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含生成森林有向图的零特征值及在编队控制中的应用

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摘要: 本文研究了含有m–生成森林有向图拉普拉斯矩阵的零特征值重数, 其中$m \geqslant 1$是一个整数. 对于这个问题, 这个图一般不含有生成树. 即使初始时具有生成树, 受到隐秘的攻击或经过障碍物造成的智能体之间的通信阻挡(如在分布式控制、分布式(在线)优化、多智能体算子等问题中)等因素后, 这个图也可能不再含有生成树了. 另外, 作为一个研究方向, 它本身亦是个有趣的科学问题. 为了解决这个问题, 本文证明了拉普拉斯矩阵的零特征值重数等于这个图中的生成森林个数, 这个结论可以看作是在带有生成树的有向图情形(即$m = 1$时)的一个推广. 再者, 结合分布式优化方法, 所得结论被应用于单积分器多智能体系统下的编队控制, 表明了达到的编队队形处在通信图拉普拉斯矩阵的核空间中. 最后给出了一个例子用以展示在编队控制中的应用.

关键词: 拉普拉斯矩阵; 多智能体网络; 有向图; 生成森林; 编队控制

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Zero eigenvalue of directed graphs with spanning forests with application to formation control

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Abstract: This paper investigates the multiplicity of zero eigenvalue of the Laplacian matrix for a directed graph, which has a spanning m-forest, where $m \geqslant 1$ is an integer. For this problem, the graph usually does not contain a spanning tree, and this scenario may occur due to insidious attacks or communication blocking by obstacles between two agents in distributed control, (online) optimization, multi-agent operators, and so on, even though it indeed has a spanning tree at the beginning. In addition, this problem is of interest as a research direction in its own right. To deal with this problem, it is shown that the multiplicity of the Laplacian's zero eigenvalue amounts to the number of spanning forests in the studied graph, which can be seen as an extension of the directed graph case with a spanning tree, in which case it has $m = 1$. Moreover, the obtained result is applied to formation control for single-integrator multi-agent systems along with distributed optimization methods, indicating that the achieved formation shape lies in the kernel space of the Laplacian matrix associated with the communication graph. Finally, an example is provided to demonstrate the applicability to formation control.

Key words: Laplacian matrix; multi-agent networks; directed graphs; spanning forest; formation control

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1 Introduction

Graph theory, as an important research topic, is of interest in its own right [1]. In the meantime, it is also well known that graph theory is one powerful tool in distributed control (e.g., formation control) [2–7], distributed (online) optimization [8–11], and multi-agent operators[12], and so forth, where a family of agents exists over a multi-agent network, who may be dispersed geographically and hold their own private information on a global mission, and the objective is for all agents to communicate with their neighbors through local information exchange in order to accomplish the global mission. In these problems, each agent is viewed as a node or vertex, the information flow is delineated by directed edges, and thereby the communication pattern among all agents can be captured by directed or undirected graphs. To ease the exposition, all those problems over multi-agent networks, where the interconnection of all agents is described as directed or undirected graphs, are collectively referred to as multi-agent problems.

For the aforementioned problems with underlying directed or undirected communication graphs, one important notion is the so-called Laplacian matrix (closely related to the adjacency matrix), which plays a pivotal role in the convergence of distributed controllers or algorithms in multi-agent problems. It is well known that all eigenvalues of the Laplacian matrix are nonnegative for undirected graphs, where all communication edges are bidirectional, and all real parts of eigenvalues of the Laplacian matrix are nonnegative for directed graphs, where each interaction edge may be unidirectional [2]. Meanwhile, an important fact is that zero must be an eigenvalue of the Laplacian for both undirected and directed graphs[2]. Furthermore, as an essential matrix, it has been shown that the multiplicity of the zero eigenvalue of the Laplacian matrix plays a significant role in determining the goal achievement for multi-agent problems. To be specific, the simplicity of its zero eigenvalue determines the achievement of consensus and so on, in which case it requires that undirected graphs (resp. directed graphs) are connected (resp. have a directed spanning tree)[2]. Regarding connected undirected graphs, the smallest positive eigenvalue is usually called the algebraic connectivity, which often determines the convergence speed for consensus [13–14], and meanwhile, as for directed graphs with a directed spanning tree, the smallest real part of its nonzero eigenvalues, in some sense, also impacts the convergence speed for consensus [15–16].

As discussed above, the connectivity for undirected graphs and directed spanning tree for directed graphs are crucial in multi-agent problems, which are necessary and sufficient for the simple multiplicity of zero eigenvalue of the Laplacian matrix associated with undirected and directed graphs, respectively [2]. However, what happens if graphs are not connected or do not have a spanning tree? For example, a connected comunication graph is attacked by insidious attackers or some communication edge is obstructed by some obstacles, and then the communication graph may be deformed to be unconnected even if it is connected at the beginning. As a result, it is important and practical to investigate the case where communication graphs are not connected or do not have a spanning tree. Along this line, for an undirected communication graph, it has been shown in [17] that the multiplicity of the zero eigenvalue of the Laplacian matrix is equal to the number of the connected components of the communication graph. However, it is yet to be studied for directed graphs, to our best knowledge, which is of also interest in its own right.

With the above motivation, this paper aims to investigate the multiplicity of the zero eigenvalue of the Laplacian matrix for directed graphs, which do not have a spanning tree in general. To address this issue, directed graphs that have m-spanning forest are taken into account for an integer $m \geqslant 1$, which include directed graphs with a directed spanning tree as a special case when $m = 1$. In summary, the contributions of this paper lie in two facets as follows.

1) This paper shows for the first time that the multiplicity of the zero eigenvalue of the Laplacian matrix for a directed graph is equal to the number of spanning forests in the graph. Generally speaking, it extends the directed graph case with a directed spanning tree to a more general case.

2) The obtained result is applied to formation control for single-integrator multi-agent systems, for which it is shown that the achieved formation shape lies in the kernel space of the Laplacian matrix associated with the communication graph, thus called kernel formation.

The rest of this paper is organized as follows. Section 2 presents some useful notions and basic yet important results. Section 3 gives the main result of this paper. Section 4 provides an application to formation control. Section 5 provides an example for formation control, and the conclusion is finally given in Section 6.

2 Preliminaries

2.1 Notations

Let us define $\mathcal{I}_k := \{1, 2, \cdots, k\}$ for an integer $k >$ 0, and denote by $\mathbb{R}^n$, $\mathbb{C}^n$ the n-dimensional real and complex Euclidean space, respectively. $\mathrm{diag}\{a_1, a_2, \cdots, a_n\}$ (resp. $\mathrm{diag}\{A_1, A_2, \cdots, A_n\}$) represents a diagonal matrix with diagonal scalar entries a_k (resp. a block-diagonal matrix with diagonal matrix entries A_k) for $k \in \mathcal{I}_n$. For a complex number z, denote by $\mathrm{Re}(z)$, $I\mathrm{m}(z)$, $|z|$ the real part, imaginary part and modulus for $z \in \mathbb{C}$, respectively. For $x \in \mathbb{R}^{n\times n}$, let $\mathrm{rank}(X)$, $\det(X)$ be the rank and determinant of X, respectively. Let I_n be the $n \times n$ identity matrix, and $\mathbf{0}_n$ (resp. $\mathbf{1}_n$) be the n-dimensional column vector of all entries 0 (resp. all entries 1) or $n \times n$ zero matrix. $||\cdot||$ and $\otimes$ stand for the standard Euclidean norm and the Kronecker product, respectively.

2.2 Graph theory

Denote by a directed graph (or digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with N nodes, consisting of the node set $\mathcal{V}$ and the edge set $\mathcal{E}$, where an element $(j, i) \in \mathcal{E}$ is called an edge, indicating the information flow from node j to node i (in this case, j is called a neighbor or parent of node i). Self-loops are not allowed. Denote the neighbor set of node i as $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}, j \in \mathcal{V}, j \neq i\}$. In a directed graph, a directed path(resp. weak path) is comprised of a sequence of adjacent edges in the form (i_1, i_2), $(i_2, i_3), \cdots, (i_{k-1}, i_k)$, abbreviated as $i_1, i_2, \cdots, i_k$, such that $(i_l, i_{l+1}) \in \mathcal{E}$ (resp. $(i_l, i_{l+1}) \in \mathcal{E}$ or $(i_{l+1}, i_l) \in \mathcal{E}$) for $l \in \mathcal{I}_{k-1}$. A directed graph is called a directed tree if except one node, called the root, every node has exactly one parent. Additionally, a directed graph is said to have a directed spanning tree if a subgraph of the directed graph, consisting of all its nodes and some edges, is exactly a directed tree.

The adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ is defined as: $a_{ij} > 0$ if *(j,i)* $\in \mathcal{E}$, and $a_{ij} = 0$ $(i \neq j)$ otherwise. Set $a_{ii} = 0$ for all *i,j* $\in \mathcal{I}_N$. Moreover, the Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{N\times N}$ is defined as $l_{ii} = \sum\limits_{j\in\mathcal{N}_i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$.

To proceed, the following concepts are necessary in this paper.

Definition 1 A directed graph is called an m-forest, if the graph consists of m disjoint components and each component is a directed tree.

Definition 2 A directed graph is said to have a spanning m-forest if some subgraph of the graph is an m-forest, and meanwhile no subgraph of the graph is a k-forest for any integer $0 < k \leqslant m-1$.

Regarding a collection of N nodes, a configuration in $\mathbb{R}^n$ is defined as $p = (p_1^{\mathrm{T}}\ p_2^{\mathrm{T}} \cdots p_N^{\mathrm{T}})^{\mathrm{T}}$, $p_i \in \mathbb{R}^n$ for $i \in \mathcal{I}_N$, (i.e., a column vector). If equipping with a graph $\mathcal{G}$, then a configuration in $\mathbb{R}^n$ is called a framework in $\mathbb{R}^n$, denoted by $\mathcal{F} = (\mathcal{G}, p)$. Moreover, a configuration p in $\mathbb{R}^n$ is called generic if any algebraic equations do not hold for all entries of $p_i, i \in \mathcal{I}_N$ over the rational numbers[18–20], that is, there does not exist any nonzero polynomial $P(z_1, z_2, \cdots, z_{nN})$ with rational coefficients such that $P(p_{11}, \cdots, p_{1n}, \cdots, p_{N1}, \cdots, p_{Nn}) = 0$, where p_{ij} is the jth entry of p_i. Intuitively speaking, a generic configuration is not degenerate (e.g., no three points are positioned on a straight line).

2.3 An useful lemma

The following lemma is useful for the ensuing analysis.

Lemma 1 [21] For a directed graph, zero is an eigenvalue of the Laplacian matrix L associated with the eigenvector $\mathbf{1}_N$ and all other eigenvalues are with positive real parts. Moreover, zero is a simple eigenvalue if and only if the graph has a spanning tree.

3 Main result

In view of Lemma 1, it is known that zero eigenvalue of the Laplacian matrix is simple if and only if the graph has a spanning tree. However, what about the situation where the graph does not have a spanning tree? Note that a graph may be degenerated due to malicious attacks or unexpected obstacles, etc. In this respect, a worse scenario than having a spanning tree, i.e., having a spanning forest, is investigated in this paper.

To answer this question, the main result of this paper is given as follows.

Theorem 1 For a directed graph with nonnegative weights, its Laplacian L has precisely m zero eigenvalues if and only if the graph has a spanning m-forest. Meanwhile, all nonzero eigenvalues of L have positive real parts.

Proof In light of Lemma 1, it is known that all nonzero eigenvalues of L have positive real parts. Therefore, it remains to show the first part of this lemma.

(Sufficiency) To start, consider a special case when the graph $\mathcal{G}$ itself is an m-forest. In this case, one can renumber the nodes in the following order. Relabeling the nodes in the first component as $k_0 = 1$ to k_1 in an arbitrary order if the first component has k_1 nodes. Similarly, one can relabel the nodes in the lth component as $\sum\limits_{i=1}^{l} k_{i-1}+1$ to $\sum\limits_{i=1}^{l} k_{i-1}+1+k_l$ if the lth component has k_l nodes, $l = 2,\cdots,m$. Denoting by L_1 the new Laplacian matrix, it is known that there exists a permutation matrix P such that $L_1 = P^{\mathrm{T}}LP$[22]. In the meantime, it is easy to see that L_1 is block-diagonal in the form $L_1 = \mathrm{diag}\{L_{11},\cdots,L_{1m}\}$, where L_{1k}, $k\in\mathcal{I}_m$ is actually the Laplacian matrix corresponding to the kth component. In view of linear algebra, the eigenvalues of L_1 exactly equal the set of eigenvalues of L_{1k} for all $k\in\mathcal{I}_m$. Note that the kth component for all $k\in\mathcal{I}_m$ is composed of a tree. By virtue of Lemma 1, each L_{1k} has exactly one zero eigenvalue and all other eigenvalues have positive real parts. Consequently, L_1 has exactly m zero eigenvalues and $\mathrm{rank}(L_1) = N-m$, so does L since they are similar.

Up to now, it has been shown that the first part of this lemma holds for an m-forest. It should be noticed that every directed graph, which has a spanning m-forest, can be obtained by consecutively adding edges to one of its m-forest subgraph. Therefore, the sufficiency part can be proved if the result holds for every new graph by adding one edge each time to previous one with one initial m-forest subgraph, denoted by $\mathcal{G}_0$, of graph $\mathcal{G}$ until to the whole graph $\mathcal{G}$.

Assume that $B = (b_{ij}) \in \mathbb{R}^{N\times N}$ is obtained by adding a new edge (r,s) to $G = (g_{ij}) \in \mathbb{R}^{N\times N}$ associated with graph $\mathcal{G}_0$. Then, one has $b_{\mathrm{sr}} = -e_{\mathrm{sr}}$, $b_{\mathrm{ss}} = g_{\mathrm{ss}} + e_{\mathrm{sr}}$ and $g_{\mathrm{sr}} = 0$, where $e_{\mathrm{sr}} > 0$ is the weight corresponding to the new edge (r,s). Define $G(\lambda) = (g_{ij}(\lambda)) := \lambda I_N - G$ and $B(\lambda) = (b_{ij}(\lambda)) := \lambda I_N - B$. Considering the characteristic polynomial of B, one can obtain that

$$\begin{aligned}&\det(B(\lambda)) = \sum_{i=1}^{N}(-1)^{s+i}b_{si}(\lambda)M_{si}(B(\lambda)) = \\ &\sum_{j=1}^{N}(-1)^{s+i}g_{si}(\lambda)M_{si}(G(\lambda)) - e_{\mathrm{sr}}M_{\mathrm{ss}}(G(\lambda)) + \\ &(-1)^{s+r}e_{\mathrm{sr}}M_{\mathrm{sr}}(G(\lambda)) = \\ &\det(G(\lambda)) + e_{\mathrm{sr}}[(-1)^{s+r}M_{\mathrm{sr}}(G(\lambda)) - M_{\mathrm{ss}}(G(\lambda))],\end{aligned} \tag{1}$$

where $M_{ij}(Q)$ means the (i,j) minor for a matrix Q, that is, it is the determinant of the submatrix of Q by removing the ith row and jth column, and we have used the fact that $b_{\mathrm{sr}}(\lambda) = g_{\mathrm{sr}}(\lambda) + e_{\mathrm{sr}}$, $b_{\mathrm{ss}}(\lambda) = g_{\mathrm{ss}}(\lambda) - e_{\mathrm{sr}}$, and $M_{si}(B(\lambda)) = M_{si}(G(\lambda))$ for all $i\in\mathcal{I}_N$.

$$T_1 = \begin{pmatrix} g_{11} & \cdots & g_{1(s-1)} & g_{1(s+1)} & \cdots & g_{1r}+g_{1s} & \cdots & g_{1N} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ g_{(s-1)1} & \cdots & g_{(s-1)(s-1)} & g_{(s-1)(s+1)} & \cdots & g_{(s-1)r}+g_{(s-1)s} & \cdots & g_{(s-1)N} \\ g_{(s+1)1} & \cdots & g_{(s+1)(s-1)} & g_{(s+1)(s+1)} & \cdots & g_{(s+1)r}+g_{(s+1)s} & \cdots & g_{(s+1)N} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ g_{N1} & \cdots & g_{N(s-1)} & g_{N(s+1)} & \cdots & g_{Nr}+g_{Ns} & \cdots & g_{NN} \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}. \tag{2}$$

Consider now the matrix T_1 given in (2) (without loss of generality, let $s<r$). Define $T_1(\lambda) = (t_{1ij}(\lambda)) := \lambda I_N - T_1$. In view of determinant's property, it is easy to see that

$$\begin{aligned}\det(T_1(\lambda)) &= M_{\mathrm{ss}}(G(\lambda)) + (-1)^{r-s-1}M_{\mathrm{sr}}(G(\lambda)) \\ &= M_{\mathrm{ss}}(G(\lambda)) - (-1)^{r+s}M_{\mathrm{sr}}(G(\lambda)),\end{aligned} \tag{3}$$

which together with (1) follows

$$\det(B(\lambda)) = \det(G(\lambda)) - e_{\mathrm{sr}}\det(T_1(\lambda)). \tag{4}$$

Observing carefully the structure of T_1, it can be seen that the graph associated with Laplacian T_1 is a m-forest. As a result, T_1 has exactly m zero eigenvalues. Consequently, applying Vieta's formulas yields that

$$\begin{aligned}&\sum_{1\leqslant i_1<\cdots<i_k\leqslant N-1}\lambda_{i_1}(T_1)\cdots\lambda_{i_k}(T_1) = \\ &(-1)^k g_{N-k-1}(T_1),\end{aligned} \tag{5}$$

where $\lambda_i(Q)$ denotes the ith eigenvalue of a matrix Q in the ascending order of real parts, and $g_k(Q)$ is the coefficient of λ^k in the characteristic polynomial of matrix Q. Thus, one has

$$(-1)^{N-m-1}g_m(T_1) \geqslant 0,\ \ g_k(T_1) = 0,\ \ \forall\, k\in\mathcal{I}_{m-1}. \tag{6}$$

Meanwhile, it is known that G has exactly m zero eigenvalues since its associated graph is an m-forest, which, invoking Vieta's formulas, implies

$$(-1)^{N-m} g_m(G) > 0, \ \ g_k(G) = 0, \quad \forall\, k \in \mathcal{I}_{m-1}. \tag{7}$$

At this point, for coefficient $g_k(B), k \in \mathcal{I}_{m-1}$ for matrix B, in view of (4) one can easily see that

$$g_k(B) = 0, \quad \forall\, k \in \mathcal{I}_{m-1}. \tag{8}$$

As for the coefficient $g_m(B)$ of λ^m for matrix B, we consider two different cases for the parity of $N-m-1$.

Case 1: $N-m-1$ is even. Inequality (6) results in $g_m(T_1) \geqslant 0$. Meanwhile, $N-m$ is odd in this case and thereby it follows from inequality (7) that $g_m(G) < 0$. Hence, in view of (4) it is easy to see that $g_m(B) < 0$.

Case 2: $N-m-1$ is odd. Similarly, one can show that $g_m(B) > 0$.

In both cases, it can be concluded that $g_m(B) \neq 0$, which together with (6) and (7) leads to that B has exactly m zero eigenvalues.

To move forward, when adding another new directed edge to the previous graph associated with B, all are kept the same except that in this step the graph associated with Laplacian T_2 (defined similarly to T_1) is with the structure: an m-forest with a new added edge, which has exactly m zero eigenvalues as shown in last step, (i.e., at least m zero eigenvalues). The above process can be continued by adding a new directed edge until to obtain the graph $\mathcal{G}$. This ends the sufficiency part.

(Necessity) Let us prove it by contradiction. If it does not contain a spanning m-forest, then there exist two cases. The first one is that the graph contains a spanning m_1-forest for some $m_1 < m$. With regard to this case, following the same line in sufficiency part one can see that the Laplacian has exactly m_1 zero eigenvalues, which is a contradiction. The second case is that the graph contains a spanning m_2-forest for some $m_2 > m$. Similarly, it can be concluded that the Laplacian has exactly m_2 zero eigenvalues, which is also a contradiction. This completes the necessity part. □

Remark 1 It should be noted that, to our best knowledge, this paper is the first to establish the result in Theorem 1, which can be actually viewed as an extension of the case where the graph has a spanning tree (see Lemma 1), i.e., when $m = 1$ in Theorem 1. Moreover, it is worth mentioning that even though a communication graph indeed has a spanning tree at the beginning, it may not be the case at other times, which may occur due to insidious attacks or communication blocking by obstacles between two agents, for example, in distributed control, (online) optimization, multi-agent operators. Additionally, this problem is of independent interest as a theoretical research direction.

4 Application to formation control

This section aims at applying the result in Theorem 1 to formation control. Towards this end, the following single-integrator multi-agent network, consisting of N agents, is addressed

$$\dot{x}_i = u_i, \ i \in \mathcal{I}_N, \tag{9}$$

where $x_i, u_i \in \mathbb{R}^n$ are the state and input (or controller) of agent i, respectively. For this network, as done in [19, 23–24] for affine formation, the controller is designed by only using the measurement of relative positions as follows:

$$u_i = -\sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j), \ i \in \mathcal{I}_N, \tag{10}$$

which leads to that network (9) can be written in a compact form

$$\dot{x} = -(L \otimes I_n)x, \tag{11}$$

where x is the concatenated state $x := (x_1^{\mathrm{T}} \ \ x_2^{\mathrm{T}} \ \cdots \ x_N^{\mathrm{T}})^{\mathrm{T}}$.

To proceed, let us concentrate on dynamic formation control for network (11). Given a desired dynamic configuration $p(t) = (p_1(t) \cdots p_N(t))^{\mathrm{T}}, p_i(t) \in \mathbb{R}^n$, the configuration $p(t)$ is said to be realizable (resp. achievable or stabilizable) under the directed graph $\mathcal{G}$ if a time-varying Laplacian $L(t)$ corresponding to $\mathcal{G}$ exists such that $(L(t) \otimes I_n)p(t) = 0$ (resp. the state of network (11) can converge to $p(t)$). It will be shown that a dynamic configuration can be achieved in some sense, if the communication graph has a spanning m-forest for $m \geqslant 2$.

In the sequel, static configuration is first considered and dynamic one will be discussed later. It is easy to see that a configuration can be achievable only if it is in the kernel of $L \otimes I_n$, which is determined by the kernel of L since zero eigenvalues only originate from L. In fact, the essence of the Laplacian approach lies in confirming the number of zero eigenvalues of the Laplacian L.

It is now ready to analyze static formation control. It is well known that all agents will achieve consensus, i.e., converging to the set $\{x_i \in \mathbb{R}^n, i \in \mathcal{I}_N : x_1 = x_2 = \cdots = x_N\}$, if the directed graph has a spanning tree. Note that having a spanning tree is equivalent to having a spanning 1-forest. What if the directed graph has a spanning 2-forest? In light of Theorem 1, the Laplacian has exactly two zero eigenvalues, and thereby the weights a_{ij} can be selected to satisfy that L has two linearly independent eigenvectors associated with zero eigenvalue. Given a graph $\mathcal{G}$ having a spanning m-forest, it is always feasible that L can be chosen for having m linearly independent eigenvectors in the eigenspace associated with zero eigenvalue (amounting to $\dim(\ker(L)) = m$, where $\dim(\cdot)$ and $\ker(\cdot)$ mean the dimension and kernel of a linear space, respectively), and the reason is as follows: 1) it is true for the Laplacian of an m-forest which can be seen from the proof of Theorem 1, i.e., $\mathrm{rank}(L) = N - m$; 2) after randomly selecting one m-forest subgraph $\mathcal{G}_0$ of $\mathcal{G}$, all weights of those edges, which do not belong to $\mathcal{G}_0$, are viewed as variables in $[0, +\infty)$, and hence the determinant of a submatrix of L becomes a polynomial with aforementioned variables; 3) it is well known that a nonzero polynomial does not equal zero almost everywhere. As a consequence, by selecting edge weights it can always be ensured that $\mathrm{rank}(L) = N - m$ if the graph has a spanning m-forest, which results in $\dim(\ker(L)) = m$, and the set of weights that make this infeasible in fact has Lebesgue measure zero.

Now, coming back to the case when the graph has a spanning 2-forest, the Laplacian L can be selected to have two linearly independent eigenvectors associated with zero eigenvalue. Since $\mathbf{1}_N$ is an eigenvector corresponding with zero eigenvalue, there exists another eigenvector $v_1 = (v_{11} \cdots v_{1N})^{\mathrm{T}} \in \mathbb{R}^N$, linearly independent of $\mathbf{1}_N$, such that $Lv_1 = 0$. As a result, the kernel of L has the form $c_1v_1 + c_2\mathbf{1}_N$ for $c_1, c_2 \in \mathbb{R}$. It should be noted that $c_2\mathbf{1}_N$ will not make an effect on the formation shape since it means physical translations of the formation shape. Consequently, the formation shape is utterly determined by c_1v_1, which together with network (11) follows that the final formation configuration is determined by $c_1v_1 \otimes \alpha$ for $\alpha \in \mathbb{R}^n$, which is actually a line-shape because $c_1v_1 \otimes \alpha = (c_1v_{11}\alpha^{\mathrm{T}} \cdots c_1v_{1N}\alpha^{\mathrm{T}})^{\mathrm{T}}$. For example, let $N = 4$ and $n = 3$, if $v_1 = (1\ 2\ 3\ 4)^{\mathrm{T}}$ and $c_1 = 1$, then $c_1v_1 \otimes \alpha = (\alpha^{\mathrm{T}}\ 2\alpha^{\mathrm{T}}\ 3\alpha^{\mathrm{T}}\ 4\alpha^{\mathrm{T}})^{\mathrm{T}}$ for a vector $\alpha \in \mathbb{R}^3$, forming a straight line if $\alpha \neq \mathbf{0}_3$. Of course, in this case it is possible for some components of v_1 to have the same value, leading to that some agents finally stay in the same position, which is simultaneously determined by the structure of the graph and the selection of edge weights.

Along the line, if one desires to achieve a formation configuration that is not a line-shape, it is necessary for the graph to have a spanning m-forest, where $m \geqslant 3$. In the following, we assume that the interaction graph $\mathcal{G}$ has a spanning m-forest, $m \geqslant 3$. Given a desired formation configuration $p = (p_1^{\mathrm{T}} \cdots p_N^{\mathrm{T}})^{\mathrm{T}} p_i \in \mathbb{R}^n$, one way to ensure that multi-agent network (11) can achieve the configuration p is to design the edge weights a_{ij}'s as follows:

$$\sum_{j\in\mathcal{N}_i} a_{ij}(p_i - p_j) = 0, \tag{12}$$

which can be done in a distributed means by resorting to distributed optimization as done in [19]. However, one drawback is that this way is not always feasible, that is, there exist configurations p's that cannot be achieved by selecting any edge weights. But this problem can be solved by using real weights (i.e., negative weights are allowed) in a generic sense (e.g., see [19]).

Regarding a graph having a spanning m-forest, $m \geqslant 3$, the Laplacian L in general has several eigenvectors in its kernel space which are linearly independent of the eigenvector $\mathbf{1}_N$. Let us denote these eigenvectors as $v_1 = \mathbf{1}_N, v_2, \cdots, v_m$ (here there are m linearly independent eigenvectors that are viable by the foregoing argument). In this regard, as shown before, the main claim of this section is:

The final formation shape is completely determined by $v_2, \cdots, v_m$ up to translations and scalings. (13)

Note that any linear combination of $v_2 \cdots v_m$ is in the kernel of L, thus called kernel formation for (11). However, the different linear combination of $v_2 \cdots v_m$ will generally give rise to different formation shapes, thus leading to different formation shapes from the desired one. That is, a desired configuration p can be achieved after properly selecting edge weights by (12), although a possibility exists for that the configuration p cannot be accurately achieved. For example, if a square shape in $\mathbb{R}^2$ space can be achieved by appropriately computing weights via (12), by the same weights the ultimate formation shape that multi-agent network (11) will converge to may be a parallelogram, instead of a square. However, the advantage is that when an underlying communication graph that has a spanning tree is attacked such that the structure of spanning trees is destroyed, resulting in a spanning forest structure, the aforementioned result provides theoretical insight into the possible deformed formation shapes.

So far, static formation control has been discussed. With regard to dynamic formation control, given a dynamic configuration $p(t) = (p_1^{\mathrm{T}}(t) \cdots p_N^{\mathrm{T}}(t))^{\mathrm{T}}, p_i^{\mathrm{T}}(t) \in \mathbb{R}^n$, under suitable conditions it can also be handled by computing the equation (12) with time-varying weights $a_{ij}(t)$'s. However, it will encounter the same problem as mentioned above.

5 An example

This section provides an example to illustrate the application to formation control, as discussed in Section 4.

Example 1 Consider the multi-agent network (11) consisting of $N = 4$ agents with the state of each agent being in $\mathbb{R}^2$ space. The communication graph is given by Fig. 1, and it is easy to see that the communication graph has a spanning 3-forest. In addition, the desired formation shape is shown in Fig. 2(a), which is bilaterally symmetrical. Specifically, one configuration of the desired shape is given in Fig. 2(b), where $p = (p_1^{\mathrm{T}} \cdots p_4^{\mathrm{T}})^{\mathrm{T}}$ with

$$p_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ p_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ p_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ p_4 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{14}$$

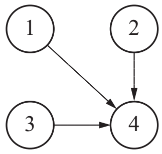

Fig. 1 The interaction graph in Example 1

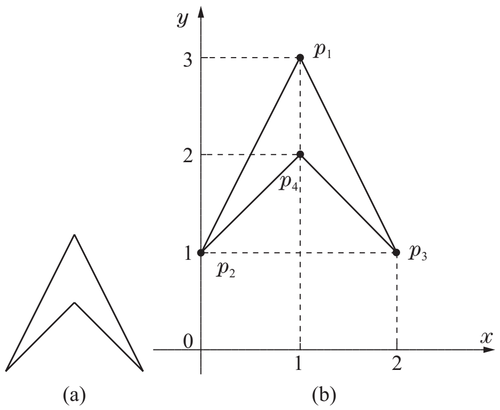

Fig. 2 (a) Desired formation shape and (b) one corresponding configuration

Inserting the configuration p into (12) yields that

$$a_{41} = 2a_{42} = 2a_{43}. \tag{15}$$

In this case, randomly select $a_{41} = 2, a_{42} = a_{43} = 1$. Note that all other weights are zero. Therefore, the Laplacian L is of the form

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 4 \end{pmatrix}. \tag{16}$$

That is to say, by the Laplacian in (16) the desired configuration p is achievable for multi-agent network (11).

After some calculations, it is easy to obtain that a basis of the subspace ker(L) can be chosen as

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \end{pmatrix}. \tag{17}$$

As a result, any linear combination of v_1, v_2, v_3 lies in the subspace $\ker(L)$, and furthermore any linear combination of $v_1 \otimes b_1, v_2 \otimes b_2, v_3 \otimes b_3$ is in the kernel of $L \otimes I_2$, where $b_i \in \mathbb{R}^2, i \in \mathcal{I}_3$ is arbitrary. For instance, let $b_1 = (1\ \ 0)^{\mathrm{T}}, b_2 = (1/2\ \ 0)^{\mathrm{T}}, b_3 = (1/2\ 1/2)^{\mathrm{T}}$, and then one has $q := v_1 \otimes b_1 + v_2 \otimes b_2 + v_3 \otimes b_3 = (1\,0\ -1/2\ 2\ 1/2\ 0\ 1/2\ 1/2)^{\mathrm{T}}$, where $q = (q_1^{\mathrm{T}} \cdots q_4^{\mathrm{T}})^{\mathrm{T}}, q_i \in \mathbb{R}^2, i \in \mathcal{I}_4$. It means that network (11) maybe converges to the configuration q with

$$q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ q_2 = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix},$$

$$q_3 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix},\ q_4 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}. \tag{18}$$

For example, if the configuration q is the initial state, then the agents will stay where they are in the configuration q, although q is different from the desired configuration p. The configuration q is shown in Fig. 3, from which it is easy to see that the spatial shape with configuration q is different from that with p, which is consistent with the theoretical discussion in Section 4.

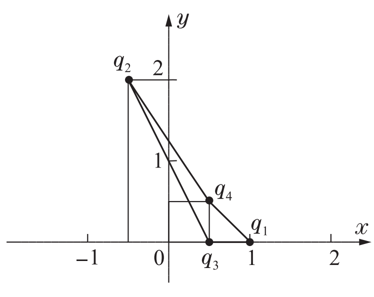

Fig. 3 The configuration q in Example 1

6 Conclusion

The multiplicity of zero eigenvalue of the Laplacian matrix has been investigated for directed graphs that have a spanning forest, a more general case than having a spanning tree. For example, this case may arise when this network undergoes insidious attacks or encounters unexpected obstacles, even though the communication graph for all agents indeed has a spanning tree at the beginning. In fact, as a scientific problem, it is of also interest in its own right, as an extension of the case where a directed graph has a spanning tree. To address this problem, a necessary and sufficient condition has been established for confirming the multiplicity of the zero eigenvalue for the Laplacian matrix. In addition, the obtained result has also been applied to formation control, ensuring that a desired formation shape can be achieved in some sense. To be specific, the final formation shape is completely determined by the kernel space of the Laplacian matrix for the communication graph, which may be different from the desired formation shape. Finally, the theoretical result has been validated by a concrete example in the plane.

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