反阻尼一维波动方程的稳定性

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摘要: 文章利用边界控制法研究了一类反阻尼一维波动方程的稳定性. 首先, 通过边界控制的backstepping方法, 引入包含有两个核函数的积分变换, 将控制系统转化为稳定的目标系统. 核函数个数的增加导致核函数方程更加复杂, 文中运用了一系列的数学计算技巧求解出核函数, 从而得到反馈控制器; 其次, 运用类似的方法找到变换的逆变换; 最后, 选择合适范数, 利用变换及逆变换的有界性证明得到闭环系统的稳定性.

关键词: 波动方程; 边界控制; 稳定性

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Stabilization of a 1-D wave equation with anti-damping

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Abstract: This paper is to stabilize a 1-D wave equation with an anti-damping by boundary control. To use the back-stepping method of boundary control, a new transformation of two kernel functions is introduced. The equations of k-ernel functions are more complicated mathematically. By some mathematical skill, solutions of the kernel equations are constructed. Finally, the inverse transformation is attained. Through boundedness of the transformation and its inverse, stability of the closed-loop system is established.

Key words: wave equation; boundary control; stability

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1 Introduction

Wave equations describe a variety of natural phenomena, such as sound waves, water waves, etc. Therefore, wave equation has a rich engineering background. Stabilization of wave equations plays a role in practical applications. Some results can be found in [1–2]. Stability of a wave equation with velocity recirculation is considered in [3]. Also, applications in deep oil drilling can be found, for example, in [4]. In recent years, boundary control of partial differential equations was developed (see, e.g., [1–13]). In [5], a wave equation with Kelvin-Voigt damping through boundary control is considered. Stability of wave equations by output feedback boundary control are also concerned (see, e.g., [8, 12]).

In [10], stability of a wave equation with an anti-damping at one boundary is addressed. Motivated by [10], we consider a 1-D wave equation with antidamping at an internal point, and stabilize it by boundary control. The control system can be written into

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + \lambda u(x_0,t), x \in (0,1), \\ \qquad\qquad t > 0, \\ u_x(0,t) = 0, \quad t > 0, \\ u(1,t) = U(t),\ t > 0, \end{cases} \tag{1}$$

where $\lambda > 0$ is a constant and $U(t)$ is the controller. The system (1) models a string vibration which is motived at the end $x = 1$ and is uncontrolled at the opposite end. The anti-damping on the internal point comes from $\lambda u(x_0,t)$. Motivated by [12–15], we will design a state feedback boundary controller $U(t)$ through backstepping to stabilize the closed-loop system.

2 Control design

The idea of control design is derived from the PDE backstepping method. It is that the control system (1) is converted into the stable target system by a bounded inverse transformation.

Firstly, to design the control input by backstepping

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method, consider the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y - \int_0^{x_0} r(x,y)u(y,t)\mathrm{d}y, \tag{2}$$

where the kernels $k(x,y)$ and $r(x,y)$ will be calculated later.

Motivated by [13–15], we introduce the backstepping transformation (2) which maps the control system (1) into the following target system:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t),\ x \in (0,1),\ t > 0, \\ w_x(0,t) = c_0 w(0,t), & t > 0, \\ w_x(1,t) = -c_1 w_t(1,t), & t > 0, \end{cases} \tag{3}$$

which is exponentially stable for $c_0 > 0$ and $c_1 > 0$ (see, e.g.,[8]).

Secondly, by computing the partial derivative of both sides of the transformation (2) with respect to x, we can attain that

$$w_x(x,t) = u_x(x,t) - \int_0^x k_x(x,y)u(y,t)\mathrm{d}y - k(x,x)u(x,t) - \int_0^{x_0} r_x(x,y)u(y,t)\mathrm{d}y. \tag{4}$$

Taking $x = 1$ in Eq. (4) and using $w_x(1,t) = -c_1 w_t(1,t)$ in Eq. (3), it gives that

$$u_x(1,t) = -c_1 w_t(1,t) + k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y + \int_0^{x_0} r_x(1,y)u(y,t)\mathrm{d}y. \tag{5}$$

By taking the partial derivative of the transformation (2) with respect to t, we get

$$w_t(x,t) = u_t(x,t) - \int_0^x k(x,y)u_t(y,t)\mathrm{d}y - \int_0^{x_0} r(x,y)u_t(y,t)\mathrm{d}y. \tag{6}$$

Then, from Eqs. (5)–(6), the controller $U(t)$ can be obtained.

$$\begin{aligned} U(t) = u(1,t) = {} & \frac{c_1}{k(1,1)}w_t(1,t) - \frac{1}{k(1,1)}\int_0^1 k_x(1,y)u(y,t)\mathrm{d}y + \\ & \frac{1}{k(1,1)}(u_x(1,t) - \int_0^{x_0} r_x(1,y)u(y,t)\mathrm{d}y) = \\ & \frac{1}{k(1,1)}(c_1u_t(1,t) + u_x(1,t)) - \\ & \frac{\int_0^1 (c_1k(1,y)u_t(y,t) + k_x(1,y)u(y,t))\mathrm{d}y}{k(1,1)} - \\ & \frac{\int_0^{x_0} (c_1r(1,y)u_t(y,t) + r_x(1,y)u(y,t))\mathrm{d}y}{k(1,1)}. \end{aligned} \tag{7}$$

Thirdly, in order to construct the stabilization of the closed-loop system (1), it is necessary to prove the boundedness and reversibility of the transformation (2). In Section 4, we will find the inverse transformation. Then, choosing the suitable norm, the stabilization of the closed-loop system is constructed by using the boundedness of the transformation.

3 Calculation of kernels

Differentiating (4) with respect to x, we obtain that

$$\begin{aligned} w_{xx}(x,t) = {} & u_{xx}(x,t) - k'(x,x)u(x,t) - \\ & k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)\mathrm{d}y - \\ & k(x,x)u_x(x,t) - \int_0^{x_0} r_{xx}(x,y)u(y,t)\mathrm{d}y, \end{aligned} \tag{8}$$

where $k'(x,x)$ represents the derivative of function $k(x,x)$ as follows:

$$k'(x,x) = k_x(x,x) + k_y(x,x)$$

and

$$k_x(x,x) = \frac{\partial k(x,y)}{\partial x}|_{y=x},\ k_y(x,x) = \frac{\partial k(x,y)}{\partial y}|_{y=x}.$$

From the system (1), differentiating Eq. (6) with respect to t, we have

$$\begin{aligned} w_{tt}(x,t) = {} & u_{tt}(x,t) - \int_0^x k(x,y)u_{tt}(y,t)\mathrm{d}y - \\ & \int_0^{x_0} r(x,y)u_{tt}(y,t)\mathrm{d}y = \\ & u_{xx}(x,t) - \int_0^x k(x,y)(u_{yy}(y,t) + \\ & \lambda u(x_0,t))\mathrm{d}y + \lambda u(x_0,t) - \\ & \int_0^{x_0} r(x,y)(u_{yy}(y,t) + \lambda u(x_0,t))\mathrm{d}y. \end{aligned}$$

Through integration by parts and $u_x(0,t) = 0$ in Eq. (1), it gives that

$$\begin{aligned} w_{tt}(x,t) = {} & u_{xx}(x,t) - (k_y(x,0) + r_y(x,0))u(0,t) - \\ & k(x,x)u_x(x,t) + k_y(x,x)u(x,t) + \\ & r_y(x,x_0)u(x_0,t) - r(x,x_0)u_x(x_0,t) - \\ & \int_0^{x_0} r_{yy}(x,y)u(y,t)\mathrm{d}y + \\ & \lambda(1 - \int_0^x k(x,y)\mathrm{d}y - \int_0^{x_0} r(x,y)\mathrm{d}y)u(x_0,t) - \\ & \int_0^x k_{yy}(x,y)u(y,t)\mathrm{d}y. \end{aligned} \tag{9}$$

By Eqs. (8)–(9), we obtain that

$$\begin{aligned} w_{tt}(x,t) - w_{xx}(x,t) = {} & 2k'(x,x)u(x,t) - \big(r_y(x,0) + k_y(x,0)\big)u(0,t) - \\ & r(x,x_0)u_x(x_0,t) + (\lambda + r_y(x,x_0))u(x_0,t) + \\ & \int_0^{x_0} \big(r_{xx}(x,y) - r_{yy}(x,y)\big)u(y,t)\mathrm{d}y + \end{aligned}$$

$$\int_0^x \big(k_{xx}(x,y)-k_{yy}(x,y)\big)u(y,t)\mathrm{d}y-$$
$$\lambda(\int_0^x k(x,y)\mathrm{d}y+\int_0^{x_0} r(x,y)\mathrm{d}y)u(x_0,t). \tag{10}$$

To satisfy the equation in Eq. (3), the kernels $k(x,y)$ and $r(x,y)$ need to satisfy the under equations

$$\begin{cases} k_{xx}(x,y)-k_{yy}(x,y)=0,\\ k_y(x,0)+r_y(x,0)=0,\\ k'(x,x)=0,\\ r_{xx}(x,y)-r_{yy}(x,y)=0,\\ r(x,x_0)=0 \end{cases} \tag{11}$$

and the compatibility condition

$$\lambda+r_y(x,x_0)-\lambda\int_0^x k(x,y)\mathrm{d}y-$$
$$\lambda\int_0^{x_0} r(x,y)\mathrm{d}y=0. \tag{12}$$

From Eq. (4) and the condition $w_x(0,t)=c_0w(0,t)$ in Eq. (3), we get

$$w_x(0,t)-c_0w(0,t)=$$
$$-(k(0,0)+c_0)u(0,t)-$$
$$\int_0^{x_0}(r_x(0,y)-c_0r(0,y))u(y,t)\mathrm{d}y=0.$$

Hence, we obtain two more conditions $r_x(0,y)=c_0r(0,y)$ and $k(0,0)=-c_0$. From $k(0,0)=-c_0$ and $k'(x,x)=0$, we get $k(x,x)=-c_0$. Therefore, the function $k(x,y)$ satisfies the following equations:

$$\begin{cases} k_{xx}(x,y)-k_{yy}(x,y)=0,\\ k_y(x,0)=-r_y(x,0),\\ k(x,x)=-c_0. \end{cases} \tag{13}$$

Solving Eq. (13), it can be obtained that (see [6] or verify directly)

$$k(x,y)=\int_0^{x-y} r_y(s,0)\mathrm{d}s-c_0. \tag{14}$$

Next, affected by the variable separation method of ODE, we suppose that the function $r(x,y)$ can be expressed as

$$r(x,y)=p(x)q(y). \tag{15}$$

From Eqs. (11) (15) and $r_x(0,y)=c_0r(0,y)$, it is obtained that

$$\begin{cases} p''(x)q(y)-p(x)q''(y)=0,\\ p'(0)=c_0p(0),\\ q(x_0)=0. \end{cases} \tag{16}$$

Now, to solve the equation (16), we search the possible solutions such that $q''(x)/q(x)=p''(x)/p(x)$ are constants. Let

$$\frac{p''(x)}{p(x)}=\frac{q''(x)}{q(x)}=a^2,$$

where $a>0$ is a constant to be determined.

Therefore, $q(y)$ and $p(x)$ satisfy the equations

$$\begin{cases} q''(y)-a^2q(y)=0,\\ q(x_0)=0 \end{cases} \tag{17}$$

and

$$\begin{cases} p''(x)-a^2p(x)=0,\\ p'(0)=c_0p(0). \end{cases} \tag{18}$$

Solving the problems (17)–(18), we have

$$p(x)=b(\mathrm{e}^{ax}+\frac{a-c_0}{a+c_0}\mathrm{e}^{-ax}), \tag{19}$$
$$q(y)=c\sinh(ay-ax_0), \tag{20}$$

where b and c are constants to be determined.

Now, checking the compatibility condition (12), we can obtain the conditions which the constants a,b,c need to satisfy. First, from Eqs. (14)–(15) and Eqs. (19)–(20), we get

$$\int_0^x k(x,y)\mathrm{d}y=\int_0^x(\int_0^{x-y} q'(0)p(\xi)\mathrm{d}\xi-c_0)\mathrm{d}y=$$
$$\int_0^x(\int_0^{x-y} b(\mathrm{e}^{a\xi}+\frac{a-c_0}{a+c_0}\mathrm{e}^{-a\xi})q'(0)\mathrm{d}\xi)\mathrm{d}y-c_0x=$$
$$\frac{bq'(0)}{a}\int_0^x \mathrm{e}^{ax-ay}\mathrm{d}y-c_0x-$$
$$\frac{bq'(0)}{a}\int_0^x\frac{(a-c_0)\mathrm{e}^{ay-ax}+2c_0}{a+c_0}\mathrm{d}y=$$
$$\frac{bq'(0)}{a^2}(\frac{(a-c_0)\mathrm{e}^{-ax}-2a}{a+c_0}+\mathrm{e}^{ax})-$$
$$\frac{2bc_0q'(0)}{a(a+c_0)}x-c_0x=$$
$$\frac{q'(0)}{a^2}p(x)-\frac{2bq'(0)}{a(a+c_0)}-\frac{2bq'(0)c_0x}{a(a+c_0)}-c_0x=$$
$$\frac{c\cosh(ax_0)p(x)}{a}-\frac{2bc\cosh(ax_0)}{a+c_0}-$$
$$(\frac{2bc\cosh(ax_0)}{a+c_0}+1)c_0x. \tag{21}$$

Next, from Eqs. (15)(20), it gives that

$$\int_0^{x_0} r(x,y)\mathrm{d}y=\int_0^{x_0} p(x)q(y)\mathrm{d}y=$$
$$\frac{cp(x)}{a}(1-\cosh(ax_0)) \tag{22}$$

and

$$r_y(x,x_0)=p(x)q'(x_0)=acp(x). \tag{23}$$

By Eqs. (21)–(23), it holds that

$$\lambda+r_y(x,x_0)-\lambda\int_0^x k(x,y)\mathrm{d}y-\lambda\int_0^{x_0} r(x,y)\mathrm{d}y=$$
$$\lambda(\frac{2bc\cosh(ax_0)}{a+c_0}+1)(c_0x+1)+\frac{a^2-\lambda}{a}cp(x).$$

To satisfy the compatibility condition (12), we take

$$a^2=\lambda, bc=-\frac{a+c_0}{2\cosh(ax_0)}. \tag{24}$$

Then, by Eqs. (15)(19)–(20) and Eq. (24), it holds that

$$r(x,y)=bc(\mathrm{e}^{ax}+\frac{a-c_0}{a+c_0}\mathrm{e}^{-ax})\sinh(ay-ax_0)=$$
$$\frac{-(\sqrt{\lambda}+c_0)}{2\cosh(\sqrt{\lambda}x_0)}(\mathrm{e}^{\sqrt{\lambda}x}+\frac{\sqrt{\lambda}-c_0}{\sqrt{\lambda}+c_0}\mathrm{e}^{-\sqrt{\lambda}x})\times$$
$$\sinh(\sqrt{\lambda}y-\sqrt{\lambda}x_0). \tag{25}$$

By Eqs. (14)(25), the function $k(x,y)$ is obtained

$$k(x,y)=\int_0^{x-y}r_y(s,0)\mathrm{d}s-c_0=$$
$$\frac{\sqrt{\lambda}-c_0}{2}\mathrm{e}^{\sqrt{\lambda}(y-x)}-\frac{\sqrt{\lambda}+c_0}{2}\mathrm{e}^{\sqrt{\lambda}(x-y)}. \tag{26}$$

According to the above calculation and analysis, the following theorem can be obtained.

Theorem 1 For any $\lambda>0$, Eq. (11) have classical solutions which are defined by Eqs. (25)–(26). And the solutions $k(x,y)$ and $r(x,y)$ are bounded on a triangle $0\leqslant y<x\leqslant 1$.

4 Stability

To establish the stabilization of the closed-loop system (1) under the controller (7), the inverse transformation of the transformation (2) is required. Then, the stabilization of the closed-loop system can be obtained by using the boundedness of the transformation.

4.1 Inverse transformation

The inverse transformation of the transformation (2) can be written as follows:

$$u(x,t)=$$
$$w(x,t)+\int_0^{x_0}h(x,y)w(y,t)\mathrm{d}y+$$
$$\int_0^{x}l(x,y)w(y,t)\mathrm{d}y, \tag{27}$$

where the functions $h(x,y)$ and $l(x,y)$ will be decided later.

Similarly, computing u_{tt} and u_{xx}, we have

$$u_{tt}(x,t)-u_{xx}(x,t)-\lambda u(x_0,t)=$$
$$h(x,x_0)w_x(x_0,t)-2l'(x,x)w(x,t)+$$
$$\big(l_y(x,0)+h_y(x,0)-c_0l(x,0)-$$
$$c_0h(x,0)\big)w(0,t)-\big(h_y(x,x_0)+\lambda\big)w(x_0,t)-$$
$$\int_0^{x}\big(l_{xx}(x,y)-l_{yy}(x,y)\big)w(y,t)\mathrm{d}y-$$
$$\lambda\int_0^{x_0}(h(x_0,y)+l(x_0,y))w(y,t)\mathrm{d}y-$$
$$\int_0^{x_0}(h_{xx}(x,y)-h_{yy}(x,y))w(y,t)\mathrm{d}y. \tag{28}$$

From Eq. (28), to satisfy the equation in Eq. (1), the functions $l(x,y)$ and $h(x,y)$ are determined by the following equations:

$$\begin{cases}l_{xx}(x,y)-l_{yy}(x,y)=0,\\ l_y(x,0)+h_y(x,0)-c_0l(x,0)-c_0h(x,0)=0,\\ l'(x,x)=0,\\ h_{xx}(x,y)-h_{yy}(x,y)+\lambda h(x_0,y)+\lambda l(x_0,y)=0,\\ h_y(x,x_0)+\lambda=0,\\ h(x,x_0)=0.\end{cases} \tag{29}$$

From Eq. (27), we can obtain

$$u_x(0,t)=\big(l(0,0)+c_0\big)w(0,t)+\int_0^{x_0}h_x(0,y)w(y,t)\mathrm{d}y.$$

To satisfy the boundary condition $u_x(0,t)=0$, take $l(0,0)=-c_0$ and $h_x(0,y)=0$. From $l'(x,x)=0$ and $l(0,0)=-c_0$, we get $l(x,x)=-c_0$. The equations which the the functions $l(x,y)$ and $h(x,y)$ satisfy are the following equations:

$$\begin{cases}l_{xx}(x,y)-l_{yy}(x,y)=0,\\ l_y(x,0)+h_y(x,0)-c_0l(x,0)-c_0h(x,0)=0,\\ l(x,x)=-c_0,\\ h_{xx}(x,y)-h_{yy}(x,y)+\lambda h(x_0,y)+\lambda l(x_0,y)=0,\\ h_y(x,x_0)+\lambda=0,\\ h(x,x_0)=0,\\ h_x(0,y)=0.\end{cases} \tag{30}$$

Now, to search a solution of Eq. (30), we consider the kernel function $l(x,y)$ is a constant. So we can obtain

$$l(x,y)=-c_0. \tag{31}$$

Then the equations of the kernel function $h(x,y)$ is expressed as follows:

$$\begin{cases}h_{xx}(x,y)-h_{yy}(x,y)+\lambda h(x_0,y)=\lambda c_0,\\ h_y(x,x_0)+\lambda=0,\\ h(x,x_0)=0,\\ h_x(0,y)=0,\\ h_y(x,0)+c_0^2-c_0h(x,0)=0.\end{cases} \tag{32}$$

Using the similar method in Section 3, we consider that the problem of $h(x,y)$ has a solution of separation variables. $h(x,y)$ is expressed as follows:

$$h(x,y)=m(x)n(y). \tag{33}$$

From Eqs. (32)–(33), we obtain

$$\begin{cases} m''(x)n(y) - m(x)n''(y) + \lambda m(x_0)n(y) = \lambda c_0, \\ m(x)n'(x_0) + \lambda = 0, \\ n(x_0) = 0, \\ m'(0) = 0, \\ m(x)n'(0) + c_0^2 - c_0 m(x)n(0) = 0. \end{cases} \tag{34}$$

For simplicity, to solve Eq. (34), we suppose that $m(x)$ is a constant. Let

$$m(x) = M, \tag{35}$$

where M is a nonzero constant to be determined. It is easy to verify that $m'(0)=0$. By Eqs. (34)–(35), we obtain that $n(y)$ needs to satisfy the following equations:

$$\begin{cases} n''(y) - \lambda n(y) + \dfrac{\lambda c_0}{M} = 0, \\ n'(x_0) = \dfrac{-\lambda}{M}, \\ n(x_0) = 0. \end{cases} \tag{36}$$

Then the constant M and the solution of Eq. (36) need to satisfy the compatibility condition

$$Mn'(0) + c_0^2 - Mc_0 n(0) = 0. \tag{37}$$

Solving the second order ordinary differential Eq. (36), we can obtain

$$n(y) = d_1 \mathrm{e}^{\sqrt{\lambda}y} + d_2 \mathrm{e}^{-\sqrt{\lambda}y} + \frac{c_0}{M}. \tag{38}$$

where the constants d_1 and d_2 need to be determined.

To calculate the constants d_1, d_2 and M, we substitute Eq. (38) into $n'(x_0) = \dfrac{-\lambda}{M}$, $n(x_0) = 0$ and the compatibility condition (37). So the constants satisfy the following equations.

$$\begin{cases} d_1 \mathrm{e}^{\sqrt{\lambda}x_0} + d_2 \mathrm{e}^{-\sqrt{\lambda}x_0} + \dfrac{c_0}{M} = 0, \\ \sqrt{\lambda} d_1 \mathrm{e}^{\sqrt{\lambda}x_0} - \sqrt{\lambda} d_2 \mathrm{e}^{-\sqrt{\lambda}x_0} = \dfrac{-\lambda}{M}, \\ M(\sqrt{\lambda}d_1 - \sqrt{\lambda}d_2) - Mc_0(d_1 + d_2) = 0. \end{cases} \tag{39}$$

Simplifying the equation (39), we obtain

$$c_0 = \sqrt{\lambda}, d_2 = 0, Md_1 = -\sqrt{\lambda}\mathrm{e}^{-\sqrt{\lambda}x_0}. \tag{40}$$

By Eqs. (33)(35)(38) and (40), we have

$$h(x,y) = -\sqrt{\lambda}\mathrm{e}^{\sqrt{\lambda}(y-x_0)} + \sqrt{\lambda}. \tag{41}$$

From Eq. (31) and $c_0 = \sqrt{\lambda}$, we get

$$l(x,y) = -\sqrt{\lambda}. \tag{42}$$

Similarly, based on the above analysis and calculation, the following theorem can be constituted.

Theorem 2 For any $\lambda > 0$, Eq. (30) have classical solutions. And the solutions $h(x,y)$ and $l(x,y)$ are bounded on a triangle $0 \leqslant y < x \leqslant 1$.

4.2 Stability

From Theorems 1 and 2, we can establish the following the theorem.(see, e.g,. [6, 10])

Theorem 3 For any $\lambda > 0$, the closed-loop system (1) with the controller (7) is exponentially stable in the sense of the norm

$$\sqrt{\int_0^1 u_x^2(x,t)\mathrm{d}x + \int_0^1 u_t^2(x,t)\mathrm{d}x + u^2(0,t)}. \tag{43}$$

Proof Firstly, to obtain the stabilization of the closed-loop system (1), we will show that the invertible transformation of Eq. (2) is Eq. (27). From Theorem 1, the transformation (2) defines a linear bounded operator P in the sense of the norm (43), that is

$$w(x,t) = Pu(x) = u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y - \int_0^{x_0} r(x,y)u(y,t)\mathrm{d}y. \tag{44}$$

By the same reason, from the transformation (27) and Theorem 2, a linear bounded operator Q is defined in the sense of the norm (43), that is

$$u(x,t) = Qw(x) = w(x,t) + \int_0^x l(x,y)w(y,t)\mathrm{d}y + \int_0^{x_0} h(x,y)w(y,t)\mathrm{d}y. \tag{45}$$

Then, we will prove that the inverse operator of P is Q. It is required that $PQ = I$ or $QP = I$, where I is the identity operator of the norm (43). This means that $(QP)u = u$. Substituting Eq. (44) into Eq. (45) and exchanging the order of integration for the twice integrals, it gives that

$$\begin{aligned} &(QPu)(x) = \\ &(Pu)(x) + \int_0^x l(x,y)(Pu)(y)\mathrm{d}y + \\ &\int_0^{x_0} h(x,y)(Pu)(y)\mathrm{d}y = \\ &u(x) - \int_0^x k(x,y)u(y)\mathrm{d}y + \int_0^x l(x,y)u(y)\mathrm{d}y - \\ &\int_0^{x_0} r(x,y)u(y)\mathrm{d}y + \int_0^{x_0} h(x,y)u(y)\mathrm{d}y - \\ &\int_0^x \int_0^y l(x,y)k(y,\xi)u(\xi)\mathrm{d}\xi\mathrm{d}y - \\ &\int_0^x \int_0^{x_0} l(x,y)r(y,\xi)u(\xi)\mathrm{d}\xi\mathrm{d}y - \\ &\int_0^{x_0} \int_0^y h(x,y)k(y,\xi)u(\xi)\mathrm{d}\xi\mathrm{d}y - \\ &\int_0^{x_0} \int_0^{x_0} h(x,y)r(y,\xi)u(\xi)\mathrm{d}\xi\mathrm{d}y = \\ &u(x) - \int_0^x k(x,\xi)u(\xi)\mathrm{d}\xi + \int_0^x l(x,\xi)u(\xi)\mathrm{d}\xi - \\ &\int_0^{x_0} r(x,\xi)u(\xi)\mathrm{d}\xi + \int_0^{x_0} h(x,\xi)u(\xi)\mathrm{d}\xi - \\ &\int_0^x \int_\xi^x l(x,y)k(y,\xi)u(\xi)\mathrm{d}y\mathrm{d}\xi - \\ &\int_0^{x_0} \int_0^x l(x,y)r(y,\xi)u(\xi)\mathrm{d}y\mathrm{d}\xi - \end{aligned}$$

$$\int_0^{x_0}\int_\xi^{x_0} h(x,y)k(y,\xi)u(\xi)\mathrm{d}y\mathrm{d}\xi-$$
$$\int_0^{x_0}\int_0^{x_0} h(x,y)r(y,\xi)u(\xi)\mathrm{d}y\mathrm{d}\xi=$$
$$u(x)+\int_0^x F(x,\xi)u(\xi)\mathrm{d}\xi-\int_0^{x_0} G(x,\xi)u(\xi)\mathrm{d}\xi, \tag{46}$$

where

$$G(x,\xi)=$$
$$r(x,\xi)-h(x,\xi)+\int_0^x l(x,y)r(y,\xi)\mathrm{d}y+$$
$$\int_\xi^{x_0} h(x,y)k(y,\xi)\mathrm{d}y+\int_0^{x_0} h(x,y)r(y,\xi)\mathrm{d}y \tag{47}$$

and

$$F(x,\xi)=$$
$$l(x,\xi)-k(x,\xi)-\int_\xi^x l(x,y)k(y,\xi)\mathrm{d}y. \tag{48}$$

Now we prove that $G(x,\xi)=0,\ F(x,\xi)=0$. From Eqs. (25)–(26) and $c_0=\sqrt{\lambda}$, the kernel functions $k(x,y)$ and $r(x,y)$ can be rewritten as

$$r(x,y)=\frac{-\sqrt{\lambda}\mathrm{e}^{\sqrt{\lambda}x}}{\cosh(\sqrt{\lambda}x_0)}\sinh(\sqrt{\lambda}y-\sqrt{\lambda}x_0), \tag{49}$$
$$k(x,y)=-\sqrt{\lambda}\mathrm{e}^{\sqrt{\lambda}(x-y)}. \tag{50}$$

Hence, from Eqs. (48)(50) and (42), we obtain

$$F(x,\xi)=l(x,\xi)-k(x,\xi)-\int_\xi^x l(x,y)k(y,\xi)\mathrm{d}y=$$
$$\sqrt{\lambda}\mathrm{e}^{\sqrt{\lambda}(x-\xi)}-\sqrt{\lambda}-\int_\xi^x \lambda\mathrm{e}^{\sqrt{\lambda}(y-\xi)}\mathrm{d}y=0. \tag{51}$$

For $G(x,\xi)$, by Eqs. (42) and (49), it gives that

$$\int_0^x l(x,y)r(y,\xi)\mathrm{d}y=$$
$$\int_0^x \frac{\lambda\mathrm{e}^{\sqrt{\lambda}y}}{\cosh(\sqrt{\lambda}x_0)}\sinh(\sqrt{\lambda}\xi-\sqrt{\lambda}x_0)\mathrm{d}y=$$
$$\frac{\sqrt{\lambda}(\mathrm{e}^{\sqrt{\lambda}x}-1)}{\cosh(\sqrt{\lambda}x_0)}\sinh(\sqrt{\lambda}\xi-\sqrt{\lambda}x_0). \tag{52}$$

By Eqs. (41)(50), it holds that

$$\int_\xi^{x_0} h(x,y)k(y,\xi)\mathrm{d}y=$$
$$\int_\xi^{x_0} \lambda(\mathrm{e}^{\sqrt{\lambda}(2y-x_0-\xi)}-\mathrm{e}^{\sqrt{\lambda}(y-\xi)})\mathrm{d}y=$$
$$\sqrt{\lambda}-\sqrt{\lambda}\cosh(\sqrt{\lambda}x_0-\sqrt{\lambda}\xi). \tag{53}$$

Then, we substitute Eqs. (41) and (49) into

$$\int_0^{x_0} h(x,y)r(y,\xi)\mathrm{d}y,$$

we have

$$\int_0^{x_0} h(x,y)r(y,\xi)\mathrm{d}y=$$
$$\frac{-\lambda\sinh(\sqrt{\lambda}\xi-\sqrt{\lambda}x_0)\int_0^{x_0}(\mathrm{e}^{\sqrt{\lambda}y}-\mathrm{e}^{\sqrt{\lambda}(2y-x_0)})\mathrm{d}y}{\cosh(\sqrt{\lambda}x_0)}=$$
$$\frac{-\sqrt{\lambda}\sinh(\sqrt{\lambda}\xi-\sqrt{\lambda}x_0)(\cosh(\sqrt{\lambda}x_0)-1)}{\cosh(\sqrt{\lambda}x_0)}. \tag{54}$$

Therefore, by Eqs. (52)–(54)(49) and (41), it gives that

$$G(x,\xi)=$$
$$\frac{-\sqrt{\lambda}\mathrm{e}^{\sqrt{\lambda}x}}{\cosh(\sqrt{\lambda}x_0)}\sinh(\sqrt{\lambda}\xi-\sqrt{\lambda}x_0)+$$
$$\sqrt{\lambda}\mathrm{e}^{\sqrt{\lambda}(\xi-x_0)}-\sqrt{\lambda}\cosh(\sqrt{\lambda}x_0-\sqrt{\lambda}\xi)+$$
$$\frac{\sqrt{\lambda}(\mathrm{e}^{\sqrt{\lambda}x}-1)}{\cosh(\sqrt{\lambda}x_0)}\sinh(\sqrt{\lambda}\xi-\sqrt{\lambda}x_0)-$$
$$\frac{\sqrt{\lambda}\sinh(\sqrt{\lambda}\xi-\sqrt{\lambda}x_0)(\cosh(\sqrt{\lambda}x_0)-1)}{\cosh(\sqrt{\lambda}x_0)}=$$
$$\sqrt{\lambda}\mathrm{e}^{\sqrt{\lambda}(\xi-x_0)}-\frac{\sqrt{\lambda}}{2}(\mathrm{e}^{\sqrt{\lambda}x_0-\sqrt{\lambda}\xi}+\mathrm{e}^{\sqrt{\lambda}\xi-\sqrt{\lambda}x_0}+$$
$$\mathrm{e}^{\sqrt{\lambda}\xi-\sqrt{\lambda}x_0}-\mathrm{e}^{\sqrt{\lambda}x_0-\sqrt{\lambda}\xi})=0. \tag{55}$$

By Eqs. (46)(51) and (55), we obtain that $(QP)u=u$. It is that the inverse operator of P is Q. So, we get that the invertible transformation of Eq. (2) is Eq. (27).

By the boundedness of the operator P and Q, there are two positive constants C_1 and C_2 such that

$$\|Pu\|\leqslant C_1\|u\|, \tag{56}$$
$$\|Qw\|=\|P^{-1}w\|\leqslant C_2\|w\|, \tag{57}$$

where $\|\cdot\|$ is the norm of Eq. (43).

Then, we will show that the closed-loop system (1) is exponentially stable under the controller (7) for the initial state $u_0(x)$, in the sense of the norm (43).

For the initial state $u_0(x)$, $u(x,t)$ is the solution of the closed-loop system (1). According to Theorem 1, the function $w(x,t)$ determined by $w(x,t)=(Pu)(x,t)$ is the solution of the target system (3) under the initial state $w_0(x)=(Pu_0)(x)$. By Eq. (56), it gives that

$$\|w_0\|=\|Pu_0\|\leqslant C_1\|u_0\|. \tag{58}$$

Meanwhile, the system (3) is exponentially stable in the sense of the norm (43)(see, e.g., [6]). Hence, it means that the solution $w(x,t)$ of Eq. (3) satisfies the following inequality for the initial state $w_0(x)$

$$\|w(t)\|\leqslant\sqrt{N}\mathrm{e}^{-\frac{t}{2N}}\|w_0\|, \tag{59}$$

where N is a large positive number.

Finally, by Eqs. (44)–(45)(57)–(59) and $Q=P^{-1}$, it holds that

$$\begin{aligned}\|u(t)\|=&\|P^{-1}w\|\leqslant C_2\|w\|\leqslant\\ &C_2\sqrt{N}\mathrm{e}^{-\frac{t}{2N}}\|w_0\|=\\ &C_2\sqrt{N}\mathrm{e}^{-\frac{t}{2N}}\|Pu_0\|\leqslant\\ &C_1C_2\sqrt{N}\mathrm{e}^{-\frac{t}{2N}}\|u_0\|.\end{aligned}$$

That is $\|u(t)\| \leqslant C_1C_2\sqrt{N}\mathrm{e}^{-\frac{t}{2N}}\|u_0\|$. Therefore, the closed-loop system (1) is exponentially stable in the sense of the norm

$$\sqrt{\int_0^1 u_x^2(x,t)\mathrm{d}x + \int_0^1 u_t^2(x,t)\mathrm{d}x + u^2(0,t)}.$$

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