tilde{\tau}[&\cosh\tilde{\tau}(1-x)-\cos\tilde{\tau}(1-x)-\\&\cosh\tilde{\tau}\cos\tilde{\tau}x+\cos\tilde{\tau}\cosh\tilde{\tau}x+\\&\sinh\tilde{\tau}\sin\tilde{\tau}x+\sin\tilde{\tau}\sinh\tilde{\tau}x].\end{aligned}$$

$\tilde{\tau}$满足如下特征方程:

$$\cos\tilde{\tau}\sinh\tilde{\tau}+\sin\tilde{\tau}\cosh\tilde{\tau}=0.$$

同样类似于引理3的讨论, 存在$\tilde{A}$的一族特征值$\{\tilde{\lambda}_n,\bar{\tilde{\lambda}}_n\}$, $\tilde{\lambda}_n=i\tilde{\tau}_n^2$具有如下渐近表示:

$$\tilde{\tau}_n=(n-\frac{1}{4})\pi+\mathcal{O}(\frac{1}{n}).\quad(25)$$

进一步地有

$$\tilde{\lambda}_n=i(n-\frac{1}{4})^2\pi^2+\mathcal{O}(\frac{1}{n}).\quad(26)$$

设$(\tilde{\phi}_n,\tilde{\lambda}_n\tilde{\phi}_n)^{\mathrm{T}}$为对应于$\tilde{\lambda}_n=i\tilde{\tau}_n^2$的特征向量, 则有

$$\begin{aligned}&G_n(x)=\frac{2\tilde{\tau}_n^{-3}}{\mathrm{e}^{\tilde{\tau}_n}}\begin{pmatrix}\tilde{\phi}_n''(x)\\ \tilde{\lambda}_n\tilde{\phi}_n(x)\end{pmatrix}=\\&\begin{pmatrix}\mathrm{e}^{-(n-\frac{1}{4})\pi x}-\sin(n-\frac{1}{4})\pi x+\cos(n-\frac{1}{4})\pi x\\ i\mathrm{e}^{-(n-\frac{1}{4})\pi x}+i\sin(n-\frac{1}{4})\pi x-i\cos(n-\frac{1}{4})\pi x\end{pmatrix}+\\&\mathcal{O}(\frac{1}{n}),\end{aligned}$$

且有$\lim\limits_{n\to\infty}\|G_n(x)\|_{L_2\times L_2}^2=2$. 不难验证

$$\{(\tilde{\phi}_n,\tilde{\lambda}_n\tilde{\phi}_n)^{\mathrm{T}}\}_{n=1}^{\infty}\cup\{(\bar{\tilde{\phi}}_n,\bar{\tilde{\lambda}}_n\bar{\tilde{\phi}}_n)^{\mathrm{T}}\}_{n=1}^{\infty}$$

构成状态空间$\mathcal{H}$的一个Riesz基.

从而, 存在充分大的自然数$N>0$使得

$$\begin{aligned}&\sum_{n>N}^{\infty}\|F_n(x)-G_n(x)\|_{L_2\times L_2}^2=\\&\sum_{n>N}^{\infty}\|\frac{2\tau_n^{-3}}{\mathrm{e}^{\tau_n}}\begin{pmatrix}\phi_n''(x)\\ \lambda_n\phi_n(x)\end{pmatrix}-\frac{2\tilde{\tau}_n^{-3}}{\mathrm{e}^{\tilde{\tau}_n}}\begin{pmatrix}\tilde{\phi}_n''(x)\\ \tilde{\lambda}_n\tilde{\phi}_n(x)\end{pmatrix}\|_{\mathcal{H}}^2=\\&\sum_{n>N}^{\infty}\mathcal{O}(\frac{1}{n^2})<\infty.\end{aligned}\quad(27)$$

对于共轭的情形具有相同的结论. 根据上面的讨论以及定理1可知, A的广义特征向量构成状态空间$\mathcal{H}$的一个Riesz基. 证毕.

定理 3 设A如式(7)所定义. 则A生成状态空间$\mathcal{H}$上一个指数稳定的C_0–半群. 因此对任意初值$(p(\cdot,0),p_t(\cdot,t))^{\mathrm{T}}\in\mathcal{H}$, 系统(5)存在唯一解并且是指数稳定的.

证 由定理2可知, A的广义特征向量构成状态空间$\mathcal{H}$的一个Riesz基, 从而谱定增长条件成立. 为了证明A生成状态空间$\mathcal{H}$上一个指数稳定的C_0–半群, 只需证明$\mathrm{Re}\lambda<0$对任意的$\lambda\in\sigma(A)$. 事实上,

$$\mathrm{Re}\langle A(\phi,\psi)^{\mathrm{T}},(\phi,\psi)^{\mathrm{T}}\rangle_{\mathcal{H}}=-c_0|\psi(1)|^2\leqslant0,\quad(28)$$

且由性质1可知$\sigma(A)=\sigma_p(A)$, 从而$\mathrm{Re}\lambda\leqslant0$对任意的$\lambda\in\sigma(A)$. 下面只需证明$\sigma(A)\cap i\mathbb{R}=\varnothing$. 若不然, 设$\lambda=i\tau^2\in\sigma(A)$, $(\phi,\psi)^{\mathrm{T}}$为相应的特征向量, 其中$\tau\in\mathbb{R}^+$. 由方程(28)可知

$$\begin{aligned}0=&\mathrm{Re}\langle i\tau^2(\phi,\psi)^{\mathrm{T}},(\phi,\psi)^{\mathrm{T}}\rangle_{\mathcal{H}}=\\&\mathrm{Re}\langle A(\phi,\psi)^{\mathrm{T}},(\phi,\psi)^{\mathrm{T}}\rangle_{\mathcal{H}}=-c_0|\psi(1)|^2,\end{aligned}\quad(29)$$

即有$\psi(1)=0$. 由性质2可知, $\psi=i\tau^2\phi(x)$且

$$\begin{cases}\phi^{(4)}(x)-\tau^4\phi(x)=0,\\ \phi(0)=\phi'(0)=\phi(1)=\phi'(1)=\phi'''(1)=0.\end{cases}\quad(30)$$

然而, 具有边界条件的方程(30)只有零解. 这是一个矛盾. 最后剩下证明方程(30)只有零解. 首先证明ϕ在$(0,1)$内存在零点. 由$\phi(0)=\phi(1)$及Roll定理可知, 存在$\xi_1\in(0,1)$使得$\phi'(\xi_1)=0$. 类似地, 由$\phi'(0)=\phi'(\xi_1)=\phi'(1)$可知, 存在$\xi_2\in(0,\xi_1)$, $\xi_3\in(\xi_1,1)$使得$\phi''(\xi_2)=\phi''(\xi_3)=0$. 从而存在$\xi_4\in(\xi_2,\xi_3)$使得$\phi'''(\xi_4)=0$. 再由$\phi'''(\xi_4)=\phi'''(1)$可知, 存在$\xi_5\in(\xi_4,1)$使得$\phi^{(4)}(\xi_5)=0$. 根据式(31)的第一个方程可知, $\phi(\xi_5)=0$. 一般地, 设ϕ在$(0,1)$内存在n个零点, 即存在$0<\alpha_1<\cdots<\alpha_n<1$使得$\phi(0)=\phi(\alpha_1)=\cdots=\phi(\alpha_n)=\phi(1)=0$. 再次根据Roll定理可知, 存在$0<\beta_1<\alpha_1<\cdots<\alpha_n<\beta_{n+1}<1$使得$\phi'(\beta_j)=0$, $j=1,2,\cdots,n+1$. 结合$\phi'(0)=\phi'(1)=0$可知, 存在$0<\gamma_1<\beta_1<\gamma_2<\beta_2<\cdots<\beta_n<\gamma_{n+1}<\beta_{n+1}<\gamma_{n+2}<1$使得$\phi''(\gamma_j)=0$, $j=1,2,\cdots,n+2$. 从而存在$0<\gamma_1<\eta_1<\gamma_2<\eta_2<\cdots<\gamma_{n+1}<\eta_{n+1}<\gamma_{n+2}<1$使得$\phi'''(\eta_j)=0$, $j=1,2,\cdots,n+1$. 再结合$\phi'''(1)=0$可知, 存在$0<\eta_1<\theta_1<\eta_2<\cdots<\eta_n<\theta_n<\eta_{n+1}<\theta_{n+1}<1$使得$\phi^{(4)}(\theta_j)=0$, $j=1,2,\cdots,n+1$. 从而$\phi(\theta_j)=0$, $j=1,2,\cdots,n+1$. 这证明了ϕ在$(0,1)$内存在$n+1$个零点. 根据数学归纳

法, ϕ在$(0,1)$内存在无穷多个相异的零点$\{x_j\}_{j=1}^{\infty}$. 设$x_0\in[0,1]$是点列$\{x_j\}_{j=1}^{\infty}$的聚点. 则根据上面数学归纳法证明过程不难看出$\phi(x_0)=\phi'(x_0)=\phi''(x_0)=\phi'''(x_0)=0$. 从而, 根据常微分方程解的存在唯一性定理以及$\phi=0$是方程(30)的一个解,可知方程(31)有唯一解$\phi=0$.　　证毕.

定理 4　对任意初值$(p(\cdot,0),p_t(\cdot,0))^{\mathrm T}\in D(A)$, 系统(5)的解满足$|p_{xx}(1,t)|\leqslant M\mathrm e^{-\mu t}$, 其中$M$, $\mu>0$.

证　设$P(0):=(p(\cdot,0),p_t(\cdot,0))^{\mathrm T}\in D(A)$, 则$AP(0)\in\mathcal H$. 根据定理3, $\mathrm e^{At}$是指数稳定的半群. 因此系统(5)有唯一的解$P(t):=(p,p_t)=\mathrm e^{At}P(0)\in C(0,\infty;D(A))$并且存在常数$M_1$, $\mu_1>0$使得

$$\begin{cases}\|\mathrm e^{At}P(0)\|_{\mathcal H}\leqslant M_1\mathrm e^{-\mu_1 t}\|Z(0)\|_{\mathcal H},\\ \|\mathrm e^{At}AP(0)\|_{\mathcal H}\leqslant M_1\mathrm e^{-\mu_1 t}\|AZ(0)\|_{\mathcal H}.\end{cases}\tag{31}$$

因此

$$\|p(\cdot,t)\|_{L^2(0,1)}\leqslant\|P(t)\|_{\mathcal H}\leqslant M_1\mathrm e^{-\mu_1 t}\|Z(0)\|_{\mathcal H}.\tag{32}$$

另一方面, $AP(t)=A\mathrm e^{At}P(0)=\mathrm e^{At}AP(0)$, 根据$A$的定义, 则有

$$\|p_{xxxx}(\cdot,t)\|_{L^2(0,1)}\leqslant\|AP(t)\|_{\mathcal H}\leqslant M_2\mathrm e^{-\mu_1 t},\tag{33}$$

其中$M_2=M_1\|AP(0)\|_{\mathcal H}$. 根据Sobolev嵌入定理有

$$\|p_{xxx}(\cdot,t)\|_{L^2(0,1)}\leqslant C_1[\|p(\cdot,t)\|_{L^2(0,1)}+\|p_{xxxx}(\cdot,t)\|_{L^2(0,1)}],$$

其中$C_1\geqslant 1$. 由Sobolev迹定理, 存在常数$C_2>0$

$$|p_{xx}(1,t)|\leqslant C_2[\|p_{xx}(\cdot,t)\|_{L^2(0,1)}+\|p_{xxx}(\cdot,t)\|_{L^2(0,1)}]\leqslant 2C_1C_2C_3M_1\mathrm e^{-\mu_1 t},$$

其中$C_3=\|P(0)\|_{\mathcal H}+\|AP(0)\|_{\mathcal H}$.　　证毕.

根据定理4,

$$\begin{aligned}0\approx p_{xx}(1,t)&=v_{xx}(1,t)-y_{xx}(1,t)=\\ &v_{xx}(1,t)+qy_{xt}(1,t)-f(y,y_t)-d(t)\approx\\ &v_{xx}(1,t)+qv_{xt}(1,t)-f(y,y_t)-d(t),\end{aligned}$$

从而有$v_{xx}(1,t)+qv_{xt}(1,t)\approx f(y,y_t)+d(t)$.

3　伺服系统

给定参考信号$y_{\mathrm{ref}}(t)$, 设计如下伺服系统:

$$\begin{cases}\hat y_{tt}(x,t)+\hat y_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ \hat y(0,t)=\hat y_x(0,t)=0,\ t\geqslant 0,\\ \hat y(1,t)=y_{\mathrm{ref}}(t),\\ \hat y_{xx}(1,t)=-q\hat y_{xt}(1,t)+v_{xx}(1,t)+qv_{xt}(1,t).\end{cases}\tag{34}$$

根据第3个边界条件可知, 如果系统(34)能够作为受控系统(1)的状态的观测器, 则受控系统能够跟踪参考信号$y_{\mathrm{ref}}(t)$. 因此, 笔者希望通过系统(34)作为系统(1)的状态的观测器来设计反馈控制器.

令$z(x,t)=\hat y(x,t)-y(x,t)$, 则有

$$\begin{cases}z_{tt}(x,t)+z_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ z(0,t)=z_x(0,t)=0,\ t\geqslant 0,\\ z_{xx}(1,t)=-qz_{xt}(1,t)+p_{xx}(1,t),\\ z_{xxx}(1,t)=\hat y_{xxx}(1,t)-u(t).\end{cases}\tag{35}$$

此时原系统的性能输出跟踪误差可表示为

$$z(1,t)=\hat y(1,t)-y(1,t)=e(t).$$

设计输出反馈控制器如下:

$$u(t)=\hat y_{xxx}(1,t)-c_0[\hat y_t(1,t)-y_t(1,t)].\tag{36}$$

系统(36)代入控制器(35)得到闭环系统

$$\begin{cases}z_{tt}(x,t)+z_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ z(0,t)=z_x(0,t)=0,\ t\geqslant 0,\\ z_{xx}(1,t)=-qz_{xt}(1,t)+p_{xx}(1,t),\\ z_{xxx}(1,t)=c_0z_t(1,t).\end{cases}\tag{37}$$

定理 5　设$p_{xx}(1,t)$由系统(5)生成, 则对任意初值$(z(\cdot,0),z_t(\cdot,t))^{\mathrm T}\in\mathcal H$, 系统(37)是指数稳定的.

证　联立系统(5)与系统(37), 并令$\alpha(x,t)=p(x,t)-z(x,t)$. 则

$$\begin{cases}\alpha_{tt}(x,t)+\alpha_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ \alpha(0,t)=\alpha_x(0,t)=0,\ t\geqslant 0,\\ \alpha_{xx}(1,t)=-q\alpha_{xt}(1,t),\\ \alpha_{xxx}(1,t)=c_0\alpha_t(1,t).\end{cases}\tag{38}$$

系统(35)可以写成如下发展方程:

$$\frac{\mathrm d}{\mathrm dt}(\alpha(\cdot,t),\alpha_t(\cdot,t))^{\mathrm T}=A_\alpha(\alpha(\cdot,t),\alpha_t(\cdot,t))^{\mathrm T},\tag{39}$$

其中算子$A_\alpha:\mathcal D(A_\alpha)(\subset\mathcal H)\to\mathcal H$按照如下方式定义:

$$\begin{cases}A_\alpha(\phi,\psi)^{\mathrm T}=(\psi,-\phi'''')^{\mathrm T},\\ \mathcal D(A_\alpha)=\\ \{(\phi,\psi)^{\mathrm T}\in(H^4(0,1)\times H_E^2(0,1))\cap\mathcal H:\\ \phi''(1)=-q\psi'(1),\ \phi'''(1)=c_0\psi(1)\}.\end{cases}\tag{40}$$

根据文献[23], 系统(37)指数稳定. 由上一节的讨论可知, $z(x,t)=p(x,t)-\alpha(x,t)$也是指数稳定的.　　证毕.

为说明伺服系统(34)的适定性, 回顾一个简单引理[25].

引理 4 假设U_1, U_2, X是3个Hilbert空间. 算子A在X中生成指数稳定的C_0半群e^{At}, 算子$B_1 \in \mathcal{L}(U_1, X_{-1})$和算子$B_2 \in \mathcal{L}(U_2, X_{-1})$对半群$\mathrm{e}^{At}$都是可允许的. $f_1 : X \times [0, \infty) \to U_1$是连续的, 并且满足全局Lipschitz条件. 则初值问题$\dot{x} = Ax + B_1 f_1(x, t) + B_2 f_2(x, t), x(0) = x_0$存在唯一解$x \in C(0, \infty; X)$.

定理 6 假设$d \in L^{\infty}(0, +\infty), y_{\mathrm{ref}} \in W^{1,\infty}(0, \infty)$ $f : \mathcal{H} \to \mathbb{R}$满足全局Lipschitz条件且有界. 对任意初值$(\widehat{y}_0, \widehat{y}_1) \in \mathcal{H}$满足相容性条件$\widehat{y}_0(1) = y_{\mathrm{ref}}(0)$, 则伺服系统(34)存在唯一解$(\widehat{y}, \widehat{y}_t) \in C(0, \infty; \mathcal{H})$且满足

$$\sup_{t\geqslant 0} E_{\widehat{y}}(t) = \sup_{t\geqslant 0} \int_0^1 [\widehat{y}_{xx}^2(x, t) + \widehat{y}_t^2(x, t)]dx < \infty.$$

进一步, 如果$f(y, y_t) = d(t) = y_{\mathrm{ref}}(t) \equiv 0$, 则式(34)的解是指数稳定的.

证 设$\beta(x, t) = \widehat{y}(x, t) - p(x, t)$, 则根据式(5)和式(34)知$\beta(x, t)$满足

$$\begin{cases} \beta_{tt}(x, t) + \beta_{xxxx}(x, t) = 0,\ x \in (0, 1),\ t > 0, \\ \beta(0, t) = \beta_x(0, t) = 0,\ t \geqslant 0, \\ \beta(1, t) = y_{\mathrm{ref}}(t) - p(1, t), \\ \beta_{xx}(1, t) = -q\beta_{xt}(1, t) + g(\beta, \beta_t) + d(t), \end{cases} \tag{41}$$

其中$g(\beta, \beta_t) = f(\beta + p - z, \beta_t + p_t - z_t)$. 注意到$\dot{y}_{\mathrm{ref}} \in L^{\infty}(0, \infty)$和$p_t(1, t) \in L^2(0, \infty)$, 可知式(41)的解也是下面系统的解:

$$\begin{cases} \beta_{tt}(x, t) + \beta_{xxxx}(x, t) = 0,\ x \in (0, 1),\ t > 0, \\ \beta(0, t) = \beta_x(0, t) = 0,\ t \geqslant 0, \\ \beta_t(1, t) = \dot{y}_{\mathrm{ref}}(t) - p_t(1, t), \\ \beta_{xx}(1, t) = -q\beta_{xt}(1, t) + g(\beta, \beta_t) + d(t). \end{cases} \tag{42}$$

下一步, 证明式(42)存在唯一解并且有界. 定义算子$A_0 : D(A_0) \to \mathcal{H}$

$$\begin{cases} A_0(\phi, \psi)^{\mathrm{T}} = (\psi, -\phi'''')^{\mathrm{T}},\ \forall (\phi, \psi) \in D(A_0), \\ D(A_0) = \{(\phi, \psi)^{\mathrm{T}} \in \mathcal{H} \cap (H^4(0, 1) \times H_L^2(0, 1)) : \\ \qquad \psi(1) = 0,\ \phi'''(1) = -q\psi'(1)\}. \end{cases} \tag{43}$$

系统(42)可以写成如下算子形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \beta \\ \beta_t \end{pmatrix} = A_0 \begin{pmatrix} \beta \\ \beta_t \end{pmatrix} + B_1(g(\beta, \beta_t) + d(t)) + B_2[\dot{y}_{\mathrm{ref}}(t) - p_t(1, t)], \tag{44}$$

其中: $B_1 = (0, -\delta'(x-1))^{\mathrm{T}}, B_2 = (0, -\delta'''(x-1))^{\mathrm{T}}$. 根据文献[23–24], A_0生成指数稳定半群, B_1和B_2对半群$\mathrm{e}^{A_0 t}$是可允许的. 根据引理4和$d, \dot{y}_{\mathrm{ref}} \in L^{\infty}(0, \infty)$以及$f : \mathcal{H} \to \mathbb{R}$有界满足全局Lipschitz条件, 故而$g : \mathcal{H} \to \mathbb{R}$有界满足全局Lipschitz条件, 知式(42)有唯一解并且有界. 当$f(y, y_t) = d(t) = y_{\mathrm{ref}}(t) \equiv 0$, 式(42)的解指数趋向零是显然的.

下一步, 证明系统(42)的解也是系统(41)的解. 事实上, 设$(\beta, \beta_t)^{\mathrm{T}}$是系统(42)的解, 根据系统(42)的边界条件, $\beta_t(1, t) = \dot{y}_{\mathrm{ref}}(t) - p_t(1, t), t \geqslant 0$, 则有$\beta(1, t) = y_{\mathrm{ref}}(t) - p(1, t) + C$, 其中$C = \beta(1, 0) - [y_{\mathrm{ref}}(0) - p(1, 0)]$. 根据相容性条件$\widehat{y}_0(1) = y_{\mathrm{ref}}(0)$和$\beta(x, 0) = \widehat{y}(x, 0) - p(x, 0)$, 知$C = \widehat{y}_0(1) - p(1, 0) - [y_{\mathrm{ref}}(0) - p(1, 0)] = 0$. 因此系统(42)的解满足系统(41)的所有边界条件, 故系统(42)的解也是系统(41)的解.

由于系统(42)的解是有界的以及定理3, 因此式(34)的解也是有界的. 当$f(y, y_t) = d(t) = y_{\mathrm{ref}}(t) \equiv 0$, 由于算子$A_0$生成指数稳定半群, 因此定理结论证明. 证毕.

4 闭环系统

对于参考信号$y_{\mathrm{ref}}(t)$, 系统(1)在输出反馈控制(36)下的闭环系统可以写成

$$\begin{cases} y_{tt}(x, t) + y_{xxxx}(x, t) = 0,\ x \in (0, 1),\ t > 0, \\ y(0, t) = y_x(0, t) = 0,\ t \geqslant 0, \\ y_{xx}(1, t) = -qy_{xt}(1, t) + f(y, y_t) + d(t), \\ y_{xxx}(1, t) = \hat{y}_{xxx}(1, t) - c_0[\hat{y}_t(1, t) - y_t(1, t)], \\ v_{tt}(x, t) + v_{xxxx}(x, t) = 0,\ x \in (0, 1),\ t > 0, \\ v(0, t) = v_x(0, t) = 0,\ t \geqslant 0, \\ v_x(1, t) = y_x(1, t), \\ v_{xxx}(1, t) = \hat{y}_{xxx}(1, t) - c_0[\hat{y}_t(1, t) - y_t(1, t)] + \\ \qquad c_0[v_t(1, t) - y_t(1, t)], \\ \hat{y}_{tt}(x, t) + \hat{y}_{xxxx}(x, t) = 0,\ x \in (0, 1),\ t > 0, \\ \hat{y}(0, t) = \hat{y}_x(0, t) = 0,\ t \geqslant 0, \\ \hat{y}(1, t) = y_{\mathrm{ref}}(t), \\ \hat{y}_{xx}(1, t) = -q\hat{y}_{xt}(1, t) + v_{xx}(1, t) + qv_{xt}(1, t), \\ e(t) = y_{\mathrm{ref}} - y_{\mathrm{out}} = y_{\mathrm{ref}}(t) - y(1, t), \end{cases} \tag{45}$$

其状态空间是$\mathcal{X} = \mathcal{H}^3$.

定理 7 假设$d \in L^{\infty}(0, +\infty), y_{\mathrm{ref}} \in W^{1,\infty}(0, \infty)$ $f : \mathcal{H} \to \mathbb{R}$满足全局Lipschitz条件且有界. 则对任意初值$(y_0, y_1, v_0, v_1, \widehat{y}_0, \widehat{y}_1)^{\mathrm{T}} \in \mathcal{X}$满足相容性条件$v_0'(1) - y_0'(1) = 0,\ \widehat{y}_0(1) = y_{\mathrm{ref}}(0)$, 闭环系统(45)存在唯一解$(y, y_t, v, v_t, \widehat{y}, \widehat{y}_t)^{\mathrm{T}} \in C(0, \infty; \mathcal{X})$. 此外, 闭环系统的解满足如下性质:

1) $\sup\limits_{t\geqslant 0}(\int_0^1 [y_{xx}^2(x, t) + y_t^2(x, t) + v_{xx}^2(x, t) + v_t^2(x, t) + \widehat{y}_{xx}^2(x, t) + \widehat{y}_t^2(x, t)]\mathrm{d}x) < +\infty;$

2) 存在常数$M,\mu>0$使得

$$\int_0^1([v_{xx}(x,t)-y_{xx}(x,t)]^2+[v_t(x,t)-y_t(x,t)]^2)\mathrm{d}x\leqslant M\mathrm{e}^{-\mu t};$$

且

$$\int_0^1([\widehat{y}_{xx}(x,t)-y_{xx}(x,t)]^2+[\widehat{y}_t(x,t)-y_t(x,t)]^2)\mathrm{d}x\leqslant M\mathrm{e}^{-\mu t};$$

3) 存在常数$M,\mu>0$使得

$$|e(t)|=|y_x(1,t)-y_{\mathrm{ref}}(t)|\leqslant M\mathrm{e}^{-\mu t},\ t\geqslant 0;$$

4) 当$y_{\mathrm{ref}}(t)=d(t)\equiv 0$, 闭环系统(45)的解$(y,y_t,z,z_t,\widehat{y},\widehat{y}_t)^{\mathrm{T}}$是指数稳定的.

证　设$p(x,t)=v(x,t)-y(x,t)$和$z(x,t)=\hat{y}(x,t)-y(x,t)$, 则闭环系统(45)等价于

$$\begin{cases}p_{tt}(x,t)+p_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ p(0,t)=p_x(0,t)=0, \qquad t\geqslant 0,\\ p_x(1,t)=0,\\ p_{xxx}(1,t)=c_0p_t(1,t),\\ z_{tt}(x,t)+z_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ z(0,t)=z_x(0,t)=0, \qquad t\geqslant 0,\\ z_{xx}(1,t)=-qz_{xt}(1,t)+p_{xx}(1,t),\\ z_{xxx}(1,t)=c_0z_t(1,t),\\ \hat{y}_{tt}(x,t)+\hat{y}_{xxxx}(x,t)=0,\ x\in(0,1),\ t>0,\\ \hat{y}(0,t)=\hat{y}_x(0,t)=0, \qquad t\geqslant 0,\\ \hat{y}(1,t)=y_{\mathrm{ref}}(t),\\ \hat{y}_{xx}(1,t)=-q\hat{y}_{xt}(1,t)+v_{xx}(1,t)+qv_{xt}(1,t),\\ e(t)=y_{\mathrm{ref}}-y_{\mathrm{out}}=y_{\mathrm{ref}}(t)-y(1,t).\end{cases} \tag{46}$$

显然(p,z)部分独立于$\hat{y}$部分, 根据定理3, 5知(p,z)部分存在唯一解, 根据定理3知$\hat{y}$部分存在唯一解, 从而系统(45)存在唯一解. 另一方面,

$$\begin{pmatrix}y(x,t)\\ v(x,t)\end{pmatrix}=\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}z(x,t)\\ p(x,t)\end{pmatrix}+\begin{pmatrix}\widehat{y}(x,t)\\ \widehat{y}(x,t)\end{pmatrix},$$

根据定理3, 5和定理6, 知性质1)成立. 由$p(x,t)=v(x,t)-y(x,t)$, $z(x,t)=\widehat{y}(x,t)-y(x,t)$和定理3知性质2)成立. 再根据Sobolev嵌入定理和性质2)知道性质3)成立. 最后性质4)由定理6得到.　　证毕.

5 结论

本文通过边界剪切力控制讨论了Euler-Bernoulli梁带有控制非同位的边界干扰输出跟踪问题. 利用自抗扰方法设计了总扰动估计器在线估计未知总扰动(内部扰动和外部扰动之和), 然后利用估计出来的总扰动设计伺服系统, 其作用是在线模拟实际受控系统. 最后根据伺服系统, 给出控制输出跟踪的反馈控制, 同时分析了跟踪误差与闭环系统的稳定性和有界性.

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