具有外部干扰的不稳定剪切梁的输入–状态稳定

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摘要: 本文针对一端受到范德华力的不稳定剪切梁方程, 考虑其输入–状态稳定性问题. 通过可逆变换把方程等价地变成一个具有反馈循环的2×2的一阶运输方程与常微分方程的耦合系统. 通过自抗扰控制方法, 给出具有时变增益的扩张状态观测器来估计干扰. 应用Backstepping变换和干扰估计量, 设计系统的反馈控制来补偿系统本身的不稳定以及消除匹配干扰. 通过C_0–半群方法证明闭环系统的适定性, 以及Lyapunov方法证明闭环系统的输入–状态稳定性. 数值仿真验证理论结果的正确性.

关键词: 不稳定剪切梁方程; 反馈控制; 自抗扰控制方法; 输入–状态稳定性

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Input-to-state stabilization of a destabilized shear beam with external disturbances

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Abstract: In this paper, the input-to-state stabilization of an unstable shear beam with van der Waals forces at one end is considered. Through an invertible transformation, the equation is transformed into a 2×2 system of first-order transport equations, which convects in opposite directions cascaded with an ordinary differential equation (ODE). Using the active disturbance rejection control (ADRC) method, an extended state observer with the time-varying gain is given to estimate the disturbance. Applying the backstepping transformation and the disturbance estimation, the feedback control of the closed-loop system is proposed to compensate for the instability of the system itself and cancel the matched disturbance. By the C_0-semigroup method and the Lyapunov method, the well-posedness and the input-to-state stability (ISS) of the closed-loop systems are proved, respectively. The validity of the theoretical results is verified by numerical simulations.

Key words: destabilized shear beam; feedback control; active disturbance rejection control; input-to-state stabilization

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1 引言

在机器人柔性机械手臂, 涡轮发动机的叶片和天线等实际应用的推动下, 柔性梁成为一个活跃的研究课题. 在线性柔性梁模型中, Euler-Bernoulli梁是最简单的梁模型, 而剪切梁模型是将剪切变形的影响添加到Euler-Bernoulli梁模型中得到的[1].

在没有干扰的情况下, 梁方程的镇定问题已经有不少的研究成果, 例如, 通过设计非同位控制, 梁方程达到了稳定[2]; 随后, 应用Backstepping方法, 输出反馈控制被设置来镇定剪切梁方程[3]; 通过将剪切梁方程转化为具有反馈循环的2×2的一阶运输方程与常微分方程(ordinary differential equation, ODE)的耦合系统, 状态反馈控制被设置使得系统达到指数稳定[4]; 最近, 通过内模原理, 不同的误差反馈控制被设计来分别解决Timoshenko梁[5]和Euler-Bernoulli梁[6]的鲁棒输出调节问题. 当外部干扰从控制端进入梁方程时,

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Guo和Jin[7]分别采用自抗扰控制(active disturbance rejection control, ADRC)和滑模控制(sliding mode control, SMC)方法来设计边界反馈控制, 使得一维Euler-Bernoulli梁方程达到稳定; Liu等人[8]运用滑模控制方法研究了一维剪切梁方程, 获得了系统的指数稳定性结果.

当外部干扰从空间域的内部进入系统时, 其对系统的影响通常很复杂. 因此, 引入了输入–状态稳定性的概念[9–10], 以描述此类干扰对系统的影响. 对于具有扰动的一维抛物型偏微分方程, Karafyllis和Krstic[11]给出了在不同范数下的输入–状态稳定性结果. Mironchenko等人[12–13]应用Lyapunov方法研究了无穷维系统的输入–状态稳定性. Zhang等人[14]讨论了所有通道带有干扰的ODE和反应扩散方程耦合系统的输入–状态稳定性. Kawan等人[15]应用小增益定理研究了由非线性系统组成的无限网络的输入–状态稳定性问题. 然而, 关于梁方程的输入–状态稳定性的研究较少. 因此, 本文研究同时具有边界和分布干扰的不稳定剪切梁方程的输入–状态稳定性问题.

本文的组织结构如下: 在第2节中, 首先介绍系统的模型, 并通过一系列变换把原系统转化为等价系统, 然后应用Backstepping方法将2×2的一阶运输方程与ODE的耦合系统转化为目标系统, 再采用自抗扰控制方法设计具有时变增益的扩张状态观测器来估计干扰, 最后设计控制来补偿系统自身的不稳定以及抵消匹配干扰; 第3节致力于证明闭环系统的适定性; 第4节证明闭环系统具有输入–状态稳定性; 第5节给出数值仿真来验证控制器的有效性; 第6节是小结.

2 问题描述

本文研究具有边界和分布干扰的剪切梁方程

$$\begin{cases} u_{tt}(x,t)=u_{xx}(x,t)-w_x(x,t)+d_1(x,t),\\ u_x(0,t)=w(0,t)-au(0,t)+d_2(t),\\ u_x(1,t)=U_1(t)+d_3(t),\\ w_{xx}(x,t)-b^2w(x,t)+b^2u_x(x,t)=b^2d_4(x,t),\\ w_x(0,t)=0,\\ w(1,t)=U_2(t)+d_5(t), \end{cases} \tag{1}$$

其中: $u(x,t)$表示梁在位置x和时间t的横向位移, w是弯曲角度, $a,b>0$是两个常数, $U_1(t)$和$U_2(t)$是控制输入. 未知外部干扰$d_1,\ d_4\in H^1_{\rm loc}(0,\infty;L^2(0,1))$, $d_2,d_3,d_5\in H^1_{\rm loc}(0,\infty)$. 令$\langle\cdot,\cdot\rangle$和$\|\cdot\|$分别表示Hilbert空间$L^2(0,1)$的内积和范数.

本文的创新点主要体现在: 1) 考虑了控制端和非控制端同时有干扰进入梁方程的情况, 针对控制端进入的匹配干扰, 通过使用ADRC方法来估计, 并进一步消除它; 针对非控制端进入的干扰, 研究其相应的输入–状态稳定性问题; 2) 本文通过设计具有时变增益的扩张状态观测器使得干扰估计量可以指数地估计干扰.

为了更简便的设计控制, 首先给出模型(1)的等价形式

$$\begin{cases} u_{tt}(x,t)=u_{xx}(x,t)+b^2u(x,t)-\\ \qquad b^2\cos h(bx)u(0,t)+d_1(x,t)+\\ \qquad b^3\displaystyle\int_0^x\sin h(b(x-s))u(s,t)\mathrm{d}s-\\ \qquad \dfrac{b\sin h(bx)}{\cos h(b)}(U_2(t)+d_5(t)+\\ \qquad b\displaystyle\int_0^1\sin h(b(1-s))u_s(s,t)\mathrm{d}s)+\\ \qquad \dfrac{b^2\sin h(bx)}{\cos h(b)}\times\\ \qquad \displaystyle\int_0^1\sin h(b(1-s))d_4(s,t)\mathrm{d}s-\\ \qquad b^2\displaystyle\int_0^x\cos h(b(x-s))d_4(s,t)\mathrm{d}s,\\ u_x(0,t)=-au(0,t)+\dfrac{1}{\cos h(b)}(U_2(t)+d_5(t)+\\ \qquad b\displaystyle\int_0^1\sin h(b(1-s))u_s(s,t)\mathrm{d}s)+\\ \qquad d_2(t)-\dfrac{b}{\cos h(b)}\times\\ \qquad \displaystyle\int_0^1\sin h(b(1-s))d_4(s,t)\mathrm{d}s,\\ u_x(1,t)=U_1(t)+d_3(t). \end{cases} \tag{2}$$

设计如下控制$U_2(t)$来抵消方程和边界上的定积分项:

$$U_2(t)=-b\int_0^1\sin h(b(1-s))u_s(s,t)\mathrm{d}s. \tag{3}$$

进一步得到如下偏微分方程:

$$\begin{cases} u_{tt}(x,t)=u_{xx}(x,t)+b^2u(x,t)-\\ \qquad b^2\cos h(bx)u(0,t)+D_1(x,t)+\\ \qquad b^3\displaystyle\int_0^x\sin h(b(x-s))u(s,t)\mathrm{d}s,\\ u_x(0,t)=-au(0,t)+D_2(t),\\ u_x(1,t)=U_1(t)+d_3(t), \end{cases} \tag{4}$$

其中:

$$A(x)=\frac{b\sin h(bx)}{\cos h(b)}, \tag{5}$$

$$B(x,t)=bA(x)\int_0^1\sin h(b(1-s))d_4(s,t)\mathrm{d}s-b^2\int_0^x\cos h(b(x-s))d_4(s,t)\mathrm{d}s, \tag{6}$$

$$C(t)=\frac{b}{\cos h(b)}\int_0^1\sin h(b(1-s))d_4(s,t)\mathrm{d}s, \tag{7}$$

$$D_1(x,t)=d_1(x,t)-A(x)d_5(t)+B(x,t), \tag{8}$$

$$D_2(t)=d_2(t)-C(t)+\frac{1}{\cos h(b)}d_5(t). \tag{9}$$

下面, 引入如下变换:

$$\begin{cases} \alpha(x,t)=u_t(x,t)+u_x(x,t),\\ \beta(x,t)=u_t(x,t)-u_x(x,t). \end{cases} \tag{10}$$

其逆变换为

$$\begin{cases} u_t(x,t)=\dfrac{\alpha(x,t)+\beta(x,t)}{2}, \\ u_x(x,t)=\dfrac{\alpha(x,t)-\beta(x,t)}{2}. \end{cases} \tag{11}$$

变换(10)把系统(4)转变为如下等价形式:

$$u_t(0,t)=au(0,t)+\alpha(0,t)-D_2(t), \tag{12}$$

$$\beta_t(x,t)=-\beta_x(x,t)+E_0(x,t)+D_1(x,t), \tag{13}$$

$$\beta(0,t)=\alpha(0,t)+2au(0,t)-2D_2(t), \tag{14}$$

$$\alpha_t(x,t)=\alpha_x(x,t)+E_0(x,t)+D_1(x,t), \tag{15}$$

$$\alpha(1,t)=u_t(1,t)+U_1(t)+d_3(t), \tag{16}$$

其中

$$E_0(x,t)=\frac{b^2}{2}\int_0^x \cos h(b(x-y))\big(\alpha(y,t)-\beta(y,t)\big)\mathrm{d}y. \tag{17}$$

此时, 系统(4)转化为一个具有相反传输方向的2×2一阶运输方程与不稳定ODE的耦合系统.

2.1 Backstepping变换

引入如下Backstepping变换:

$$\xi(x,t)=\alpha(x,t)-\int_0^x k(x,y)\alpha(y,t)\mathrm{d}y-\int_0^x l(x,y)\beta(y,t)\mathrm{d}y+m(x)u(0,t), \tag{18}$$

其逆变换为

$$\alpha(x,t)=\xi(x,t)+\int_0^x p(x,y)\xi(y,t)\mathrm{d}y+\int_0^x q(x,y)\beta(y,t)\mathrm{d}y-n(x)u(0,t). \tag{19}$$

对于任意$(x,y)\in\varOmega=\{(x,y)\in\mathbb{R}^2|0\leqslant y\leqslant x\leqslant 1\}$, 核函数$(k(x,y),\ l(x,y),\ m(x)),(p(x,y),q(x,y),n(x))$在$\varOmega\times\varOmega\times[0,1]$中满足如下方程组:

$$\begin{cases} k_x(x,y)=-k_y(x,y)-\dfrac{b^2}{2}\cos h(b(x-y))+\dfrac{b^2}{2}\displaystyle\int_y^x k(x,s)\cos h(b(s-y))\mathrm{d}s+\dfrac{b^2}{2}\displaystyle\int_y^x l(x,s)\cos h(b(s-y))\mathrm{d}s, \\ l_x(x,y)=l_y(x,y)+\dfrac{b^2}{2}\cos h(b(x-y))-\dfrac{b^2}{2}\displaystyle\int_y^x l(x,s)\cos h(b(s-y))\mathrm{d}s-\dfrac{b^2}{2}\displaystyle\int_y^x k(x,s)\cos h(b(s-y))\mathrm{d}s, \\ l(x,x)=0, \\ k(x,0)=l(x,0)-m(x), \\ m'(x)=am(x)-2al(x,0), \\ m(0)=a+c \end{cases} \tag{20}$$

和

$$\begin{cases} p_x(x,y)=-p_y(x,y)-\dfrac{b^2}{2}\cos h(b(x-y))+\dfrac{b^2}{2}\displaystyle\int_y^x q(x,s)\cos h(b(s-y))\mathrm{d}s-\dfrac{b^2}{2}\displaystyle\int_y^x \cos h(b(x-s))p(s,y)\mathrm{d}s+\dfrac{b^2}{2}\displaystyle\int_y^x\int_y^s q(x,s)\times\cos h(b(s-\tau))p(\tau,y)\mathrm{d}\tau\mathrm{d}s, \\ q_y(x,y)=q_x(x,y)-\dfrac{b^2}{2}\cos h(b(x-y))+\dfrac{b^2}{2}\displaystyle\int_y^x \cos h(b(x-s))q(s,y)\mathrm{d}s+\dfrac{b^2}{2}\displaystyle\int_y^x q(x,s)\cos h(b(s-y))\mathrm{d}s-\dfrac{b^2}{2}\displaystyle\int_y^x\int_y^s q(x,s)\times\cos h(b(s-\tau))q(\tau,y)\mathrm{d}\tau\mathrm{d}s, \\ p(x,0)=q(x,0)-n(x), \\ q(x,x)=0, \\ n'(x)=-cn(x)-(a-c)q(x,0)-\dfrac{b^2}{2}\displaystyle\int_0^x \cos h(b(x-y))n(y)\mathrm{d}y+\dfrac{b^2}{2}\displaystyle\int_0^x\int_y^x q(x,s)\times\cos h(b(s-y))n(y)\mathrm{d}s\mathrm{d}y, \\ n(0)=a+c, \end{cases} \tag{21}$$

其中$c>2a$. 因此, 得到如下目标系统:

$$u_t(0,t)=-cu(0,t)+\xi(0,t)-D_2(t), \tag{22}$$

$$\beta_t(x,t)=-\beta_x(x,t)+D_1(x,t)+\frac{b^2}{2}\int_0^x f(x,y)\xi(y,t)\mathrm{d}y-\frac{b^2}{2}\int_0^x \cos h(b(x-y))n(y)u(0,t)\mathrm{d}y+\frac{b^2}{2}\int_0^x v(x,y)\beta(y,t)\mathrm{d}y, \tag{23}$$

$$\beta(0,t)=\xi(0,t)+(a-c)u(0,t)-2D_2(t), \tag{24}$$

$$\xi_t(x,t)=\xi_x(x,t)+D_3(x,t), \tag{25}$$

$$\xi(1,t)=u_t(1,t)+U_1(t)+d_3(t)-\int_0^1 k(1,y)\alpha(y,t)\mathrm{d}y+m(1)u(0,t)-\int_0^1 l(1,y)\beta(y,t)\mathrm{d}y, \tag{26}$$

其中:

$$f(x,y)=\cos h(b(x-y))+$$

$$\int_y^x \cos h(b(x-s))p(s,y)\mathrm{d}s, \tag{27}$$

$$v(x,y)=\int_y^x \cos h(b(x-s))q(s,y)\mathrm{d}s-\cos h(b(x-y)), \tag{28}$$

$$D_3(x,t)=D_1(x,t)+(2l(x,0)-m(x))D_2(t)-\int_0^x (k(x,y)+l(x,y))D_1(y,t)\mathrm{d}y. \tag{29}$$

命题 1 方程组(20)–(21)分别具有光滑解

$(k(x,y),l(x,y),m(x)),(p(x,y),q(x,y),n(x))\in C^1(\Omega)\times C^1(\Omega)\times C^1([0,1])$.

证 通过特征线法和连续逼近方法[16]求解方程组(20)–(21), 证明了方程组存在唯一解[4]. 证毕.

接下来设计控制U_1如下:

$$\begin{aligned}U_1(t)&=U_{11}(t)+U_{12}(t)=\\&\int_0^1(k(1,y)+l(1,y))u_t(y,t)\mathrm{d}y+\\&\int_0^1(l_y(1,y)-k_y(1,y))u(y,t)\mathrm{d}y+\\&k(1,1)u(1,t)-u_t(1,t)+U_{12}(t),\end{aligned} \tag{30}$$

其中U_{12}是连续且待定的. 把控制(30)代入系统(26), 可得

$$u_t(0,t)=-cu(0,t)+\xi(0,t)-D_2(t), \tag{31}$$

$$\begin{aligned}\beta_t(x,t)=&-\beta_x(x,t)+D_1(x,t)+\\&\frac{b^2}{2}\int_0^x f(x,y)\xi(y,t)\mathrm{d}y-\\&\frac{b^2}{2}\int_0^x \cos h(b(x-y))n(y)u(0,t)\mathrm{d}y+\\&\frac{b^2}{2}\int_0^x v(x,y)\beta(y,t)\mathrm{d}y,\end{aligned} \tag{32}$$

$$\beta(0,t)=\xi(0,t)+(a-c)u(0,t)-2D_2(t), \tag{33}$$

$$\xi_t(x,t)=\xi_x(x,t)+D_3(x,t), \tag{34}$$

$$\xi(1,t)=U_{12}(t)+d_3(t). \tag{35}$$

2.2 自抗扰控制

下面使用自抗扰控制方法[17–19]来估计干扰.

定义线性算子$\mathcal{A}_\xi: D(\mathcal{A}_\xi)\to L^2(0,1)$如下:

$$\begin{cases}\mathcal{A}_\xi g=g',\\ D(\mathcal{A}_\xi)=\{g\in H^1(0,1)|\ g(1)=0\}.\end{cases} \tag{36}$$

系统(34)–(35)可以被写为

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(\cdot,t)=\mathcal{A}_\xi\xi(\cdot,t)+ID_3(\cdot,t)+\mathcal{B}_\xi[U_{12}(t)+d_3(t)], \tag{37}$$

其中: I是单位算子, $\mathcal{B}_\xi=\delta(x-1)$. 通过直接计算得到$\mathcal{A}_\xi$的共轭算子

$$\begin{cases}\mathcal{A}_\xi^*k=-k',\\ D(\mathcal{A}_\xi^*)=\{k\in H^1(0,1)|\ k(0)=0\}.\end{cases} \tag{38}$$

因此,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\xi(\cdot,t),k\rangle=&\langle\xi(\cdot,t),\mathcal{A}_\xi^*k\rangle+\langle ID_3(\cdot,t),k\rangle+\\&[U_{12}(t)+d_3(t)]\mathcal{B}_\xi^*k,\end{aligned} \tag{39}$$

其中$k(x)$是测试函数. 特别地, 取$k(x)=x\in D(\mathcal{A}_\xi^*)$, 系统(39)等价于

$$\dot{y}_1(t)=y_2(t)+U_{12}(t)+D_4(t), \tag{40}$$

其中:

$$y_1(t)=\int_0^1 x\xi(x,t)\mathrm{d}x, \tag{41}$$

$$y_2(t)=-\int_0^1 \xi(x,t)\mathrm{d}x, \tag{42}$$

$$D_4(t)=\int_0^1 xD_3(x,t)\mathrm{d}x+d_3(t). \tag{43}$$

接下来设计系统(40)的具有时变增益的扩张状态观测器(extended state observer, ESO), 即

$$\begin{cases}\dot{\hat{y}}_1(t)=y_2(t)+U_{12}(t)+\hat{D}_4(t)-\\ \qquad g(t)[\hat{y}_1(t)-y_1(t)],\\ \dot{\hat{D}}_4(t)=-g^2(t)[\hat{y}_1(t)-y_1(t)],\end{cases} \tag{44}$$

其中$g(t)$是一个时变增益函数并且满足

$$\begin{cases}g(t)>0,\ \dot{g}(t)>0,\ \forall t\geqslant 0,\ g(t)\to\infty\ (t\to\infty),\\ \sup\limits_{t\in[0,\infty)}|\dfrac{\dot{g}(t)}{g(t)}|=\bar{M}<\infty,\ \text{其中}\ \bar{M}>0,\\ \text{且}\ \lim\limits_{t\to\infty}\dfrac{\mathrm{e}^{\mu t}}{g(t)}=0.\end{cases} \tag{45}$$

下面证明$\hat{D}_4(t)$可以用来估计干扰$D_4(t)$.

引理 1 不妨设$\hat{y}_1, \hat{D}_4$是方程组(44)的解, 假设$|\dot{D}_4(t)|\leqslant C\mathrm{e}^{\mu t}, \forall t\geqslant 0$, 其中$C,\mu>0$是两个正常数, 以及$g(t)=k\mathrm{e}^{\nu t}, k>0, \nu>\mu$, 则存在正常数$k_0$和$\kappa$使得

$$g^2(t)|\hat{y}_1(t)-y_1(t)|^2+|\hat{D}_4(t)-D_4(t)|^2\leqslant k_0\mathrm{e}^{-\kappa t}$$

成立.

证 令$\tilde{y}_1(t)=g(t)[\hat{y}_1(t)-y_1(t)]$, $\tilde{D}_4(t)=\hat{D}_4(t)-D_4(t)$, 有

$$\begin{cases}\dot{\tilde{y}}_1(t)=-g(t)[\tilde{y}_1(t)-\tilde{D}_4(t)]+\dfrac{\dot{g}(t)}{g(t)}\tilde{y}_1(t),\\ \dot{\tilde{D}}_4(t)=-g(t)\tilde{y}_1(t)-\dot{D}_4(t).\end{cases} \tag{46}$$

定义

$$V_y(t)=\tilde{y}_1^2(t)+\frac{3}{2}\tilde{D}_4^2(t)-\tilde{y}_1(t)\tilde{D}_4(t), \tag{47}$$

则

$$\frac{1}{2}V_y(t) \leqslant \tilde{y}_1^2(t) + \tilde{D}_4^2(t) \leqslant 2V_y(t). \tag{48}$$

对$V_y(t)$沿着系统(46)的解求导, 可得

$$\dot{V}_y(t) \leqslant -\frac{1}{2}\gamma(t)V_y(t) + 4\sqrt{2}|\dot{D}_4(t)|\sqrt{V_y(t)}, \tag{49}$$

其中

$$\gamma(t) = g(t) - \sup_{t\in[0,\infty)}\{3\frac{\dot{g}(t)}{g(t)}\} \to \infty,\ 随着\ t \to \infty. \tag{50}$$

因此, 存在$t_0 > 0$使得$\gamma(t) > 0, \forall t \geqslant t_0$. 结合式(49), 有

$$\frac{\mathrm{d}\sqrt{V_y(t)}}{\mathrm{d}t} \leqslant -\frac{1}{4}\gamma(t)\sqrt{V_y(t)} + 2\sqrt{2}C\mathrm{e}^{\mu t},\ \forall t \geqslant t_0, \tag{51}$$

即

$$\sqrt{V_y(t)} \leqslant \sqrt{V_y(t_0)}\mathrm{e}^{-\frac{1}{4}\int_{t_0}^t \gamma(\tau)\mathrm{d}\tau} + 2\sqrt{2}C\int_{t_0}^t \mathrm{e}^{\mu s-\frac{1}{4}\int_s^t \gamma(\tau)\mathrm{d}\tau}\mathrm{d}s. \tag{52}$$

显然, 不等式(52)右端第1项随着$t\to\infty$收敛于0. 接下来看右端第2项, 由于随着$t \to \infty$, $\mathrm{e}^{\frac{1}{4}\int_{t_0}^t \gamma(\tau)\mathrm{d}\tau} \to \infty$, 应用L'Hospital规则和式(45) 可得

$$\lim_{t\to\infty}\int_{t_0}^t \mathrm{e}^{\mu s-\frac{1}{4}\int_s^t \gamma(\tau)\mathrm{d}\tau}\mathrm{d}s = \lim_{t\to\infty}\frac{4\mathrm{e}^{\mu t}}{\gamma(t)} = 0. \tag{53}$$

由条件(45)可知, 存在常数$\varepsilon > 0$和$\mu_1 > 0$使得

$$\varepsilon^{-1}\mathrm{e}^{\mu t} \leqslant g(t) \leqslant \mathrm{e}^{\mu_1 t}, \tag{54}$$

这意味着$g(t)$是一个指数类型的函数. 不失一般性, 假设$g(t) = k\mathrm{e}^{\nu t}$, 其中$k > 0$, $\nu > \mu \geqslant 0$. 将$g(t)$代入式(50)可得

$$\gamma(t) = k\mathrm{e}^{\nu t} - 3\nu. \tag{55}$$

接下来将式(55)代入式(53)有

$$\int_{t_0}^t \mathrm{e}^{\mu s-\frac{1}{4}\int_s^t \gamma(\tau)\mathrm{d}\tau}\mathrm{d}s = \int_{t_0}^t \mathrm{e}^{\mu s-\frac{1}{4}\int_s^t (k\mathrm{e}^{\nu\tau}-3\nu)\mathrm{d}\tau}\mathrm{d}s = O(\mathrm{e}^{-(\nu-\mu)t}). \tag{56}$$

另一方面

$$\mathrm{e}^{-\frac{1}{4}\int_{t_0}^t \gamma(\tau)\mathrm{d}\tau} = o(\mathrm{e}^{-(\nu-\mu)t}). \tag{57}$$

因此, 通过式(52)可得, 存在一个常数M_0使得

$$\sqrt{V_y(t)} \leqslant M_0\mathrm{e}^{-(\nu-\mu)t} \tag{58}$$

成立. 进而得到

$$\tilde{y}_1^2(t) + \tilde{D}_4^2(t) \leqslant k_0\mathrm{e}^{-\kappa t}, \tag{59}$$

对$\forall t > t_0$成立, 其中$k_0 = 2M_0^2$, $\kappa = 2(\nu-\mu)$.

证毕.

根据引理1, 设计如下反馈控制:

$$U_{12}(t) = -\hat{D}_4(t). \tag{60}$$

因此, 系统(4)的闭环系统为

$$u_{tt}(x,t) = u_{xx}(x,t) + b^2u(x,t) + D_1(x,t) - b^2\cos h(bx)u(0,t) + b^3\int_0^x \sin h(b(x-s))u(s,t)\mathrm{d}s, \tag{61}$$

$$u_x(0,t) = -au(0,t) + D_2(t), \tag{62}$$

$$u_x(1,t) = -u_t(1,t) + k(1,1)u(1,t) + \int_0^1 (k(1,y)+l(1,y))u_t(y,t)\mathrm{d}y + \int_0^1 (l_y(1,y)-k_y(1,y))u(y,t)\mathrm{d}y - \hat{D}_4(t) + d_3(t), \tag{63}$$

$$\dot{\hat{y}}_1(t) = y_2(t) - g(t)[\hat{y}_1(t) - y_1(t)], \tag{64}$$

$$\dot{\hat{D}}_4(t) = -g^2(t)[\hat{y}_1(t) - y_1(t)], \tag{65}$$

其中:

$$\begin{cases} y_1(t) = \int_0^1 x\xi(x,t)\mathrm{d}x,\ y_2(t) = -\int_0^1 \xi(x,t)\mathrm{d}x, \\ \xi(x,t) = \alpha(x,t) - \int_0^x k(x,y)\alpha(y,t)\mathrm{d}y - \int_0^x l(x,y)\beta(y,t)\mathrm{d}y + m(x)u(0,t), \\ \alpha(x,t) = u_t(x,t) + u_x(x,t), \\ \beta(x,t) = u_t(x,t) - u_x(x,t). \end{cases} \tag{66}$$

3 闭环系统的适定性

本节证明闭环系统(61)–(65)在空间$\mathcal{H}_1 = H^1(0,1) \times L^2(0,1) \times \mathbb{R}^2$中存在唯一温和解. 利用误差变量$\tilde{y}_1$, $\tilde{D}_4$和可逆变换(10)(18), 得到系统(61)–(65)的等价系统如下:

$$u_t(0,t) = -cu(0,t) + \xi(0,t) - D_2(t), \tag{67}$$

$$\beta_t(x,t) = -\beta_x(x,t) + D_1(x,t) + \frac{b^2}{2}\int_0^x f(x,y)\xi(y,t)\mathrm{d}y - \frac{b^2}{2}\int_0^x \cos h(b(x-y))n(y)u(0,t)\mathrm{d}y + \frac{b^2}{2}\int_0^x v(x,y)\beta(y,t)\mathrm{d}y, \tag{68}$$

$$\beta(0,t) = \xi(0,t) + (a-c)u(0,t) - 2D_2(t), \tag{69}$$

$$\xi_t(x,t) = \xi_x(x,t) + D_3(x,t), \tag{70}$$

$$\xi(1,t) = -\hat{D}_4(t) + d_3(t), \tag{71}$$

$$\dot{\tilde{y}}_1(t) = -g(t)[\tilde{y}_1(t) - \tilde{D}_4(t)] + \frac{\dot{g}(t)}{g(t)}\tilde{y}_1(t), \tag{72}$$

$$\dot{\tilde{D}}_4(t) = -g(t)\tilde{y}_1(t) - \dot{D}_4(t). \tag{73}$$

系统(67)–(73)的“$(\tilde{y}_1, \tilde{D}_4)$部分”是常微分方程组, 因此只需要证明系统(67)–(71)的解的存在性.

在状态空间$\mathcal{H}_2 = \mathbb{R} \times L^2(0,1) \times L^2(0,1)$中研究系统(67)–(71), 其内积定义为

$$\begin{aligned}&\langle (X_1, f_1, g_1)^{\mathrm{T}}, (X_2, f_2, g_2)^{\mathrm{T}} \rangle_{\mathcal{H}_2} = \\ &cX_1X_2 + \int_0^1 f_1(x)f_2(x)\mathrm{d}x + \\ &c_1\int_0^1 g_1(x)g_2(x)\mathrm{d}x,\end{aligned} \tag{74}$$

其中$(X_i, f_i, g_i)^{\mathrm{T}} \in \mathcal{H}_2\ (i = 1, 2)$, $c_1 \geqslant 2$.

定义系统(67)–(71)的算子$\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}_2$为

$$\begin{cases}\mathcal{A}(X, h, g) = (-cX + g(0), -h', g'),\\ D(\mathcal{A}) = \{(X, h, g) \in \mathbb{R}\times \\ \qquad H^1(0,1)\times H^1(0,1) | h(0) = \\ \qquad g(0) + (a-c)X,\ g(1) = 0\}.\end{cases} \tag{75}$$

直接计算得到$\mathcal{A}$的共轭算子

$$\begin{cases}\mathcal{A}^*(Y, f, k) = (-cY + \dfrac{a-c}{c}f(0), f', -k'),\\ D(\mathcal{A}^*) = \{(Y, f, k) \in \mathbb{R}\times \\ \qquad H^1(0,1)\times H^1(0,1) | f(1) = 0,\\ \qquad k(0) = cY + f(0)\}.\end{cases} \tag{76}$$

因此系统(67)–(71)可以写为如下发展方程的形式:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Z(\cdot,t) = &\mathcal{A}Z(\cdot,t) + \mathcal{B}Z(\cdot,t) + \\ &I(-D_2(t), D_1(\cdot,t), D_3(\cdot,t))^{\mathrm{T}} + \\ &\mathcal{B}_1(-2D_2(t)) + \mathcal{B}_2(-\hat{D}_4(t) + d_3(t)),\end{aligned} \tag{77}$$

其中: $Z(\cdot,t) = (u(0,t)\ \ \beta(\cdot,t)\ \ \xi(\cdot,t))^{\mathrm{T}}$, I是一个单位算子, 并且

$$\begin{aligned}\mathcal{B}Z(\cdot,t) = (0, &\frac{b^2}{2}\int_0^x v(x,y)\beta(y,t)\mathrm{d}y + \\ &\frac{b^2}{2}\int_0^x f(x,y)\xi(y,t)\mathrm{d}y - \frac{b^2}{2}\times \\ &\int_0^x \cos h(b(x-y))n(y)u(0,t)\mathrm{d}y, 0),\end{aligned} \tag{78}$$

$$\mathcal{B}_1 = (0, \delta(x), 0),\ \mathcal{B}_2 = (0, 0, \delta(x-1)).$$

定理 1 对于任给的$T > 0$以及初值$(u(0,0), \beta(\cdot,0), \xi(\cdot,0)) \in \mathcal{H}_2$, 系统(67)–(71)存在唯一解

$$(u(0,\cdot), \beta, \xi) \in C(0,T; \mathcal{H}_2).$$

证 由定义(75)可得

$$\begin{aligned}&\mathrm{Re}\langle \mathcal{A}(X,h,g)^{\mathrm{T}}, (X,h,g)^{\mathrm{T}}\rangle_{\mathcal{H}_2} \leqslant \\ &-(\frac{c^2}{2} - a^2 + ac)X^2 - \frac{1}{2}h^2(1) - \\ &(\frac{c_1}{2} - 1)g^2(0) \leqslant 0.\end{aligned} \tag{79}$$

因此$\mathcal{A}$是耗散算子. 下面证明$\mathcal{A}^{-1}$存在. 对任意给定的$(Y, f, k) \in \mathcal{H}_2$, 解$\mathcal{A}(X, h, g) = (Y, k, f)$可得

$$\begin{cases}X = -\dfrac{1}{c}(Y + \displaystyle\int_0^1 f(\tau)\mathrm{d}\tau),\\ g(x) = -\displaystyle\int_x^1 f(\tau)\mathrm{d}\tau,\\ h(x) = -\dfrac{a}{c}\displaystyle\int_0^1 f(\tau)\mathrm{d}\tau - \\ \qquad \dfrac{a-c}{c}Y - \displaystyle\int_0^x k(\tau)\mathrm{d}\tau.\end{cases} \tag{80}$$

由式(80)有$h(0) = g(0) + (a-c)X$和$g(1) = 0$. 因此, $(X, h, g) \in D(\mathcal{A})$并且$\mathcal{A}^{-1}$存在. 由Sobolev嵌入定理知, $\mathcal{A}^{-1}$在$\mathcal{H}_2$上是紧的. 根据Lumer-Phillips定理[20]可得, 算子$\mathcal{A}$在$\mathcal{H}_2$上生成一个压缩C_0–半群$\mathrm{e}^{\mathcal{A}t}$.

另一方面, 通过命题1和式(78)可知$\mathcal{B}$是一个有界算子. 因此, 为了获得系统(67)–(71)的适定性, 只需要证明算子$\mathcal{B}_1$和$\mathcal{B}_2$关于$\mathrm{e}^{\mathcal{A}t}$都是可允许的.

考虑系统(67)–(71)的对偶系统

$$\begin{cases}\dot{Y}(t) = -cY(t) + \dfrac{a-c}{c}\phi(0,t),\\ \phi_t(x,t) = \phi_x(x,t),\\ \phi(1,t) = 0,\\ \psi_t(x,t) = -\psi_x(x,t),\\ \psi(0,t) = cY(t) + \phi(0,t),\\ Y_1(t) = \phi(0,t),\ Y_2(t) = \psi(1,t).\end{cases} \tag{81}$$

“ϕ”子方程的解为

$$\phi(x,t) = \begin{cases}\phi_0(x+t), & 0 \leqslant x+t < 1,\\ 0, & x+t \geqslant 1,\end{cases} \tag{82}$$

其中$\phi_0(x)$是初始条件. 通过计算, 存在$T > 0$, 使得

$$\begin{aligned}&\int_0^T |Y_1(t)|^2\mathrm{d}t = \int_0^T |\phi(0,t)|^2\mathrm{d}t = \\ &\|\phi_0\|^2_{L^2(0,1)} \leqslant C_T\|\phi_0\|^2_{L^2(0,1)}\end{aligned} \tag{83}$$

对于$C_T \geqslant 1$成立.

定义系统(81)的能量函数为

$$\begin{aligned}E^*(t) = &\frac{c_1}{2}\int_0^1 \phi^2(x,t)\mathrm{d}x + \\ &\frac{1}{2}\int_0^1 \psi^2(x,t)\mathrm{d}x + \frac{1}{2}cY^2(t).\end{aligned} \tag{84}$$

对$E^*(t)$关于时间t沿着系统(81)的解求导可得

$$\dot{E}^*(t) \leqslant -\frac{1}{2}\psi^2(1,t). \tag{85}$$

那么

$$\int_0^T |Y_2(t)|^2\mathrm{d}t = \int_0^T |\psi(1,t)|^2\mathrm{d}t \leqslant 2E^*(0). \tag{86}$$

另一方面, 对于任意给定的$(\theta, \varPhi, \varPsi) \in \mathcal{H}_2$, 有

$$\mathcal{A}^{*-1}(\theta, \varPhi, \varPsi)^{\mathrm{T}} =$$

$$\begin{pmatrix} -\frac{1}{c}(\theta+\frac{a-c}{c}\int_0^1 \Phi(\tau)\mathrm{d}\tau) \\ -\int_x^1 \Phi(\tau)\mathrm{d}\tau \\ -\theta-\frac{a}{c}\int_0^1 \Phi(\tau)\mathrm{d}\tau-\int_0^x \Psi(\tau)\mathrm{d}\tau \end{pmatrix}, \tag{87}$$

以及

$$\mathcal{B}_1^*\mathcal{A}^{*-1}(\theta,\Phi,\Psi)^{\mathrm{T}}=-\int_0^1 \Phi(\tau)\mathrm{d}\tau, \tag{88}$$

$$\mathcal{B}_2^*\mathcal{A}^{*-1}(\theta,\Phi,\Psi)^{\mathrm{T}}=-\theta-\frac{a}{c}\int_0^1 \Phi(\tau)\mathrm{d}\tau-\int_0^1 \Psi(\tau)\mathrm{d}\tau. \tag{89}$$

因此, $\mathcal{B}_1^*\mathcal{A}^{*-1}$, $\mathcal{B}_2^*\mathcal{A}^{*-1}$都是从$\mathcal{H}_2$到$\mathbb{R}$的有界算子. 结合式(83)(86)表明$\mathcal{B}_1$, $\mathcal{B}_2$对于$\mathrm{e}^{\mathcal{A}^*t}$是可允许的, 从而对于$\mathrm{e}^{\mathcal{A}t}$也是可允许的. 因此, 对于任意的初值$(u(0,0), \beta(\cdot,0),\xi(\cdot,0))\in\mathcal{H}_2$, 系统(67)–(71)存在唯一温和解$(u(0,\cdot),\beta,\xi)\in C((0,T);\mathcal{H}_2)$. 　证毕.

最后, 通过系统(61)–(65)与系统(67)–(73)的等价性, 可以得到系统(61)–(65)的适定性.

定理 2　对任意$T>0$以及初值$(u(\cdot,0), u_t(\cdot,0), \hat{y}_1(0), \hat{D}_4(0))\in\mathcal{H}_1$, 系统(61)–(65)存在唯一温和解

$$(u,u_t,\hat{y}_1,\hat{D}_4)\in C(0,T;\mathcal{H}_1).$$

4　输入–状态稳定性

下面考虑系统(61)–(63)的输入–状态稳定性.

定理 3　假设时变增益$g(t)$满足条件(45), 那么, 对于任意的干扰$d_1, d_4\in H^1_{\mathrm{loc}}(0,\infty;L^2(0,1))$, d_2, d_3, $d_5\in H^1_{\mathrm{loc}}(0,\infty)$, 以及$t_0$时刻的值$(u(\cdot,t_0),u_t(\cdot,t_0))\in\mathcal{H}_3=H^1(0,1)\times L^2(0,1)$, 系统(61)–(63)是输入–状态稳定的, 即存在与干扰无关的正常数$E_i(i=1,2,3,4,5,6)$, 使得

$$\begin{aligned}&\|(u(\cdot,t),u_t(\cdot,t))\|^2_{\mathcal{H}_3}\leqslant\\ &E_1\mathrm{e}^{-E_2t}\|(u(\cdot,t_0),u_t(\cdot,t_0))\|^2_{\mathcal{H}_3}+\\ &E_3\sup_{t_0\leqslant s\leqslant t}(\|d_1(\cdot,s)\|^2)+E_4\sup_{t_0\leqslant s\leqslant t}(|d_2(s)|^2)+\\ &E_5\sup_{t_0\leqslant s\leqslant t}(\|d_4(\cdot,s)\|^2)+E_6\sup_{t_0\leqslant s\leqslant t}(|d_5(s)|^2),\end{aligned} \tag{90}$$

对于$\forall t>t_0$成立. 特别地, 当$d_i(i=1,2,4,5)=0$时, 闭环系统(61)–(63)是指数稳定的.

证　系统(67)–(71)的能量为

$$E(t)=c_1\int_0^1 \beta^2(x,t)\mathrm{d}x+\int_0^1 \xi^2(x,t)\mathrm{d}x+cu^2(0,t). \tag{91}$$

定义Lyapunov–Krasovskii泛函为

$$V(t)=\frac{1}{2}cu^2(0,t)+\frac{1}{2}\int_0^1 \lambda\mathrm{e}^{-\rho x}\beta^2(x,t)\mathrm{d}x+\frac{1}{2}\int_0^1 \sigma\mathrm{e}^{rx}\xi^2(x,t)\mathrm{d}x, \tag{92}$$

其中$\lambda,\rho,r,\sigma>0$是设计参数. 利用Cauchy-Schwarz不等式和Young's不等式, 有

$$\delta_1E(t)\leqslant V(t)\leqslant\delta_2E(t), \tag{93}$$

其中: $\delta_1=\dfrac{\min\{\frac{1}{2},\frac{\sigma}{2},\frac{\lambda\mathrm{e}^{-\rho}}{2}\}}{\max\{c,c_1+1\}}$, $\delta_2=\dfrac{\max\{\frac{1}{2},\frac{\lambda}{2},\frac{\sigma\mathrm{e}^{r}}{2}\}}{\min\{1,c,c_1\}}$.

对$V(t)$关于t沿着系统(67)–(71)的解求导可得

$$\begin{aligned}&\dot{V}(t)=\\ &-c^2u^2(0,t)+cu(0,t)\xi(0,t)-cu(0,t)D_2(t)-\\ &\int_0^1 \lambda\mathrm{e}^{-\rho x}\beta(x,t)\beta_x(x,t)\mathrm{d}x+\\ &\frac{b^2}{2}\int_0^1\int_0^x \lambda\mathrm{e}^{-\rho x}\beta(x,t)f(x,y)\xi(y,t)\mathrm{d}y\mathrm{d}x+\\ &\frac{b^2}{2}\int_0^1\int_0^x \lambda\mathrm{e}^{-\rho x}\beta(x,t)v(x,y)\beta(y,t)\mathrm{d}y\mathrm{d}x+\\ &\int_0^1 \lambda\mathrm{e}^{-\rho x}\beta(x,t)D_1(x,t)\mathrm{d}x-\\ &\frac{b^2}{2}\int_0^1\int_0^x \lambda\mathrm{e}^{-\rho x}\beta(x,t)\cos h(b(x-y))\times\\ &n(y)u(0,t)\mathrm{d}y\mathrm{d}x+\int_0^1 \sigma\mathrm{e}^{rx}\xi(x,t)\xi_x(x,t)\mathrm{d}x+\\ &\int_0^1 \sigma\mathrm{e}^{rx}\xi(x,t)D_3(x,t)\mathrm{d}x.\end{aligned} \tag{94}$$

不妨设

$$\begin{aligned}M_1>\max\{&\cos h(b)\max_{0\leqslant y\leqslant x\leqslant1}\{|n(y)|\},\\ &\max_{0\leqslant y\leqslant x\leqslant1}\{|v(x,y)|\},\ \max_{0\leqslant y\leqslant x\leqslant1}\{|f(x,y)|\}\},\end{aligned} \tag{95}$$

可得

$$\begin{aligned}\dot{V}(t)\leqslant&-\eta V(t)+4\lambda\int_0^1 D_1^2(x,t)\mathrm{d}x+\\ &\frac{5\sigma\mathrm{e}^r}{2\mu}\int_0^1 D_3^2(x,t)\mathrm{d}x+(2+3\lambda)\,D_2^2(t)+\\ &\frac{\sigma}{2}\,\mathrm{e}^rk_0\mathrm{e}^{-\kappa t},\ \forall t>t_0,\end{aligned} \tag{96}$$

其中

$$\begin{aligned}\eta=\min\{&\frac{3}{2}c-\frac{3\lambda(a-c)^2}{c}-\frac{2b^2\lambda M_1}{c\rho},\\ &\rho\lambda-\frac{3b^2\lambda M_1}{8}-\frac{2b^2\lambda M_1}{\rho}-\frac{\lambda}{8},\\ &\frac{3r}{4}-\frac{2b^2\lambda M_1}{\sigma\rho}\}>0\end{aligned} \tag{97}$$

以及

$$\frac{\sigma}{2}-\frac{3\lambda}{2}-2>0 \tag{98}$$

对于足够大的$\rho,\sigma>0$和足够小的$\lambda>0$成立.

进一步, 由于$D_i(i=1,2,3)$是依赖于$d_i\ (i=1,2,4,5)$的干扰, 存在正常数$A_i>0\ (i=1,2,3,4)$使得

$$\begin{aligned}\dot{V}(t)\leqslant&-\eta V(t)+A_1\|d_1(\cdot,t)\|^2+A_2|d_2(t)|^2+\\ &A_3\|d_4(\cdot,t)\|^2+A_4|d_5(t)|^2+\end{aligned}$$

$$\frac{\sigma}{2}\,\mathrm{e}^{r}k_0\mathrm{e}^{-\kappa t},\ \forall t>t_0. \tag{99}$$

那么, 对于$\forall t>t_0$, 有

$$\begin{aligned}V(t)\leqslant{}&\mathrm{e}^{-\eta t}V(t_0)+\frac{A_1}{\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_1(\cdot,s)\|^2)+\\&\frac{A_2}{\eta}\sup_{t_0\leqslant s\leqslant t}(|d_2(s)|^2)+\frac{A_4}{\eta}\sup_{t_0\leqslant s\leqslant t}(|d_5(s)|^2)+\\&\frac{A_3}{\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_4(\cdot,s)\|^2)+\\&\frac{\sigma}{2\eta}\mathrm{e}^{r}k_0\frac{1}{\eta-\kappa}(\mathrm{e}^{-\kappa t}-\mathrm{e}^{(\eta-\kappa)t_0}\mathrm{e}^{-\eta t}),\end{aligned} \tag{100}$$

则存在常数$A_5>0$, 使得

$$\begin{aligned}V(t)\leqslant{}&A_5V(t_0)\mathrm{e}^{-\sigma t}+\frac{A_1}{\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_1(\cdot,s)\|^2)+\\&\frac{A_2}{\eta}\sup_{t_0\leqslant s\leqslant t}(|d_2(s)|^2)+\frac{A_4}{\eta}\sup_{t_0\leqslant s\leqslant t}(|d_5(s)|^2)+\\&\frac{A_3}{\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_4(\cdot,s)\|^2),\end{aligned} \tag{101}$$

其中$\sigma=\min\{\eta,\kappa\}$. 因此,

$$\begin{aligned}E(t)\leqslant{}&\frac{\delta_2}{\delta_1}A_5\mathrm{e}^{-\sigma t}E(t_0)+\frac{A_1}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_1(\cdot,s)\|^2)+\\&\frac{A_2}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(|d_2(s)|^2)+\frac{A_4}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(|d_5(s)|^2)+\\&\frac{A_3}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_4(\cdot,s)\|^2),\end{aligned} \tag{102}$$

即

$$\begin{aligned}&\|\beta(\cdot,t)\|^2+\|\xi(\cdot,t)\|^2+c|u(0,t)|^2\leqslant\\&\frac{\delta_2}{\delta_1}A_5\mathrm{e}^{-\sigma t}(\|\beta(\cdot,t_0)\|^2+\|\xi(\cdot,t_0)\|^2+c|u(0,t_0)|^2)+\\&\frac{A_1}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_1(\cdot,s)\|^2)+\\&\frac{A_2}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(|d_2(s)|^2)+\frac{A_4}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(|d_5(s)|^2)+\\&\frac{A_3}{\delta_1\eta}\sup_{t_0\leqslant s\leqslant t}(\|d_4(\cdot,s)\|^2).\end{aligned} \tag{103}$$

由于变换(18)的有界可逆性, 存在常数$c_i(i=1,2,3,4,5,6)$使得

$$\begin{cases}\|\xi(\cdot,t)\|^2\leqslant c_1\|\alpha(\cdot,t)\|^2+c_2\|\beta(\cdot,t)\|^2+\\\qquad c_3|u(0,t)|^2,\\\|\alpha(\cdot,t)\|^2\leqslant c_4\|\xi(\cdot,t)\|^2+c_5\|\beta(\cdot,t)\|^2+\\\qquad c_6|u(0,t)|^2.\end{cases} \tag{104}$$

因此,

$$\begin{aligned}&c_7(\|\xi(\cdot,t)\|^2+\|\beta(\cdot,t)\|^2+c|u(0,t)|^2)\leqslant\\&\|\beta(\cdot,t)\|^2+\|\alpha(\cdot,t)\|^2+c|u(0,t)|^2\leqslant\\&c_8(\|\xi(\cdot,t)\|^2+\|\beta(\cdot,t)\|^2+c|u(0,t)|^2),\end{aligned} \tag{105}$$

其中:

$$c_7=\frac{1}{\max\{c_1,c_2+1,\dfrac{c_3+c}{c}\}}, \tag{106}$$

$$c_8=\max\{c_4,c_5+1,\frac{c_6+c}{c}\}. \tag{107}$$

进一步由变换(10)–(11)可得

$$\begin{aligned}&\frac{1}{4}(\|\alpha(\cdot,t)\|^2+\|\beta(\cdot,t)\|^2+c|u(0,t)|^2)\leqslant\\&\|(u(\cdot,t),u_t(\cdot,t))\|^2_{\mathcal{H}_3}\leqslant\\&\|\alpha(\cdot,t)\|^2+\|\beta(\cdot,t)\|^2+c|u(0,t)|^2.\end{aligned} \tag{108}$$

根据式(105)(108)可得

$$\begin{aligned}&\|(u(\cdot,t),u_t(\cdot,t))\|^2_{\mathcal{H}_3}\leqslant\\&E_1\mathrm{e}^{-E_2t}\|(u(\cdot,t_0),u_t(\cdot,t_0))\|^2_{\mathcal{H}_3}+\\&E_3\sup_{t_0\leqslant s\leqslant t}(\|d_1(\cdot,s)\|^2)+E_4\sup_{t_0\leqslant s\leqslant t}(|d_2(s)|^2)+\\&E_5\sup_{t_0\leqslant s\leqslant t}(\|d_4(\cdot,s)\|^2)+\\&E_6\sup_{t_0\leqslant s\leqslant t}(|d_5(s)|^2),\end{aligned} \tag{109}$$

其中:

$$E_1=\frac{4\delta_2c_8}{\delta_1c_7}A_5,\ E_2=\sigma,\ E_3=\frac{A_1c_8}{\delta_1\eta}, \tag{110}$$

$$E_4=\frac{A_2c_8}{\delta_1\eta},\ E_5=\frac{A_3c_8}{\delta_1\eta},\ E_6=\frac{A_4c_8}{\delta_1\eta}. \tag{111}$$

最后, 当$d_i(i=1,2,4,5)=0$时, 通过式(109)可知, 闭环系统(61)–(63)是指数稳定的. 证毕.

5 数值仿真

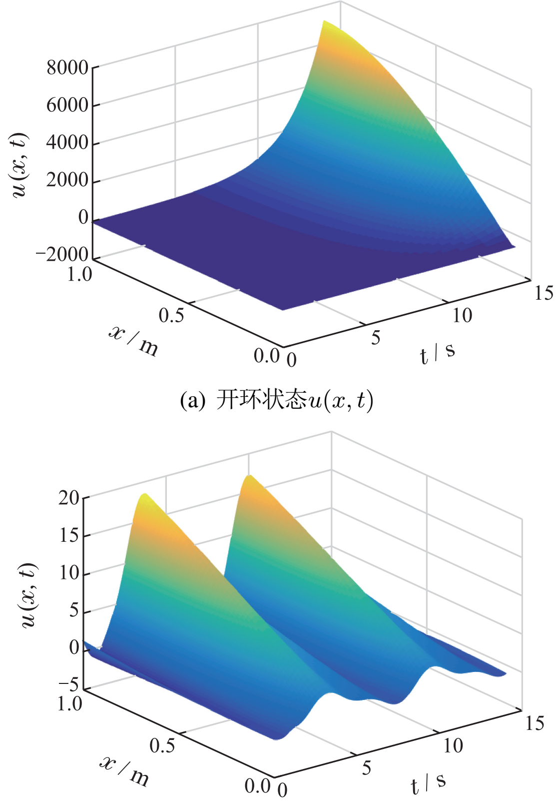
本节给出主要结论的数值模拟结果. 使用空间步长为0.02以及时间步长为0.002的有限差分方法来离散方程. 系统参数取为$a=0.9$, $b=0.6$. 设计参数取为$c=8$, $k_1=3$, $\nu=0.3$. 假设干扰为

$$\begin{cases}d_1(x,t)=\mathrm{e}^{-xt}+2,\ d_2(t)=\cos t,\\d_3(t)=2\cos t,\ d_5(t)=0.5\sin(2t),\\d_4(x,t)=\dfrac{x}{5}+\dfrac{1}{(t+1)^2}.\end{cases} \tag{112}$$

初值选为

$$\begin{cases}u(x,0)=4x^3,\\u_t(x,0)=0.5x^4-10x^3+4.5x^2+5x.\end{cases} \tag{113}$$

由开环仿真图1(a)可以看出, 对于选取的参数, 系统是不稳定的. 另一方面, 若不抵消匹配干扰, 即不通过ADRC方法对干扰进行估计, 则在控制(30)和(60)的作用下, 系统(4)的状态如图1(b)所示. 然而, 如果利用ADRC方法对干扰进行估计并进一步通过控制消除它, 那么具有干扰的系统(4)在控制(30)和控制(60)的作用下的状态呈现在图1(c). 图2表明干扰估计器$\hat{D}_4$很好地跟踪到了干扰D_4的真实值.



(a) 开环状态$u(x,t)$

(b) 未抵消匹配干扰的闭环状态$u(x,t)$

(c) 闭环状态$u(x,t)$

图 1　系统(4)的状态

Fig. 1　The state of system (4)

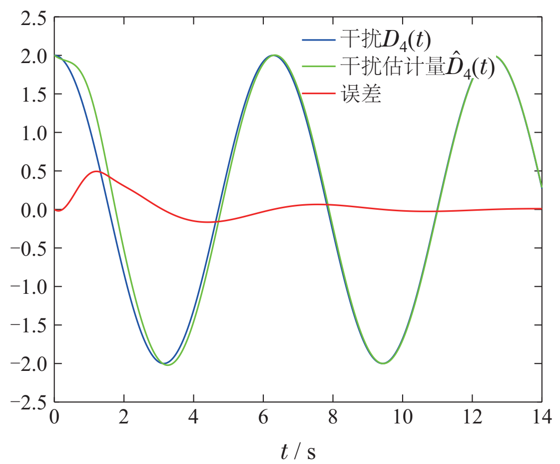


图 2　干扰$D_4(t)$、干扰估计量$\hat{D}_4(t)$以及误差

Fig. 2　Disturbance $D_4(t)$, its estimation $\hat{D}_4(t)$ and the error

6　小结

本章考虑了具有外部干扰的不稳定剪切梁的输入–状态稳定性, 其中梁的一端受到局部的范德华力. 由于范德华力对于系统的作用导致开环系统不稳定, 因此, 本文通过在梁的控制端构造反馈控制使得系统自身达到指数镇定. 对于外部的匹配干扰, 构造了具有时变增益的扩张状态观测器用于估计干扰并在反馈环节进行消除. 最后证明了系统在外部干扰下的输入–状态稳定性.

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