基于g-期望的部分可观测非零和随机微分博弈

杨碧璇, 郭铁信, 吴锦标†
(中南大学 数学与统计学院, 湖南 长沙 410083)

摘要: 本文研究了g-期望下的部分可观测非零和随机微分博弈系统, 该系统的状态方程由Itô-Lévy过程驱动, 成本函数由g-期望描述. 根据Girsanov定理和凸变分技巧, 本文得到了最大值原理和验证定理. 为对所获结果进行说明, 本文讨论了关于资产负债管理的博弈问题.

关键词: 随机微分博弈; g-期望; 正倒向随机微分方程; 最大值原理; 验证定理

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Partially observed nonzero-sum stochastic differential games with g-expectations

YANG Bi-xuan, GUO Tie-xin, WU Jin-biao†
(School of Mathematics and Statistics, Central South University, Changsha Hunan 410083, China)

Abstract: This paper is concerned with a partially observed nonzero-sum stochastic differential game system under g-expectation, where the state is governed by a Itô-Lévy process and the cost functionals are described by g-expectations. Based on Girsanov's theorem and convex variation techniques, we derive a maximum principle and a verification theorem. An asset-liability management game problem is discussed to illustrate the results.

Key words: stochastic differential game; g-expectation; forward-backward stochastic differential equation; maximum principle; verification theorem

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1 Introduction

With the increasing demand of researchers in today's technological revolution, stochastic differential game (SDG) theory has emerged to better grasp of the real world and played a distinguished role in many fields, especially in economics, finance, control theory and behavioral science. The pioneering work of SDGs was established by Ho[1]. Over recent years, SDG theory has became a very active area of research, such as An and Øksendal[2], Wang and Yu[3], Zhu and Zhang[4], and Wu and Liu[5].

Because of the continuing global financial crisis in recent years, some investigators have questioned whether current theories of risk management are appropriate and paid more attention to develop prudent methods of assessing risks. The theory of g-expectations is a fairly new research topic to avoid risks in mathematical finance and was first introduced by Peng[6] as particular nonlinear expectations depending on backward stochastic differential equations. As an application, the model of risk minimizing portfolios was studied by Øksendal and Sulem[7], where the risk is represented in terms of g-expectations. For a comprehensive survey of theories on g-expectations and relevant applications, one can refer to the paper by Peng[8]. In fact, combining SDG systems with cost functionals defined by g-expectations, one can naturally obtain forward-backward stochastic differential games (FBSDGs).

The theory of FBSDGs has got a rapid development of late years due to its widely applications in risk measures, for example, the optimal portfolio-consumption problem under model uncertainty[9]. The FBSDG systems are given by forward-backward stochastic differential equations (FBSDEs), which include stochastic differential equations (SDEs) as a special case. Yu[10] dealt with a linear-quadratic nonzero-sum FBS-

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†Corresponding author. E-mail: wujinbiao@csu.edu.cn.
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DG problem and derived an explicit form of the unique Nash equilibrium point. Hui and Xiao[11] considered both zero-sum and nonzero-sum FBSDGs and obtained the maximum principles and the verification theorems. An and Øksendal[12] discussed the sufficient maximum principles for both zero-sum and nonzero-sum SDGs of Itô-Lévy processes with g-expectations and partial information.

In practice, the controllers generally can not observe complete information, but they are able to get the related information, which is called the correlated noise, for instance, the recursive utility optimization problem[13]. Inspired by this phenomenon, many investigators have set out to study partially observed systems. Wu[14] first devoted to the maximum principle for partially observed forward-backward stochastic control systems. As a generalization of results of [14], Xiao[15] considered a partially observed forward-backward stochastic optimal control system with jumps and obtained the necessary maximum principle and the sufficient verification theorem. Xiong et al.[16] analyzed a necessary and sufficient maximum principle for partially observed nonzero-sum differential game system of FBSDEs.

To the best of our knowledge, the maximum principle and the verification theorem for a partially observed nonzero-sum SDG system with g-expectation have not been established in earlier work, and are entirely new. The main contributions are described as follows. On the one hand, our work extends the results of [12] to a partially observed nonzero-sum differential game, where the state is described by a Itô-Lévy process and the cost functionals are defined by g-expectations, i.e., FBSDEs. On the other hand, for the partially observed game system, we suppose that each player has his own observation process to serve as the available information, which is distinguished from the model of partial information in [12]. What's more, we solve a partially observed asset-liability management game problem, where the information filtration can be generated by observable stock price processes.

The rest of this paper is organized as follows: In Section 2, we introduce some notions and formulate the game system; In Sections 3 and 4, we establish a maximum principle and a verification theorem for the game system, respectively; Section 5 provides an example of the partially observed asset-liability management game model; Some conclusions are drawn in Section 6.

2 Statement of the game problem

Let $T > 0$ be a finite time duration and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete filtered probability space equipped with three mutually independent 1-dimensional standard Brownian motions $W(\cdot)$, $Y_1(\cdot)$ and $Y_2(\cdot)$ defined on $[0,T]$ and an independent Poisson random measure $N(\mathrm{d}t, \mathrm{d}\eta)$ defined on $[0,T] \times \mathbb{R}_0$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Denote the compensated Poisson random measure by $\tilde{N}(\mathrm{d}t, \mathrm{d}\eta) := N(\mathrm{d}t, \mathrm{d}\eta) - \nu(\mathrm{d}\eta)\mathrm{d}t$, where ν is the Lévy measure of N satisfying $\int_{\mathbb{R}_0}(1\wedge |\eta|^2)\nu(\mathrm{d}\eta) < \infty$. In addition, let $\mathcal{F}_t^W$, $\mathcal{F}_t^1$, $\mathcal{F}_t^2$ and $\mathcal{F}_t^N$ be the P-completed natural filtration generated by $W(\cdot)$, $Y_1(\cdot)$, $Y_2(\cdot)$ and $N(\cdot,\cdot)$, respectively. We assume that $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \mathcal{F}_t^N \vee \mathcal{N}$, $\mathcal{F} := \mathcal{F}_T$, where $\mathcal{N}$ denotes the totality of P-null sets.

Let $\mathbb{R}$ be the 1-dimensional Euclidean space, $|\cdot|$ the Euclidean norm. In the sequel, we denote by $L^2(\mathcal{F}_T; \mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_T$-measurable random variables ξ such that $E[|\xi|^2] < \infty$, by $L^2_{\mathcal{F}}(s_1, s_2; \mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $(l(t))_{t\in[s_1,s_2]}$ such that $E[\int_{s_1}^{s_2} |l(t)|^2\mathrm{d}t] < \infty$, by $L^2(\nu)$ the space of integrable functions $k : \mathbb{R}_0 \to \mathbb{R}$ with norm $||k(\eta)||_\nu^2 := \int_{\mathbb{R}_0} |k(\eta)|^2\nu(\mathrm{d}\eta) < \infty$, and by $F_\nu^2(s_1, s_2; \mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_t$-predictable processes $(l(t,\eta))_{t\in[s_1,s_2]}$ such that $E[\int_{s_1}^{s_2}\int_{\mathbb{R}_0} |l(t,\eta)|^2\nu(\eta)\mathrm{d}t] < \infty$.

Suppose that the state of a stochastic game system is described by the following jump-diffusion SDE:

$$\begin{cases} \mathrm{d}x(t) = b(t, x(t), v_1(t), v_2(t))\mathrm{d}t + \\ \qquad \sigma(t, x(t), v_1(t), v_2(t))\mathrm{d}W(t) + \\ \qquad \int_{\mathbb{R}_0} \gamma(t, x(t), v_1(t), v_2(t), \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ \qquad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $v_1 : \Omega\times[0,T] \mapsto U_1$, and $v_2 : \Omega\times[0,T] \mapsto U_2$ are control processes of Player 1 and Player 2, respectively. Here, U_1 and U_2 are nonempty convex subsets of $\mathbb{R}$. b, $\sigma : \Omega \times [0,T] \times \mathbb{R} \times U_1 \times U_2 \mapsto \mathbb{R}$, and $\gamma : \Omega \times [0,T] \times \mathbb{R} \times U_1 \times U_2 \times \mathbb{R}_0 \mapsto \mathbb{R}$ are given mappings, which satisfy

A1) The functions b, σ and γ are continuously differentiable with respect to (x, v_1, v_2); b and σ have a linear growth in (x, v_1, v_2), and their partial derivatives are uniformly bounded and Lipschitz continuous; there exists a constant $C > 0$ such that $(\int_{\mathbb{R}_0} |\gamma(t, x, v_1, v_2, \eta)|^2\nu(\eta))^{\frac{1}{2}}$ is bounded by $C(1 + |x| + |v_1| + |v_2|)$, and $\int_{\mathbb{R}_0} |\frac{\partial\gamma}{\partial x}(t, x, v_1, v_2, \eta)|^2\nu(\eta)$ and $\int_{\mathbb{R}_0} |\frac{\partial\gamma}{\partial v_i}(t, x, v_1, v_2, \eta)|^2\nu(\eta)$ $(i = 1, 2)$ are uniformly bounded and Lipschitz continuous; for any $(x, v_1, v_2) \in \mathbb{R} \times U_1 \times U_2$, $b(\cdot, x, v_1, v_2), \sigma(\cdot, x, v_1, v_2) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$, and $\gamma(\cdot, x, v_1, v_2, \cdot) \in F_\nu^2(0, T; \mathbb{R})$.

We suppose that the state $x(\cdot)$ can not be observed directly, but Player i can observe his own related process $Y_i(\cdot)$, which is governed by:

$$\begin{cases}\mathrm{d}Y_i(t) = \varrho_i(t, x(t), v_1(t), v_2(t))\mathrm{d}t + \mathrm{d}W_i^{v_1,v_2}(t), \\ Y_i(0) = 0,\ i = 1, 2,\end{cases} \tag{2}$$

where $W_1^{v_1,v_2}(\cdot), W_2^{v_1,v_2}(\cdot)$ are $\mathbb{R}$-valued stochastic processes depending on $v_1(\cdot)$ and $v_2(\cdot)$. $\varrho_i : \Omega\times[0,T]\times\mathbb{R}\times U_1\times U_2 \mapsto \mathbb{R}$ is a continuous function, which satisfies

A2) The function ϱ_i is continuously differentiable with respect to (x, v_1, v_2), and its partial derivatives and ϱ_i are all uniformly bounded.

The admissible control sets for each player are given by:

$$\mathcal{A}_i = \{v_i(\cdot) \in U_i \mid v_i(\cdot) \text{ is an } \mathcal{F}_t^i\text{-adapted process and satisfies } \sup_{0\leqslant t\leqslant T} E|v_i(t)|^2 < \infty\},\ \ i = 1, 2.$$

Every element of $\mathcal{A}_i$ is called an admissible control for Player $i (i = 1, 2)$. $\mathcal{A}_1 \times \mathcal{A}_2$ is said to be the set of admissible controls for the players.

For any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$, A1) implies that (1) admits a unique strong solution $x(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ (see Tang and Li[17]). Now, we define a new probability measure P^{v_1,v_2} by $\frac{\mathrm{d}P^{v_1,v_2}}{\mathrm{d}P}\big|_{\mathcal{F}_t} = Z^{v_1,v_2}(t)$, where

$$\begin{cases}\mathrm{d}Z^{v_1,v_2}(t) = \\ \varrho_1(t, x(t), v_1(t), v_2(t))Z^{v_1,v_2}(t)\mathrm{d}Y_1(t) + \\ \varrho_2(t, x(t), v_1(t), v_2(t))Z^{v_1,v_2}(t)\mathrm{d}Y_2(t), \\ Z^{v_1,v_2}(0) = 1,\end{cases} \tag{3}$$

that is

$$Z^{v_1,v_2}(t) = \exp\Big(\sum_{m=1}^{2}\int_0^t \varrho_m(s, x(s), v_1(s), v_2(s)) \cdot \mathrm{d}Y_m(s) - \frac{1}{2}\sum_{m=1}^{2}\int_0^t \varrho_m^2(s, x(s), v_1(s), v_2(s))\mathrm{d}s\Big).$$

Based on Girsanov's theorem and A2), $(W(\cdot), W_1^{v_1,v_2}(\cdot), W_2^{v_1,v_2}(\cdot))$ is a 3-dimensional standard Brownian motion and $\tilde{N}(\cdot,\cdot)$ is still a compensated Poisson random measure defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P^{v_1,v_2})$.

The cost functional of Player i is defined by

$$J_i(v_1(\cdot), v_2(\cdot)) = E^{v_1,v_2}\Big[\int_0^T f_i(t, x(t), v_1(t), v_2(t))\mathrm{d}t + \psi_i(x(T))\Big],\quad i = 1, 2, \tag{4}$$

where E^{v_1,v_2} is the expectation with respect to P^{v_1,v_2}. $f_i : \Omega\times[0,T]\times\mathbb{R}\times U_1\times U_2 \mapsto \mathbb{R}$, and $\psi_i : \Omega\times\mathbb{R} \mapsto \mathbb{R}$ satisfy

A3) The functions f_i and ψ_i are continuously differentiable with respect to (x, v_1, v_2) and x, respectively, and partial derivatives of f_i and derivative of ψ_i have a linear growth in (x, v_1, v_2) and x, respectively; f_i and ψ_i are uniformly Lipschitz with respect to (x, v_1, v_2) and x, respectively; for any $(x, v_1, v_2) \in \mathbb{R}\times U_1\times U_2$ and $x \in \mathbb{R}$, $f_i(\cdot, x, v_1, v_2) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$, and $\psi_i(x) \in L^2(\mathcal{F}_T; \mathbb{R})$.

It is well known that the linear expectation E^{v_1,v_2} in (4) does not express investors' performances (see Chen and Epstein[18]). In what follows, we introduce a nonlinear expectation (i.e., a g-expectation) to replace E^{v_1,v_2}.

We consider the following backward SDEs with random jumps under $\theta_i \in L^2(\mathcal{F}_T; \mathbb{R})$:

$$\begin{cases}-\mathrm{d}y_i(t) = g_i(t, y_i(t), z_i(t), k_i(t,\cdot))\mathrm{d}t - \\ \qquad z_i(t)\mathrm{d}W(t) - \int_{\mathbb{R}_0} k_i(t,\eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ \qquad t \in [0, T], \\ y_i(T) = \theta_i,\ i = 1, 2,\end{cases} \tag{5}$$

where $g_i : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times L^2(\nu) \mapsto \mathbb{R}$ is a given mapping, which satisfies

A4) The function g_i is continuously differentiable with respect to (y_i, z_i, k_i), and the partial derivatives of g_i are uniformly bounded and Lipschitz continuous; $g_i(\cdot, 0, 0, 0) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$.

From Theorem 2.1 in [19] and A4), we know that (5) exists a unique strong solution $(y_i(\cdot), z_i(\cdot), k_i(\cdot,\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times F^2_{\nu}(0, T; \mathbb{R})$. If $g_i(\cdot, y_i, 0, 0) \equiv 0$ for any $y_i \in \mathbb{R}$, then we define the g-expectation $\mathcal{E}_{g_i}^{v_1,v_2}$ of θ_i related to g_i by

$$\mathcal{E}_{g_i}^{v_1,v_2}(\theta_i) = y_i(0),\ i = 1, 2.$$

With the g-expectations, we introduce the new cost functionals J_{g_i} $(i = 1, 2)$ as follows:

$$J_{g_i}(v_1(\cdot), v_2(\cdot)) = \mathcal{E}_{g_i}^{v_1,v_2}\Big[\int_0^T f_i(t, x(t), v_1(t), v_2(t))\mathrm{d}t + \psi_i(x(T))\Big].$$

Thus, the partially observed nonzero-sum differential game problem with g-expectation is to find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\begin{cases}J_{g_1}(u_1(\cdot), u_2(\cdot)) = \min\limits_{v_1(\cdot)\in\mathcal{A}_1} J_{g_1}(v_1(\cdot), u_2(\cdot)), \\ J_{g_2}(u_1(\cdot), u_2(\cdot)) = \min\limits_{v_2(\cdot)\in\mathcal{A}_2} J_{g_2}(u_1(\cdot), v_2(\cdot)).\end{cases}$$

The pair of admissible controls $(u_1(\cdot), u_2(\cdot))$ is called a Nash equilibrium point of the game system.

From the theory of backward SDEs and the definition of g-expectations, we can reformulate the partially observed game problem as follows: let $(\beta_i(\cdot), \iota_i(\cdot), \varsigma_i(\cdot,\cdot))$ be the adapted solution of the following backward SDE:

$$\begin{cases} -\mathrm{d}\beta_i(t) = g_i(t, \beta_i(t), \iota_i(t), \varsigma_i(t, \cdot))\mathrm{d}t - \\ \qquad \iota_i(t)\mathrm{d}W(t) - \int_{\mathbb{R}_0} \varsigma_i(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ \qquad t \in [0, T], \\ \beta_i(T) = \theta_i(x, \upsilon_1, \upsilon_2), \end{cases}$$

where

$$\theta_i(x, \upsilon_1, \upsilon_2) = \int_0^T f_i(t, x(t), \upsilon_1(t), \upsilon_2(t))\mathrm{d}t + \psi_i(x(T)).$$

For any $t \in [0, T]$, we define

$$\begin{cases} y_i(t) = \beta_i(t) - \int_0^t f_i(s, x(s), \upsilon_1(s), \upsilon_2(s))\mathrm{d}s, \\ z_i(t) = \iota_i(t),\ k_i(t, \eta) = \varsigma_i(t, \eta). \end{cases}$$

It is easy to obtain that $(y_i(\cdot), z_i(\cdot), k_i(\cdot, \cdot))$ is the unique solution of the following backward SDE:

$$\begin{cases} -\mathrm{d}y_i(t) = [g_i(t, y_i(t), z_i(t), k_i(t, \cdot)) + \\ \qquad f_i(t, x(t), \upsilon_1(t), \upsilon_2(t))]\mathrm{d}t - \\ \qquad z_i(t)\mathrm{d}W(t) - \int_{\mathbb{R}_0} k_i(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ \qquad t \in [0, T], \\ y_i(T) = \psi_i(x(T)). \end{cases}$$

Hence, the state equations can be rewritten by the following FBSDEs:

$$\begin{cases} \mathrm{d}x(t) = b(t, x(t), \upsilon_1(t), \upsilon_2(t))\mathrm{d}t + \\ \qquad \sigma(t, x(t), \upsilon_1(t), \upsilon_2(t))\mathrm{d}W(t) + \\ \qquad \int_{\mathbb{R}_0} \gamma(t, x(t), \upsilon_1(t), \upsilon_2(t), \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ -\mathrm{d}y_i(t) = [g_i(t, y_i(t), z_i(t), k_i(t, \cdot)) + \\ \qquad f_i(t, x(t), \upsilon_1(t), \upsilon_2(t))]\mathrm{d}t - \\ \qquad z_i(t)\mathrm{d}W(t) - \int_{\mathbb{R}_0} k_i(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ \qquad t \in [0, T], \\ x(0) = x_0,\ \ y_i(T) = \psi_i(x(T)),\ i = 1, 2, \end{cases} \tag{6}$$

and observation processes $Y_i(\cdot)$ $(i = 1, 2)$ satisfy (2). The cost functionals J_{g_i} $(i = 1, 2)$ are given by:

$$\begin{aligned} & J_{g_i}(\upsilon_1(\cdot), \upsilon_2(\cdot)) = \\ & E^{\upsilon_1, \upsilon_2}[\int_0^T (f_i(t, x(t), \upsilon_1(t), \upsilon_2(t)) + \\ & g_i(t, y_i(t), z_i(t), k_i(t, \cdot)))\mathrm{d}t + \psi_i(x(T))] = \\ & E[\int_0^T Z^{\upsilon_1, \upsilon_2}(t)(f_i(t, x(t), \upsilon_1(t), \upsilon_2(t)) + \\ & g_i(t, y_i(t), z_i(t), k_i(t, \cdot)))\mathrm{d}t + Z^{\upsilon_1, \upsilon_2}(T)\psi_i(x(T))]. \end{aligned}$$

The game problem is to find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\begin{cases} J_{g_1}(u_1(\cdot), u_2(\cdot)) = \min\limits_{\upsilon_1(\cdot) \in \mathcal{A}_1} J_{g_1}(\upsilon_1(\cdot), u_2(\cdot)), \\ J_{g_2}(u_1(\cdot), u_2(\cdot)) = \min\limits_{\upsilon_2(\cdot) \in \mathcal{A}_2} J_{g_2}(u_1(\cdot), \upsilon_2(\cdot)). \end{cases} \tag{7}$$

We denote by $(\hat{x}(\cdot), \hat{y}_1(\cdot), \hat{z}_1(\cdot), \hat{k}_1(\cdot, \cdot), \hat{y}_2(\cdot), \hat{z}_2(\cdot), \hat{k}_2(\cdot, \cdot))$ and $Z(\cdot)$ the corresponding state processes along with the optimal controls $(u_1(\cdot), u_2(\cdot))$.

3 Maximum principle

In this section, we prove a maximum principle for the game system expressed by Theorem 1.

For any $(\epsilon, \upsilon_1(\cdot), \upsilon_2(\cdot)) \in [0, 1] \times \mathcal{A}_1 \times \mathcal{A}_2$, we take the perturbations $u_1^\epsilon(\cdot) = u_1(\cdot) + \epsilon\upsilon_1(\cdot)$ and $u_2^\epsilon(\cdot) = u_2(\cdot) + \epsilon\upsilon_2(\cdot)$. Since both U_1 and U_2 are convex sets, $(u_1^\epsilon(\cdot), u_2^\epsilon(\cdot))$ is an element of $\mathcal{A}_1 \times \mathcal{A}_2$. Suppose that the processes $(x^{\epsilon_1}(\cdot), y_i^{\epsilon_1}(\cdot), z_i^{\epsilon_1}(\cdot), k_i^{\epsilon_1}(\cdot, \cdot))$ $((x^{\epsilon_2}(\cdot), y_i^{\epsilon_2}(\cdot), z_i^{\epsilon_2}(\cdot), k_i^{\epsilon_2}(\cdot, \cdot)))$ $(i = 1, 2)$ and $Z^{\epsilon_1}(\cdot)$ $(Z^{\epsilon_2}(\cdot))$ are the solutions of (6) and (3) along with $(u_1^\epsilon(\cdot), u_2(\cdot))$ $((u_1(\cdot), u_2^\epsilon(\cdot)))$, respectively.

For simplicity, we employ some notations as follows:

$$\begin{aligned} & \chi(t) = \chi(t, \hat{x}(t), u_1(t), u_2(t)),\ \chi = b, \sigma, f_i, \varrho_i, \\ & \gamma(t) = \gamma(t, \hat{x}(t), u_1(t), u_2(t), \cdot), \\ & g_i(t) = g_i(t, \hat{y}_i(t), \hat{z}_i(t), \hat{k}_i(t, \cdot)),\ i = 1, 2, \\ & \frac{\partial b}{\partial x}(t) = [\frac{\partial b}{\partial x}(t, x, u_1(t), u_2(t))]_{x = \hat{x}(t)}. \end{aligned}$$

We introduce the variational equations:

$$\begin{cases} \mathrm{d}x^i(t) = [\frac{\partial b}{\partial x}(t)x^i(t) + \frac{\partial b}{\partial \upsilon_i}(t)\upsilon_i(t)]\mathrm{d}t + \\ \qquad [\frac{\partial \sigma}{\partial x}(t)x^i(t) + \frac{\partial \sigma}{\partial \upsilon_i}(t)\upsilon_i(t)]\mathrm{d}W(t) + \\ \qquad \int_{\mathbb{R}_0} [\frac{\partial \gamma}{\partial x}(t, \eta)x^i(t) + \frac{\partial \gamma}{\partial \upsilon_i}(t, \eta)\upsilon_i(t)] \cdot \\ \qquad \tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ -\mathrm{d}y_j^i(t) = [\frac{\partial g_j}{\partial y_j}(t)y_j^i(t) + \frac{\partial g_j}{\partial z_j}(t)z_j^i(t) + \\ \qquad \int_{\mathbb{R}_0} \frac{\mathrm{d}\nabla_{k_j} g_j}{\mathrm{d}\nu}(t, \eta)k_j^i(t, \eta)\nu(\mathrm{d}\eta) + \\ \qquad \frac{\partial f_j}{\partial x}(t)x^i(t) + \frac{\partial f_j}{\partial \upsilon_i}(t)\upsilon_i(t)]\mathrm{d}t - \\ \qquad z_j^i(t)\mathrm{d}W(t) - \int_{\mathbb{R}_0} k_j^i(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ \qquad t \in [0, T], \\ x^i(0) = 0,\ y_j^i(T) = \psi_j'(\hat{x}(T))x^i(T), \\ i, j = 1, 2, \end{cases} \tag{8}$$

where $\frac{\mathrm{d}\nabla_{k_j} g_j}{\mathrm{d}\nu}(t, \eta)$ is the Radom-Nikodym derivative of $\nabla_{k_j} g_j(t, \eta)$ with respect to $\nu(\eta)$. Here, $\nabla_{k_j} g_j(t, \eta)$ stands for the Fréchet derivative of g_j with respect to $k_j \in L^2(\nu)$, and we assume that $\nabla_{k_j} g_j(t, \eta)$ as a random measure is absolutely continuous with respect to ν.

$$\begin{cases}\mathrm{d}Z^i(t)=\sum\limits_{m=1}^{2}[Z^i(t)\varrho_m(t)+Z(t)(\dfrac{\partial\varrho_m}{\partial x}(t)x^i(t)+\\ \qquad\dfrac{\partial\varrho_m}{\partial\upsilon_i}(t)\upsilon_i(t))]\mathrm{d}Y_m(t),\ \ t\in[0,T],\\ Z^i(0)=0,\ i=1,2.\end{cases}\tag{9}$$

Since (8) and (9) are a linear FBSDE with random jumps and a linear SDE, we can easily derive that both of them exist a unique adapted solution, respectively, under A1)–A4) and for any $(\upsilon_1(\cdot),\upsilon_2(\cdot))\in\mathcal{A}_1\times\mathcal{A}_2$ (see Wu[19] and Øksendal[20]).

Similarly to Lemmas 1–3 in [15], we can obtain the following Lemmas 1–2. Thus, we omit the details for simplicity.

Lemma 1 Under A1)–A4), for $i,j=1,2$, we have

$$\lim_{\epsilon\to0}\sup_{0\leqslant t\leqslant T}E|\frac{x^{\epsilon_i}(t)-\hat{x}(t)}{\epsilon}-x^i(t)|^2=0,$$

$$\lim_{\epsilon\to0}\sup_{0\leqslant t\leqslant T}E|\frac{y_j^{\epsilon_i}(t)-\hat{y}_j(t)}{\epsilon}-y_j^i(t)|^2=0,$$

$$\lim_{\epsilon\to0}E\int_0^T|\frac{z_j^{\epsilon_i}(t)-\hat{z}_j(t)}{\epsilon}-z_j^i(t)|^2\mathrm{d}t=0,$$

$$\lim_{\epsilon\to0}E\int_0^T\int_{\mathbb{R}_0}|\frac{k_j^{\epsilon_i}(t,\eta)-\hat{k}_j(t,\eta)}{\epsilon}-k_j^i(t,\eta)|^2\nu(\mathrm{d}\eta)\mathrm{d}t=0,$$

$$\lim_{\epsilon\to0}\sup_{0\leqslant t\leqslant T}E|\frac{Z^{\epsilon_i}(t)-Z(t)}{\epsilon}-Z^i(t)|^2=0.$$

Since $(u_1(\cdot),u_2(\cdot))$ is a Nash equilibrium point of the game problem (7), it is clear that

$$\begin{cases}\epsilon^{-1}[J_{g_1}(u_1^\epsilon(\cdot),u_2(\cdot))-J_{g_1}(u_1(\cdot),u_2(\cdot))]\geqslant0,\\ \epsilon^{-1}[J_{g_2}(u_1(\cdot),u_2^\epsilon(\cdot))-J_{g_2}(u_1(\cdot),u_2(\cdot))]\geqslant0.\end{cases}\tag{10}$$

Besides, let $\tilde{Z}^i(\cdot)=Z^{-1}(\cdot)Z^i(\cdot)\ (i=1,2)$. For the optimal controls $(u_1(\cdot),u_2(\cdot))$, we have

$$\begin{cases}\mathrm{d}\tilde{Z}^i(t)=\sum\limits_{m=1}^{2}[\dfrac{\partial\varrho_m}{\partial x}(t)x^i(t)+\dfrac{\partial\varrho_m}{\partial\upsilon_i}(t)\upsilon_i(t)]\cdot\\ \qquad\mathrm{d}W_m^{u_1,u_2}(t),\ t\in[0,T],\\ \tilde{Z}^i(0)=0,\ i=1,2.\end{cases}$$

According to the inequality (10), and by Lemma 1 and Taloy's expansion, we derive the following inequalities.

Lemma 2 Suppose that A2)–A4) hold and $(u_1(\cdot),u_2(\cdot))$ is a Nash equilibrium point. Then, it yields the variational inequalities as follows:

$$E^{u_1,u_2}[\int_0^T((f_i(t)+g_i(t))\tilde{Z}^i(t)+\frac{\partial f_i}{\partial x}(t)x^i(t)+\frac{\partial f_i}{\partial\upsilon_i}(t)\upsilon_i(t)+\frac{\partial g_i}{\partial y_i}(t)y_i^i(t)+\frac{\partial g_i}{\partial z_i}(t)z_i^i(t)+\int_{\mathbb{R}_0}\frac{\mathrm{d}\nabla_{k_i}g_i}{\mathrm{d}\nu}(t,\eta)k_i^i(t,\eta)\nu(\mathrm{d}\eta))\mathrm{d}t+\psi_i(\hat{x}(T))\cdot\tilde{Z}^i(T)+\psi_i'(\hat{x}(T))x^i(T)]\geqslant0,\ i=1,2.\tag{11}$$

The Hamiltonian functions $H_i:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times L^2(\nu)\times U_1\times U_2\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times L^2(\nu)\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}\ (i=1,2)$ are defined by

$$H_i(t,x,y_i,z_i,k_i,\upsilon_1,\upsilon_2;p_i,q_i,s_i,\mu_i,\beta_{1i},\beta_{2i})=(g_i(t,y_i,z_i,k_i)+f_i(t,x,\upsilon_1,\upsilon_2))(1-p_i)+b(t,x,\upsilon_1,\upsilon_2)q_i+\sigma(t,x,\upsilon_1,\upsilon_2)s_i+\int_{\mathbb{R}_0}\gamma(t,x,\upsilon_1,\upsilon_2,\eta)\mu_i(\eta)\nu(\mathrm{d}\eta)+\varrho_1(t,x,\upsilon_1,\upsilon_2)\beta_{1i}+\varrho_2(t,x,\upsilon_1,\upsilon_2)\beta_{2i}.$$

To establish the maximum principle, we introduce the adjoint equations as follows:

$$\begin{cases}-\mathrm{d}L_i(t)=[g_i(t)+f_i(t)]\mathrm{d}t-\sum\limits_{m=1}^{2}\beta_{mi}(t)\cdot\\ \qquad\mathrm{d}W_m^{u_1,u_2}(t),\ t\in[0,T],\\ L_i(T)=\psi_i(\hat{x}(T)),\ i=1,2,\end{cases}\tag{12}$$

$$\begin{cases}\mathrm{d}p_i(t)=-\dfrac{\partial H_i}{\partial y_i}(t)\mathrm{d}t-\dfrac{\partial H_i}{\partial z_i}(t)\mathrm{d}W(t)-\\ \qquad\displaystyle\int_{\mathbb{R}_0}\frac{\mathrm{d}\nabla_{k_i}H_i}{\mathrm{d}\nu}(t,\eta)\tilde{N}(\mathrm{d}t,\mathrm{d}\eta),\\ -\mathrm{d}q_i(t)=\dfrac{\partial H_i}{\partial x}(t)\mathrm{d}t-s_i(t)\mathrm{d}W(t)-\\ \qquad\displaystyle\int_{\mathbb{R}_0}\mu_i(t,\eta)\tilde{N}(\mathrm{d}t,\mathrm{d}\eta),\ t\in[0,T],\\ p_i(0)=0,\\ q_i(T)=(1-p_i(T))\psi_i'(\hat{x}(T)),\ i=1,2,\end{cases}\tag{13}$$

where

$$\frac{\partial H_i}{\partial y_i}(t)=[\frac{\partial H_i}{\partial y_i}(t,\hat{x}(t),y_i,\hat{z}_i(t),\hat{k}_i(t,\cdot),u_1(t),u_2(t),p_i(t),q_i(t),s_i(t),\mu_i(t,\cdot),\beta_{1i}(t),\beta_{2i}(t))]_{y_i=\hat{y}_i(t)}.$$

If A1)–A4) hold, then (12) and (13) admit a unique adapted solution, respectively (see Wu [19]). Note that $1-p_i(\cdot)$ is a geometric Lévy process with initial value $1-p_i(0)=1\ (i=1,2)$.

We state the maximum principle for the game system.

Theorem 1 Suppose that A1)–A4) hold and $(u_1(\cdot),u_2(\cdot))$ is a Nash equilibrium point of the nonzero-sum game problem (7) with the corresponding state process $(\hat{x}(\cdot),\hat{y}_1(\cdot),\hat{z}_1(\cdot),\hat{k}_1(\cdot,\cdot),\hat{y}_2(\cdot),\ \hat{z}_2(\cdot),\hat{k}_2(\cdot,\cdot))$. Let $(L_i(\cdot),\beta_{1i}(\cdot),\beta_{2i}(\cdot))$ and $(p_i(\cdot),\ q_i(\cdot),s_i(\cdot),\mu_i(\cdot,\cdot))\ (i=1,2)$ be the solutions of (12) and (13), respectively. Then we have

$$E^{u_1,u_2}[\frac{\partial H_1}{\partial \upsilon_1}(t)(\upsilon_1 - u_1(t))|\mathcal{F}_t^1] \geqslant 0,$$
$$E^{u_1,u_2}[\frac{\partial H_2}{\partial \upsilon_2}(t)(\upsilon_2 - u_2(t))|\mathcal{F}_t^2] \geqslant 0,$$

for any $(\upsilon_1, \upsilon_2) \in U_1 \times U_2$, a.e. $t \in [0, T]$, P^{u_1,u_2} – a.s..

Proof We only consider the case $i = 1$. Applying Itô's formula to $\tilde{Z}^1(t)L_1(t) + x^1(t)q_1(t) + y_1^1(t)p_1(t)$, we get

$$E^{u_1,u_2}[\psi_1(\hat{x}(T))\tilde{Z}^1(T) + \psi_1'(\hat{x}(T))x^1(T)] = E^{u_1,u_2}\int_0^T [\sum_{m=1}^{2} \beta_{m1}(t)\frac{\partial \varrho_m}{\partial \upsilon_1}(t)\upsilon_1(t) - \tilde{Z}^1(t)\cdot (g_1(t)+f_1(t)) - p_1(t)\frac{\partial f_1}{\partial \upsilon_1}(t)\upsilon_1(t) + q_1(t)\frac{\partial b}{\partial \upsilon_1}(t)\cdot \upsilon_1(t) + s_1(t)\frac{\partial \sigma}{\partial \upsilon_1}(t)\upsilon_1(t) + \int_{\mathbb{R}_0} \mu_1(t,\eta)\frac{\partial \gamma}{\partial \upsilon_1}(t,\eta)\cdot \upsilon_1(t)\nu(\mathrm{d}\eta) - x^1(t)\frac{\partial f_1}{\partial x}(t) - y_1^1(t)\frac{\partial g_1}{\partial y_1}(t) - z_1^1(t)\cdot \frac{\partial g_1}{\partial z_1}(t) - \int_{\mathbb{R}_0} k_1^1(t,\eta)\frac{\partial g_1}{\partial k_1}(t,\eta)\nu(\mathrm{d}\eta)]\mathrm{d}t. \tag{14}$$

Substituting (14) into (11), we derive that

$$E^{u_1,u_2}\int_0^T \frac{\partial H_1}{\partial \upsilon_1}(t)\upsilon_1(t)\mathrm{d}t \geqslant 0, \tag{15}$$

for any $\upsilon_1(\cdot)$ such that $u_1(\cdot) + \upsilon_1(\cdot) \in \mathcal{A}_1$. Let $\pi_1(\cdot) = u_1(\cdot) + \upsilon_1(\cdot)$. From (15), it implies that

$$E^{u_1,u_2}[\frac{\partial H_1}{\partial \upsilon_1}(t)(\pi_1(t) - u_1(t))] \geqslant 0, \text{a.e.}. \tag{16}$$

Moreover, for any $\upsilon_1 \in U_1$, $A \in \mathcal{F}_t^1$, we suppose that $\chi_1(t) = \upsilon_1 I_A + u_1(t)I_{A^c}$. It is obvious that $\chi_1(\cdot) \in \mathcal{A}_1$. Thus, inserting χ_1 into (16) yields

$$E^{u_1,u_2}[\frac{\partial H_1}{\partial \upsilon_1}(t)(\upsilon_1 - u_1(t))I_A] \geqslant 0, \text{a.e.},$$

for any $A \in \mathcal{F}_t^1$. Therefore, we have

$$E^{u_1,u_2}[\frac{\partial H_1}{\partial \upsilon_1}(t)(\upsilon_1 - u_1(t))|\mathcal{F}_t^1] \geqslant 0,$$

a.e., P^{u_1,u_2} – a.s.. QED.

4 Verification theorem

In this section, we build a sufficient verification theorem for the game problem under some convexity conditions.

Theorem 2 Let A1)–A4) hold. Let $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$, and $(\hat{x}(\cdot), \hat{y}_1(\cdot), \hat{z}_1(\cdot), \hat{k}_1(\cdot,\cdot), \hat{y}_2(\cdot), \hat{z}_2(\cdot), \hat{k}_2(\cdot,\cdot))$ be the corresponding trajectory. Suppose that $(L_i(\cdot), \beta_{1i}(\cdot), \beta_{2i}(\cdot))$ and $(p_i(\cdot), q_i(\cdot), s_i(\cdot), \mu_i(\cdot,\cdot))$ $(i = 1, 2)$ satisfy (12) and (13), respectively. Furthermore, suppose that for all $t \in [0, T]$, $H_i(t,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot; p_i(t), q_i(t), s_i(t), \mu_i(t,\cdot), \beta_{1i}(t), \beta_{2i}(t))$ and $\psi_i(\cdot)$ are convex respect to the corresponding variables, respectively, and the following conditions hold:

$$\begin{aligned} E[H_1(t)|\mathcal{F}_t^1] &= \min_{\upsilon_1 \in U_1} E[H_1^{\upsilon_1}(t)|\mathcal{F}_t^1], \\ E[H_2(t)|\mathcal{F}_t^2] &= \min_{\upsilon_2 \in U_2} E[H_2^{\upsilon_2}(t)|\mathcal{F}_t^2], \end{aligned} \tag{17}$$

where

$$H_i(t) = H_i(t, \hat{x}(t), \hat{y}_i(t), \hat{z}_i(t), \hat{k}_i(t,\cdot), u_1(t), u_2(t); p_i(t), q_i(t), s_i(t), \mu_i(t,\cdot), \beta_{1i}(t), \beta_{2i}(t)), \quad i = 1, 2,$$
$$H_1^{\upsilon_1}(t) = H_1(t, x^{\upsilon_1}(t), y_1^{\upsilon_1}(t), z_1^{\upsilon_1}(t), k_1^{\upsilon_1}(t,\cdot), \upsilon_1(t), u_2(t); p_1(t), q_1(t), s_1(t), \mu_1(t,\cdot), \beta_{11}(t), \beta_{21}(t)),$$
$$H_2^{\upsilon_2}(t) = H_2(t, x^{\upsilon_2}(t), y_2^{\upsilon_2}(t), z_2^{\upsilon_2}(t), k_2^{\upsilon_2}(t,\cdot), u_1(t), \upsilon_2(t); p_2(t), q_2(t), s_2(t), \mu_2(t,\cdot), \beta_{12}(t), \beta_{22}(t))$$

and $(x^{\upsilon_1}(\cdot), y_1^{\upsilon_1}(\cdot), z_1^{\upsilon_1}(\cdot), k_1^{\upsilon_1}(\cdot,\cdot))$ is the corresponding solution of (6) along with $(\upsilon_1(\cdot), u_2(\cdot))$ and similarly with $(x^{\upsilon_2}(\cdot), y_2^{\upsilon_2}(\cdot), z_2^{\upsilon_2}(\cdot), k_2^{\upsilon_2}(\cdot,\cdot))$.

Then, $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point for the nonzero-sum game system.

Proof We only consider the case $i = 1$. Let

$$o^{\upsilon_1}(t) = o(t, x^{\upsilon_1}(t), \upsilon_1(t), u_2(t)), \text{ for } o = b, \sigma, f_1$$

and similarly with $\gamma^{\upsilon_1}(t)$, $g_1^{\upsilon_1}(t)$, $\varrho_m^{\upsilon_1}(t)$ $(m = 1, 2)$. By the definition of J_{g_1}, we deduce that

$$J_{g_1}(\upsilon_1(\cdot), u_2(\cdot)) - J_{g_1}(u_1(\cdot), u_2(\cdot)) = R_1 + R_2 + R_3,$$

where

$$R_1 = E^{\upsilon_1,u_2}[\psi_1(x^{\upsilon_1}(T)) - \psi_1(\hat{x}(T))],$$
$$R_2 = E\int_0^T (g_1(t) + f_1(t))(Z^{\upsilon_1,u_2}(t) - Z(t))\mathrm{d}t + E[\psi_1(\hat{x}(T))(Z^{\upsilon_1,u_2}(T) - Z(T))],$$
$$R_3 = E^{\upsilon_1,u_2}\int_0^T (g_1^{\upsilon_1}(t) - g_1(t) + f_1^{\upsilon_1}(t) - f_1(t))\mathrm{d}t.$$

From (13), we have

$$R_1 = E^{\upsilon_1,u_2}[\psi_1(x^{\upsilon_1}(T)) - \psi_1(\hat{x}(T))] - E^{\upsilon_1,u_2}[p_1(0)(y_1^{\upsilon_1}(0) - \hat{y}_1(0))].$$

Using Itô's formula to $p_1(t)(y_1^{\upsilon_1}(t) - \hat{y}_1(t))$, and by the convexity of ψ_1 with noticing that $1 - p_1(T) > 0$, we get

$$R_1 \geqslant E^{\upsilon_1,u_2}[\psi_1'(\hat{x}(T))(1 - p_1(T))(x^{\upsilon_1}(T) - \hat{x}(T))] - R_4, \tag{18}$$

where

$$R_4 = E^{\upsilon_1,u_2}\int_0^T p_1(t)(g_1^{\upsilon_1}(t) - g_1(t) + f_1^{\upsilon_1}(t) - f_1(t))\mathrm{d}t + E^{\upsilon_1,u_2}\int_0^T \frac{\partial H_1}{\partial y_1}(t)(y_1^{\upsilon_1}(t) - \hat{y}_1(t))\mathrm{d}t +$$

$$E^{\upsilon_1,u_2}\int_0^T\frac{\partial H_1}{\partial z_1}(t)(z_1^{\upsilon_1}(t)-\hat{z}_1(t))\mathrm{d}t+$$
$$E^{\upsilon_1,u_2}\int_0^T\int_{\mathbb{R}_0}\frac{\mathrm{d}\nabla_{k_1}H_1}{\mathrm{d}\nu}(t,\eta)(k_1^{\upsilon_1}(t,\eta)-$$
$$\hat{k}_1(t,\eta))\nu(\mathrm{d}\eta)\mathrm{d}t. \tag{19}$$

Applying Itô's formula to $q_1(t)(x^{\upsilon_1}(t)-\hat{x}(t))$ leads to

$$E^{\upsilon_1,u_2}[\psi_1'(\hat{x}(T))(1-p_1(T))(x^{\upsilon_1}(T)-\hat{x}(T))]=$$
$$E^{\upsilon_1,u_2}\int_0^T q_1(t)(b^{\upsilon_1}(t)-b(t))\mathrm{d}t+$$
$$E^{\upsilon_1,u_2}\int_0^T s_1(t)(\sigma^{\upsilon_1}(t)-\sigma(t))\mathrm{d}t+$$
$$E^{\upsilon_1,u_2}\int_0^T\int_{\mathbb{R}_0}\mu_1(t,\eta)(\gamma^{\upsilon_1}(t,\eta)-\gamma(t,\eta))\nu(\mathrm{d}\eta)\mathrm{d}t-$$
$$E^{\upsilon_1,u_2}\int_0^T\frac{\partial H_1}{\partial x}(t)(x^{\upsilon_1}(t)-\hat{x}(t))\mathrm{d}t. \tag{20}$$

Using Itô's formula to $L_1(t)(Z^{\upsilon_1,u_2}(t)-Z(t))$, we obtain

$$R_2=\sum_{m=1}^{2}E^{\upsilon_1,u_2}\int_0^T\beta_{m1}(t)(\varrho_m^{\upsilon_1}(t)-\varrho_m(t))\mathrm{d}t. \tag{21}$$

By the definition and convexity of H_1, we derive that

$$R_3\geqslant$$
$$E^{\upsilon_1,u_2}\int_0^T[\frac{\partial H_1}{\partial x}(t)(x^{\upsilon_1}(t)-\hat{x}(t))+\frac{\partial H_1}{\partial y_1}(t)\cdot$$
$$(y_1^{\upsilon_1}(t)-\hat{y}_1(t))+\frac{\partial H_1}{\partial z_1}(t)(z_1^{\upsilon_1}(t)-\hat{z}_1(t))+$$
$$\frac{\partial H_1}{\partial \upsilon_1}(t)(\upsilon_1(t)-u_1(t))+\int_{\mathbb{R}_0}\frac{\mathrm{d}\nabla_{k_1}H_1}{\mathrm{d}\nu}(t,\eta)\cdot$$
$$(k_1^{\upsilon_1}(t,\eta)-\hat{k}_1(t,\eta))\nu(\mathrm{d}\eta)]\mathrm{d}t+E^{\upsilon_1,u_2}\int_0^T[p_1(t)\cdot$$
$$(g_1^{\upsilon_1}(t)-g_1(t)+f_1^{\upsilon_1}(t)-f_1(t))-q_1(t)(b^{\upsilon_1}(t)-$$
$$b(t))-s_1(t)(\sigma^{\upsilon_1}(t)-\sigma(t))-\int_{\mathbb{R}_0}\mu_1(t,\eta)\cdot$$
$$(\gamma^{\upsilon_1}(t,\eta)-\gamma(t,\eta))\nu(\mathrm{d}\eta)-\sum_{m=1}^{2}\beta_{m1}(t)(\varrho_m^{\upsilon_1}(t)-$$
$$\varrho_m(t))]\mathrm{d}t. \tag{22}$$

Combining (18)–(22), we have

$$J_{g_1}(\upsilon_1(\cdot),u_2(\cdot))-J_{g_1}(u_1(\cdot),u_2(\cdot))\geqslant$$
$$E^{\upsilon_1,u_2}\int_0^T\frac{\partial H_1}{\partial \upsilon_1}(t)(\upsilon_1(t)-u_1(t))\mathrm{d}t=$$
$$E\int_0^T Z^{\upsilon_1,u_2}(t)E[\frac{\partial H_1}{\partial \upsilon_1}(t)(\upsilon_1(t)-u_1(t))|\mathcal{F}_t^1]\mathrm{d}t.$$

From (17), we deduce that

$$E[\frac{\partial H_1}{\partial \upsilon_1}(t)(\upsilon_1(t)-u_1(t))|\mathcal{F}_t^1]\geqslant 0.$$

Since $Z^{\upsilon_1,u_2}(\cdot)>0$, we conclude that

$$J_{g_1}(u_1(\cdot),u_2(\cdot))=\min_{\upsilon_1(\cdot)\in\mathcal{A}_1}J_{g_1}(\upsilon_1(\cdot),u_2(\cdot)).$$

In the same way, we obtain

$$J_{g_2}(u_1(\cdot),u_2(\cdot))=\min_{\upsilon_2(\cdot)\in\mathcal{A}_2}J_{g_2}(u_1(\cdot),\upsilon_2(\cdot)).$$

Hence, $(u_1(\cdot),u_2(\cdot))$ is a Nash equilibrium point.

QED.

5 Application to finance

Motivated by Huang et al.[21], Xiong and Zhou[22], we consider a partially observed game problem about the asset-liability management of a firm. Suppose that the liability process $F(\cdot)$ of the firm is described by

$$-\mathrm{d}F(t)=[b_1(t)\upsilon_1(t)+b_2(t)\upsilon_2(t)-b(t)]\mathrm{d}t+\sigma(t)\mathrm{d}W(t)+\int_{\mathbb{R}_0}\gamma(t,\eta)\tilde{N}(\mathrm{d}t,\mathrm{d}\eta),$$

where $\upsilon_1(t)$ and $\upsilon_2(t)$ are the rates of capital injection or withdrawal, and serve as the control strategies of two policymakers; $b(t)>0$ is the expected liability rate; $\sigma(t)>0$ and $\gamma(t,\eta)>0$ are the liability risks; $b_1(t)>0$ and $b_2(t)>0$ are bounded coefficients.

We introduce the cash balance process $x(\cdot)$ deduced from the liability process $F(\cdot)$ as follows:

$$x(t)=\mathrm{e}^{\int_0^t b_0(s)\mathrm{d}s}(x_0-\int_0^t\mathrm{e}^{-\int_0^s b_0(r)\mathrm{d}r}\mathrm{d}F(s)).$$

It can be written in the following form:

$$\begin{cases}\mathrm{d}x(t)=[b_0(t)x(t)+b_1(t)\upsilon_1(t)+b_2(t)\upsilon_2(t)-\\ \qquad b(t)]\mathrm{d}t+\sigma(t)\mathrm{d}W(t)+\\ \qquad\int_{\mathbb{R}_0}\gamma(t,\eta)\tilde{N}(\mathrm{d}t,\mathrm{d}\eta),\\ \qquad t\in[0,T],\\ x(0)=x_0,\end{cases}$$

where x_0 is the initial investment of the firm in a money account, and $b_0(t)>0$ is the compounded interest rate.

Then, the observation equations are governed by

$$\begin{cases}\mathrm{d}Y_i(t)=c_i(t)b(t)\mathrm{d}t+\mathrm{d}W_i^{\upsilon_1,\upsilon_2}(t),\\ Y_i(0)=1,\ i=1,2,\end{cases} \tag{23}$$

where $c_i(t)$ is a bounded and deterministic function.

We define a new probability measure $P^{\upsilon_1,\upsilon_2}$ by $\frac{\mathrm{d}P^{\upsilon_1,\upsilon_2}}{\mathrm{d}P}|_{\mathcal{F}_t}=Z^{\upsilon_1,\upsilon_2}(t)$, where

$$\begin{cases}\mathrm{d}Z^{\upsilon_1,\upsilon_2}(t)=\sum_{i=1}^{2}Z^{\upsilon_1,\upsilon_2}(t)c_i(t)b(t)\mathrm{d}Y_i(t),\\ Z^{\upsilon_1,\upsilon_2}(0)=1.\end{cases}$$

Hence, $(W(\cdot),W_1^{\upsilon_1,\upsilon_2}(\cdot),W_2^{\upsilon_1,\upsilon_2}(\cdot))$ is a 3-dimensional standard Brownian motion and $\tilde{N}(\cdot,\cdot)$ is a compensated Poisson random measure defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P^{\upsilon_1,\upsilon_2})$.

Assume that the two policymakers can only observe the related stock price processes by their own:

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)[(\alpha_i c_i(t)b(t) + \frac{1}{2}\alpha_i^2)\mathrm{d}t + \\ \qquad \alpha_i \mathrm{d}W_i^{v_1,v_2}(t)], \\ S_i(0) = 1,\ i = 1, 2, \end{cases}$$

where $\alpha_i c_i(t)b(t) + \frac{1}{2}\alpha_i^2$ is an appreciation rate of the stock, and $\alpha_i > 0$ is a volatility coefficient of the stock.

Thus, $\sigma\{S_i(s); 0 \leqslant s \leqslant t\}$ is the information filtration for policymaker $i (i = 1, 2)$ at time t. Since $\mathrm{d}Y_i(t) = \frac{1}{\alpha_i} d\log S_i(t)$, we get

$$\mathcal{F}_t^i = \sigma\{Y_i(s); 0 \leqslant s \leqslant t\} = \sigma\{S_i(s); 0 \leqslant s \leqslant t\}.$$

The cost functionals J_{g_i} $(i = 1, 2)$ are defined by

$$J_{g_i}(v_1(\cdot), v_2(\cdot)) = \mathcal{E}_{g_i}^{v_1,v_2}[\int_0^T f_i(t, v_1(t), v_2(t))\mathrm{d}t - x(T)].$$

Now, we suppose that g_i is independent of y_i. That is to say, $g_i = g_i(t, z_i, k_i)$. By the similar method in Section 2, we can rewrite the cost functional J_{g_i} as follows:

$$J_{g_i}(v_1(\cdot), v_2(\cdot)) = E^{v_1,v_2}[\int_0^T (f_i(t, v_1(t), v_2(t)) + g_i(t, z_i(t), k_i(t, \cdot)))\mathrm{d}t - x(T)],\ i = 1, 2$$

with the corresponding state equations

$$\begin{cases} \mathrm{d}x(t) = [b_0(t)x(t) + b_1(t)v_1(t) + b_2(t)v_2(t) - \\ \qquad b(t)]\mathrm{d}t + \sigma(t)\mathrm{d}W(t) + \\ \qquad \int_{\mathbb{R}_0} \gamma(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ -\mathrm{d}y_i(t) = [g_i(t, z_i(t), k_i(t, \cdot)) + f_i(t, v_1(t), \\ \qquad v_2(t))]\mathrm{d}t - z_i(t)\mathrm{d}W(t) - \\ \qquad \int_{\mathbb{R}_0} k_i(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta), \\ \qquad t \in [0, T], \\ x(0) = x_0,\ \ y_i(T) = -x(T),\ i = 1, 2, \end{cases}$$

where $g_i : \Omega \times [0, T] \times \mathbb{R} \times L^2(\nu) \mapsto \mathbb{R}$ is convex with respect to z_i and k_i, and satisfies $\frac{\mathrm{d}\nabla_{k_i} g_i}{\mathrm{d}\nu}(t, \eta) > -1$ for all t, η a.s.; $f_i : \Omega \times [0, T] \times U_1 \times U_2 \mapsto \mathbb{R}$ is convex and quadratic differentiable with respect to v_1 and v_2.

Our aim is to find a pair of $\mathcal{F}_t^1 \vee \mathcal{F}_t^2$-adapted and square integrable processes $(u_1(\cdot), u_2(\cdot))$ such that

$$\begin{cases} J_{g_1}(u_1(\cdot), u_2(\cdot)) = \min\limits_{v_1(\cdot)\in\mathcal{A}_1} J_{g_1}(v_1(\cdot), u_2(\cdot)), \\ J_{g_2}(u_1(\cdot), u_2(\cdot)) = \min\limits_{v_2(\cdot)\in\mathcal{A}_2} J_{g_2}(u_1(\cdot), v_2(\cdot)). \end{cases} \tag{24}$$

The Hamiltonian functions $H_i (i = 1, 2)$ are given by

$$H_i(t, x, y_i, z_i, k_i, v_1, v_2; p_i, q_i, s_i, \mu_i, \beta_{1i}, \beta_{2i}) = (g_i(t, z_i, k_i) + f_i(t, v_1, v_2))(1 - p_i) + (b_0(t)x + b_1(t)v_1 + b_2(t)v_2 - b(t))q_i + \sigma(t)s_i + \int_{\mathbb{R}_0} \gamma(t, \eta) \cdot \mu_i(\eta)\nu(\mathrm{d}\eta) + c_1(t)b(t)\beta_{1i} + c_2(t)b(t)\beta_{2i}.$$

The adjoint process $(p_i(\cdot), q_i(\cdot), s_i(\cdot), \mu_i(\cdot, \cdot))$ satisfies:

$$\begin{cases} \mathrm{d}p_i(t) = (p_i(t) - 1)[\frac{\partial g_i}{\partial z_i}(t)\mathrm{d}W(t) + \\ \qquad \int_{\mathbb{R}_0} \frac{\mathrm{d}\nabla_{k_i} g_i}{\mathrm{d}\nu}(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta)], \\ -\mathrm{d}q_i(t) = b_0(t)q_i(t)\mathrm{d}t - s_i(t)\mathrm{d}W(t) - \\ \qquad \int_{\mathbb{R}_0} \mu_i(t, \eta)\tilde{N}(\mathrm{d}t, \mathrm{d}\eta),\ t \in [0, T], \\ p_i(0) = 0,\ q_i(T) = p_i(T) - 1,\ i = 1, 2. \end{cases} \tag{25}$$

Since $1 - p_i(t)$ is a geometric Lévy process, we derive the solution of the forward equation in (25):

$$p_i(t) = 1 - \exp\{-\frac{1}{2}\int_0^t |\frac{\partial g_i}{\partial z_i}(s)|^2\mathrm{d}s + \int_0^t \frac{\partial g_i}{\partial z_i}(s) \cdot \mathrm{d}W(s) + \int_0^t \int_{\mathbb{R}_0} [\ln(1 + \frac{\mathrm{d}\nabla_{k_i} g_i}{\mathrm{d}\nu}(s, \eta)) - \frac{\mathrm{d}\nabla_{k_i} g_i}{\mathrm{d}\nu}(s, \eta)]\nu(\mathrm{d}\eta)\mathrm{d}s + \int_0^t \int_{\mathbb{R}_0} \ln(1 + \frac{\mathrm{d}\nabla_{k_i} g_i}{\mathrm{d}\nu}(s, \eta))\tilde{N}(\mathrm{d}s, \mathrm{d}\eta)\},\ i = 1, 2.$$

Suppose

$$q_i(t) = \lambda_i(t)(p_i(t) - 1),$$

where $\lambda_i(t)$ is deterministic, and $\lambda_i(T) = 1$. Then, applying Itô's formula to $q_i(t)$, we derive

$$\mathrm{d}q_i(t) = \lambda_i'(t)(p_i(t) - 1)\mathrm{d}t + \lambda_i(t)(p_i(t) - 1) \cdot (\frac{\partial g_i}{\partial z_i}(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0} \frac{\mathrm{d}\nabla_{k_i} g_i}{\mathrm{d}\nu}(t, \eta) \cdot \tilde{N}(\mathrm{d}t, \mathrm{d}\eta)). \tag{26}$$

Comparing (26) with the backward equation in (25) by equating the $\mathrm{d}t$ coefficient, we have

$$\begin{cases} \lambda_i'(t) + b_0(t)\lambda_i(t) = 0, \\ \lambda_i(T) = 1. \end{cases}$$

The above equation admits the following solution:

$$\lambda_i(t) = \mathrm{e}^{\int_t^{\mathrm{T}} b_0(s)\mathrm{d}s}.$$

The solution of the backward equation in (25) is given by

$$q_i(t) = \mathrm{e}^{\int_t^{\mathrm{T}} b_0(s)\mathrm{d}s}(p_i(t) - 1),\ i = 1, 2.$$

From Theorem 1, if $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point, then for $i = 1, 2$, we get

$$E^{u_1,u_2}[(1-p_i(t))(\frac{\partial f_i}{\partial v_i}(t)-b_i(t)\mathrm{e}^{\int_t^{\mathrm{T}} b_0(s)\mathrm{d}s})|\mathcal{F}_t^i]=0. \quad (27)$$

Since

$$\frac{\partial^2 H_i}{\partial v_i^2}(t) = (1-p_i(t))\frac{\partial^2 f_i}{\partial v_i^2}(t) \geqslant 0,$$

based on Theorem 2, we conclude that $(u_1(\cdot), u_2(\cdot))$ is indeed a Nash equilibrium point for the game problem.

Proposition 1 For the partially observed asset-liability management game problem (24), a Nash equilibrium point $(u_1(\cdot), u_2(\cdot))$ satisfies (27).

Remark 1 There is few general filtering results for FBSDEs with jumps, and the generators (i.e., f and g) are nonlinear functions, so we only study the case that the observation processes are independent of the state in (23).

6 Conclusions

This paper discussed the maximum principle and the verification theorem for a partially observed nonzero-sum SDG with g-expectation. Owing to the complexity of computing the optimal filtering of adjoint processes, we solved a special case for the asset-liability management game problem. It would be desirable to research the general filtering theory for FBSDEs with jumps in future work.

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作者简介:

杨碧璇 博士研究生, 目前研究方向为随机控制、随机微分博弈、数理金融, E-mail: bixuanyang@126.com;

郭铁信 博士生导师, 教授, 中南大学数学与统计学院副院长, 目前研究方向为随机泛函分析、数理金融;

吴锦标 硕士生导师, 副教授, 目前研究方向为随机偏微分方程、随机控制、随机网络, E-mail: wujinbiao@csu.edu.cn.