DOI: 10.7641/CTA.2013.12046

基于动态补偿的线性系统最优干扰抑制

刘 蕾[1†], 张国山[2]

(1. 北京大学 工学院 力学与工程科学系, 北京 100871; 2. 天津大学 电气与自动化工程学院, 天津 300072)

摘要: 本文针对受外部干扰的线性时不变系统研究了基于动态补偿的最优干扰抑制问题, 其中干扰信号为已知动态特性的扰动信号. 首先, 将原系统与扰动系统联立构成增广系统, 进而转化为无扰动的标准线性二次最优问题. 其次, 给出了经具有适当动态阶的补偿器补偿后的闭环系统渐近稳定并且相关的Lyapunov方程正定对称解存在的条件, 进一步给定的二次性能指标可写成一个与该解和闭环系统初值相关的表达式. 为了得到系统的最优解, 将该Lyapunov方程转化为一个双线性矩阵不等式形式, 并给出了相应的路径跟踪算法以求得性能指标最小值以及补偿器参数. 最后, 通过数值算例说明应用本文方法可以不仅能够最小化线性二次指标, 而且能够使得系统的干扰得到抑制.

关键词: 线性系统; 动态补偿; 线性二次最优控制; 干扰抑制; 双线性矩阵不等式; 路径跟踪算法

中图分类号: TP273 **文献标识码**: A

Optimal disturbance rejection via dynamic compensation for linear systems

LIU Lei[1†], ZHANG Guo-shan[2]

(1. Department of Mechanics and Engineering Science, College of Engineering, Peking University, Beijing 100871, China;
2. School of Electrical Engineering and Automation, Tianjin University, Tianjin 300072, China)

Abstract: We investigate the linear-quadratic optimal control by using dynamic compensation for the linear time-invariant system affected by external persistent disturbances with known dynamic characteristics. By combining the system with the disturbance system, we transform this optimal disturbance rejection problem into the standard linear quadratic optimal control problem without disturbance, and develop the dynamic compensator with appropriate order to make the closed-loop system asymptotically stable with associated Lyapunov equation having a symmetric positive-definite solution. The quadratic performance index is formulated as a simple expression related to the symmetric positive-definite solution to the Lyapunov equation as well as the initial value of the closed-loop system. In order to solve the optimal control problem for the system, we transform the Lyapunov equation to a bilinear matrix inequality and develop a corresponding path-following algorithm to minimize the quadratic performance index and obtain the optimal dynamic compensator. Finally, a numerical example is provided to show that the proposed method can minimize the linear quadratic performance index and reject the system disturbances.

Key words: linear systems; dynamic compensator; linear-quadratic (LQ) optimal control; disturbance rejection; bilinear matrix inequality (BMI); path-following method

1 Introduction

The linear-quadratic (LQ) optimal control problem has been well studied for many years owing to its comprehensive practical applications [1–2]. Most of the results are obtained based on static output feedback [3–7] and dynamic compensation [8–9] for the systems without disturbance. However, many control problems involve designing a controller capable of stabilizing a given system while minimizing the worst-case response to some exogenous disturbances. This problem is called optimal disturbance rejection. It is a subject of recurrent interest. For the unknown random disturbance, [10] shows that the persistent disturbance rejection achieved by any stabilizing state-feedback linear dynamic controller can be also achieved using a memoryless variable structure controller. [11] has presented the solutions to the linear quadratic regulator (LQR) design with the worst case disturbance rejection. And [12] presents a method which combines quasi-robust linear programming concept with a well-known L_1-optimal controller synthesis for the linear systems with persistent disturbance signals, while [13] has addressed the optimal disturbance attenuation problem by output feedback for linear systems with delayed input. A necessary and sufficient condition to guarantee the existence of an

Received 6 March 2013; revised 11 July 2012.
†Corresponding author. E-mail: liulei1223@pku.edu.cn; Tel.: +86 15210790145.
This work was supported by the National Natural Science Foundation of China (Nos. 60674019, 61074088).

optimal solution is provided using the geometric approach. [14] proposes an asymptotic rejection algorithm for nonlinear systems with unknown disturbances. For the known the dynamic characteristics of disturbance, [15] has presented a feedforward and feedback optimal controller with sinusoidal disturbance, while [16] has proposed combination of the reduced-order observer and the optimal feedback controller to reject the disturbance. However, the problem for the optimal disturbance rejection based on dynamic compensation has not been investigated yet.

So we will consider the LQ optimal control for linear time-invariant system with exogenous disturbance signal based on dynamic compensation in this paper. We are interested in the case where the exact disturbance can be formulated in terms of state space description. By taking the disturbance signal as a part of the state vector, an augmented system without disturbance can be obtained. First we will give a dynamic compensator with a proper dynamic order such that the closed-loop system is asymptotically stable, and its associated Lyapunov equation has a symmetric positive-definite solution. Then the given quadratic performance index can be derived to be a simple expression related to the solution to the Lyapunov equation and the initial value of the closed-loop system. In order to solve the optimization problem for the given quadratic performance index numerically, an iterative algorithm is proposed in the light of the path-following algorithm [17–18]. By applying this algorithm, we can derive an optimal dynamic compensator and the minimum value of the quadratic performance index. Finally, a numerical example is provided to show that the proposed method can minimize the linear quadratic performance index and make the system's disturbances rejected.

2 Problem formulation and preliminary

Consider the following linear time-invariant (LTI) system with disturbance signal:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D_1 v(t), \\ y(t) = Cx(t) + D_2 v(t) \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^q$ is the output vector, and $v(t) \in \mathbb{R}^l$ is the exogenous disturbance signal. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, $D_1 \in \mathbb{R}^{n\times l}$, $D_2 \in \mathbb{R}^{q\times l}$ are constant matrices. We assume that the realization $\{A, B, C\}$ is both controllable and observable.

The exogenous disturbance signal $v(t)$ can be represented by the exosystem:

$$\begin{cases} \dot{w}(t) = Gw(t), \\ v(t) = Fw(t), \\ w(0) = w_0, \end{cases} \tag{2}$$

where $w(t) \in \mathbb{R}^k$ is the state vector of the exosystem (2), G and F are constant matrices with appropriate dimensions. We assume that (G, F) is observable and the eigenvalues of the matrix G satisfies

$$\mathrm{Re}[\lambda_i(G)] < 0,\ i = 1, 2, \cdots, k, \tag{3}$$

As the exosystem is asymptotically stable, we consider the following performance index with the linear quadratic form

$$J = \frac{1}{2}\int_0^\infty [x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)]\mathrm{d}t, \tag{4}$$

where $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ are weight matrices and

$$Q = Q^{\mathrm{T}} \geqslant 0,\ R = R^{\mathrm{T}} > 0.$$

Remark 1 If

$$\mathrm{Re}[\lambda_i(G)] = 0, i = 1, 2, \cdots, k, \tag{5}$$

the steady state solutions to the state vector and the control vector are impossible to tend to zero. Therefore, the quadratic cost functional (4) may not be applied to the optimal control. In this case, we shall consider the minimization of a quadratic performance index as

$$J = \lim_{T\to\infty}\frac{1}{T}\int_0^T [x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)]\mathrm{d}t, \tag{6}$$

such that the performance index is convergent (see the reference [15]). The method for solving the optimal control problem in terms of performance index (6) is similar to that of the performance index (4). The only difference is that the closed-loop system is stable in sense of Lyapunov, that is $|x(t)| < \varepsilon$, $\varepsilon > 0$, and the aim of the optimal disturbance rejection is reducing ε as much as possible.

In this paper, we mainly discuss the exosystem (2) satisfied the condition (3), and in the numerical example we will give a disturbance with sinusoidal specification satisfied the condition (5).

Combine system (1) with the exosystem (2), and take $w(t)$ as a part of the state vector, then an equivalent system without disturbance can be obtained. And $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ is both controllable and observable.

$$\begin{cases} \dot{\eta}(t) = \tilde{A}\eta(t) + \tilde{B}u(t), \\ y(t) = \tilde{C}\eta(t), \\ \eta(0) = \eta_0, \end{cases} \tag{7}$$

where

$$\eta(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix},\ \tilde{A} = \begin{bmatrix} A & D_1F \\ 0 & G \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tilde{C} = \begin{bmatrix} C & D_2F \end{bmatrix}.$$

On the basis of system (7), we consider the dynamic compensator

$$\begin{cases} \dot{x}_{\mathrm{c}}(t) = A_{\mathrm{c}}x_{\mathrm{c}}(t) + B_{\mathrm{c}}y(t), \\ u(t) = C_{\mathrm{c}}x_{\mathrm{c}}(t) + D_{\mathrm{c}}y(t), \\ x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \end{cases} \tag{8}$$

where $x_{\mathrm{c}}(t) \in \mathbb{R}^{n_{\mathrm{c}}}$ is the state vector of dynamic compensator.

$$A_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}\times n_{\mathrm{c}}},\ B_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}\times q},$$
$$C_{\mathrm{c}} \in \mathbb{R}^{m\times n_{\mathrm{c}}},\ D_{\mathrm{c}} \in \mathbb{R}^{m\times q}$$

are matrices of dynamic compensator which are to be solved.

The aim of this paper is to design dynamic compensator (8) with proper dynamic order n_{c} for the system (7) such that the closed-loop system is asymptotically stable and the linear quadratic performance index (4) is minimized.

3 Main results

3.1 Optimal control based on dynamic compensation

The resultant closed-loop system from system (7) and its dynamic compensator (8) is

$$\begin{cases} \dot{\zeta}(t)=\begin{bmatrix} A+BD_{\rm c}C & D_1F+BD_{\rm c}D_2F & BC_{\rm c} \\ 0 & G & 0 \\ B_{\rm c}C & B_{\rm c}D_2F & A_{\rm c}\end{bmatrix}\zeta(t), \\ y(t)=\begin{bmatrix} C & D_2F & 0\end{bmatrix}\zeta(t), \\ \zeta(0)=\zeta_0, \end{cases} \tag{9}$$

where $\zeta(t)=\begin{bmatrix} x^{\rm T}(t) & w^{\rm T}(t) & x_{\rm c}^{\rm T}(t)\end{bmatrix}^{\rm T}$.

We define

$$\bar{A}:=\begin{bmatrix} A+BD_{\rm c}C & D_1F+BD_{\rm c}D_2F & BC_{\rm c} \\ 0 & G & 0 \\ B_{\rm c}C & B_{\rm c}D_2F & A_{\rm c}\end{bmatrix}$$

and let

$$\hat{A}=\begin{bmatrix} A & D_1F & 0 \\ 0 & G & 0 \\ 0 & 0 & 0\end{bmatrix},\ \hat{B}=\begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & I\end{bmatrix},$$
$$\hat{C}=\begin{bmatrix} C & D_2F & 0 \\ 0 & 0 & I\end{bmatrix},\ K=\begin{bmatrix} D_{\rm c} & C_{\rm c} \\ B_{\rm c} & A_{\rm c}\end{bmatrix},$$

and $\bar{A}=\hat{A}+\hat{B}K\hat{C}$, then the closed system with a dynamic compensator of order $n_{\rm c}$ is brought back to the static output feedback controller case.

Here, the quadratic performance index (4) is described by

$$J=\frac{1}{2}\int_0^{\infty}[\zeta^{\rm T}(t)\bar{Q}\zeta(t)]{\rm d}t, \tag{10}$$

where

$$\bar{Q}=\hat{Q}+\hat{C}^{\rm T}K^{\rm T}\hat{R}K\hat{C},$$
$$\hat{Q}=\begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix},\hat{R}=\begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}.$$

Remark 2 Applying $\hat{Q}$, $\hat{R}$, K and C to $\bar{Q}$, we obtain

$$\bar{Q}=\begin{bmatrix} \varXi_{11} & C^{\rm T}D_{\rm c}^{\rm T}RD_{\rm c}D_2F & C^{\rm T}D_{\rm c}^{\rm T}RC_{\rm c} \\ * & D_2^{\rm T}D_{\rm c}^{\rm T}RD_{\rm c}D_2F & F^{\rm T}D_2^{\rm T}D_{\rm c}^{\rm T}RC_{\rm c} \\ * & * & C_{\rm c}^{\rm T}RC_{\rm c}\end{bmatrix},$$

where "$*$" stood for the symmetric terms in a symmetric matrix and

$$\varXi_{11}=Q+C^{\rm T}D_{\rm c}^{\rm T}RD_{\rm c}C.$$

Apparently, $\bar{Q}\geqslant 0$.

Remark 3 Since $\zeta(t)$ contains disturbance signal $w(t)$, one can find out that the smaller (10) means the less impact for the system (1) response to the disturbance (2). Then the disturbance can be rejected.

Lemma 1[19] Let the system be defined as in (7), there exists dynamic compensator (8) with dynamic order $n_{\rm c}\geqslant n-\max\{m,q\}$, such that the closed-loop system (9) is asymptotically stable.

Theorem 1 Consider the system (7), if there exists a dynamic compensator (8) with dynamic order $n_{\rm c}\geqslant n-\max\{m,q\}$, such that the closed-loop system (9) is asymptotically stable and $\bar{Q}\geqslant 0$. Then the following Lyapunov equation

$$\bar{A}^{\rm T}P+P\bar{A}+\bar{Q}=0 \tag{11}$$

has symmetric positive-definite solution P, and the performance index $J=1/2\zeta_0^{\rm T}P\zeta_0$.

Proof From Lemma 1 we know that there exists dynamic compensator (8) with order $n_{\rm c}\geqslant n-\max\{m,q\}$ such that the closed-loop system (9) is asymptotically stable and the Lyapunov equation (11) has the symmetric positive-definite solution P since $\bar{A}$ is stable and $\bar{Q}\geqslant 0$. So we choose a Lyapunov function as

$$V(\zeta,t)=\zeta^{\rm T}(t)P\zeta(t). \tag{12}$$

It is obvious that $V(\zeta,t)$ is positive-definite, and the time-derivative of $V(\zeta,t)$ along the solution to (9) is given by

$$\dot{V}(\zeta,t)=2\dot{\zeta}^{\rm T}(t)P\zeta(t)=\zeta^{\rm T}[\bar{A}^{\rm T}P+P\bar{A}]\zeta(t). \tag{13}$$

Then from (11),

$$\dot{V}(\zeta,t)=-\zeta^{\rm T}(t)\bar{Q}\zeta(t) \tag{14}$$

is negative-semi-definite.

From (12) and (14), we obtain

$$\zeta^{\rm T}(t)\bar{Q}\zeta(t)=-\frac{\rm d}{{\rm d}t}[\zeta^{\rm T}(t)P\zeta(t)]. \tag{15}$$

Substituting (15) into (10)

$$J=\frac{1}{2}\int_0^{\infty}\zeta^{\rm T}(t)\bar{Q}\zeta(t){\rm d}t=-\frac{1}{2}\zeta^{\rm T}(t)P\zeta(t)|_0^{\infty}=$$
$$-\frac{1}{2}\zeta^{\rm T}(\infty)P\zeta(\infty)+\frac{1}{2}\zeta_0^{\rm T}P\zeta_0. \tag{16}$$

Because the poles of the closed system are in the open left-half-plane, ${\rm Re}[\lambda(\bar{A})]<0$, and $\zeta(\infty)\to 0$, the following equation can be obtained:

$$J=\frac{1}{2}\zeta_0^{\rm T}P\zeta_0. \tag{17}$$

The matrix P is the solution to the Lyapunov equation (11). The proof is completed.

In order to obtain the optimal performance index, we should solve the minimization problem described by

$$\min J=\frac{1}{2}\zeta_0^{\rm T}P\zeta_0,$$
$$\text{s.t.}\begin{cases}\bar{A}^{\rm T}P+P\bar{A}+\bar{Q}=0, \\ P>0.\end{cases} \tag{18}$$

In the next section, we will give an iterative algorithm based on the path-following method.

3.2 Solution to the minimization problem

In order to solve the Lyapunov equation (11) conveniently, we should add a small positive slack factor $\epsilon_1>0$ and transform that Lyapunov equation (11) into the following inequality:

$$|\bar{A}^{\rm T}P+P\bar{A}+\bar{Q}|<\epsilon_1 I, \tag{19}$$

where I is an identity matrix, which has the same dimension with $\bar{A}$, and $|X|<\epsilon_1 I$ is interpreted as

$$-\epsilon_1 I<X<\epsilon_1 I.$$

Equation (19) represents

$$\bar{A}^{\rm T}P+P\bar{A}+\bar{Q}=M,$$

where $|M| < \epsilon_1 I$. That is

$$\bar{A}^{\mathrm{T}}P + P\bar{A} + \bar{Q} - M = 0.$$

Let $\bar{Q}_{\text{approx}} = \bar{Q} - M$, then $\bar{Q} = \bar{Q}_{\text{approx}} + M$, the corresponding performance index is

$$J = \int_0^{\infty} \zeta^{\mathrm{T}}(t)(\bar{Q}_{\text{approx}} + M)\zeta(t)\mathrm{d}t = \zeta_0^{\mathrm{T}}(P_{\text{approx}} + M)\zeta_0 \approx \zeta_0^{\mathrm{T}} P_{\text{approx}}\zeta_0.$$

So, by adding a small positive slack factor $\epsilon_1 > 0$ to transform equation (11) into inequality (19), one can see that the performance index is approximately obtained. If ϵ_1 is small enough, then the approximation is the performance index to be expected.

As K in $\bar{A}$ and P are both unknown matrix variables, the inequality (19) is actually BMI, which cannot be solved directly by LMI. In the following, we present a path-following method for solving this BMI (19). In fact, this BMI is linearized by using a perturbation approximation, and then it becomes a LMI. The detailed algorithm is as follows.

Algorithm 1

Step 1 Let $j = 1$. Select an initial feedback gain K_j satisfying that $\bar{A}$ is stable.

Step 2 Solve the following LMI problem

$$\min J_j = \zeta_0^{\mathrm{T}} P_j \zeta_0,$$

$$\text{s.t.} \begin{cases} |(\hat{A} + \hat{B}K_j\hat{C})^{\mathrm{T}}P_j + P_j(\hat{A} + \hat{B}K_j\hat{C}) + \\ \hat{Q} + \hat{C}^{\mathrm{T}}K_j^{\mathrm{T}}\hat{R}K_j\hat{C}| < \epsilon_1 I, \\ P_j > 0. \end{cases} \tag{20}$$

We obtain P_j and J_j. If this LMI optimal problem has solution go to Step 3. Otherwise, go to Step 1.

Step 3 Substituting $P_j = P_j + \delta P, K_j = K_j + \delta K$ into (20), one can assume that δP and δK are small and therefore by neglecting the second order terms we can obtain the following optimization problem:

$$\begin{aligned} &|A^{\mathrm{T}}P_j + P_jA + P_jBK_jC + C^{\mathrm{T}}K_j^{\mathrm{T}}B^{\mathrm{T}}P_j + \\ &P_jB\delta KC + C^{\mathrm{T}}\delta K^{\mathrm{T}}B^{\mathrm{T}}P_j + \delta PA + \\ &A^{\mathrm{T}}\delta P + \delta PBK_jC + C^{\mathrm{T}}K_j^{\mathrm{T}}B^{\mathrm{T}}\delta P + Q + \\ &C^{\mathrm{T}}K_j^{\mathrm{T}}RK_jC + C^{\mathrm{T}}K_j^{\mathrm{T}}R\delta KC + \\ &C^{\mathrm{T}}\delta K^{\mathrm{T}}RK_jC| < \epsilon_2 I. \end{aligned} \tag{21}$$

Note that the constraints of δP and δK are

$$|\delta P| < I, \tag{22}$$

$$|\delta K^{\mathrm{T}}\delta K| < I. \tag{23}$$

Suppose that ϵ_2 is a small positive scalar, then we can obtain $\delta P, \delta K$. If this LMI problem has a solution, go to Step 4. Otherwise, go to Step 1.

Step 4 Let

$$j = j + 1,\ P_j = P_{j-1} + \delta P,\ K_j = K_{j-1} + \delta K,$$

compute $J_j = (1/2)\zeta_0^{\mathrm{T}} P_j \zeta_0$. If

$$J_j < J_{j-1},\ J_{j-1} - J_j > \epsilon_3,\ j < N$$

(ϵ_3 is a given small positive scalar, N is the upper bound for the iteration number), then go back to Step 2. Otherwise, stop. Then the optimal performance index is obtained.

This iterative algorithm ends until a desired performance is achieved, or the performance cannot be improved further. The choice of initial values of K_1 is important for convergence to an acceptable solution [17]. The numerical example is shown that as long as we can find a K_1 such that the closed-loop system is asymptotically stable, we conclude that K_1 can be adjusted iteratively using the free variable δK and the optimal performance index J can be obtained accordingly.

4 Numerical example

Since the performance index is related to the initial value of the closed-loop system, we set $x_{c0} = 0$ in the following example so as to obtain the minimal index for convenient comparisons.

Example [13] Consider a linear system (1) with the the following given parameters

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$D_2 = 1$ and the initial value of the state vectors

$$x(0) = [1\ \ 0]^{\mathrm{T}}.$$

i) Set the exogenous disturbance system (2) with the the following given parameters

$$G = \begin{bmatrix} -0.4 & 0.5 \\ -0.1 & 0 \end{bmatrix}, F = [1\ \ 0].$$

Now $v(t)$ represents disturbance signal with damped specification. Since G satisfies (3), then choose the performance index (4) and the weight matrices are

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ R = 1.$$

- We first design a state feedback controller:

$$u(t) = K_0 x(t), \tag{24}$$

where K_0 is the state feedback controller gain.

Using the lqr($\cdot$) function in MATLAB, we can obtain

$$K = [-0.2361\ \ -2.5723],\ J^* = 1.5331.$$

The trajectories of the state vectors are shown in Fig. 1.

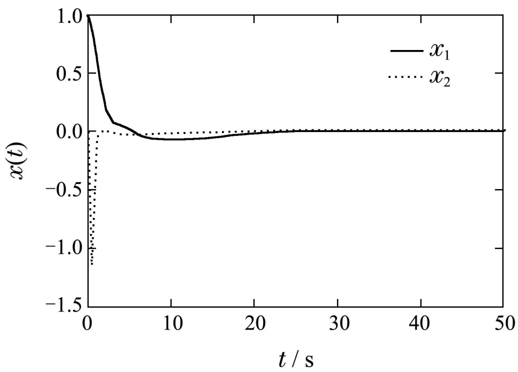

Fig. 1 Curves of state vectors by (24)

- Secondly, we design a state feedback controller:

$$u(t) = K_1 x(t) + K_2 v(t), \tag{25}$$

where K_1 and K_2 are the state feedback controller gain.

Using the lqr($\cdot$) and lyp($\cdot$) function in MATLAB, we can obtain

$$K_1 = [-0.2361\ \ -2.5723],\ K_2 = -0.2014$$

and $J^* = 1.4932$. The trajectories of the state vectors are shown in Fig. 2.

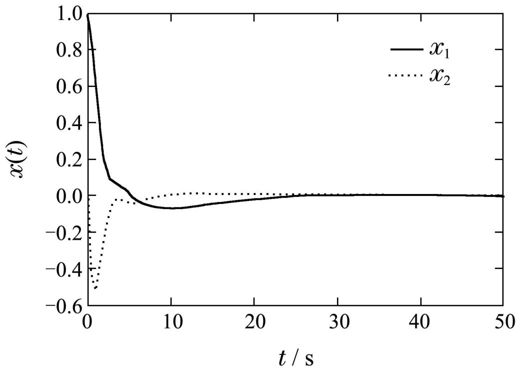

Fig. 2 Curves of state vectors by (25)

• Thirdly, we design a state feedback controller:

$$u(t) = K_{\rm x}x(t) + K_{\rm w}w(t), \tag{26}$$

where $K_{\rm x}$, $K_{\rm w}$ are the state feedback controller gain.

Using the lqr($\cdot$) function in MATLAB, we can obtain

$$K_{\rm x} = [-0.2361\ \ -2.5723],$$
$$K_{\rm w} = [-0.4507\ \ 0.3850],$$

and $J^* = 1.4540$. The trajectories of the state vectors are shown in Fig. 3.

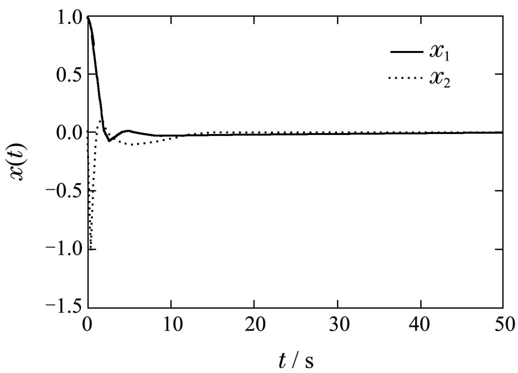

Fig. 3 Curves of state vectors by (26)

• Finally, we design a dynamic compensator (8). The controller with first order, that is $n_{\rm c} = 1$. In light of Algorithm 1 and MATLAB LMI Toolbox, we can obtain the optimal dynamic compensator with the gain:

$$A_{\rm c} = -2.2913,\ B_{\rm c} = -0.9473,$$
$$C_{\rm c} = -8.1658,\ D_{\rm c} = -2.6050,\ J^* = 1.2667$$

Define

$$\xi(t) = \left[x^{\rm T}(t)\ \ x_{\rm c}^{\rm T}(t)\right]^{\rm T}.$$

Now the trajectories of the state vectors are shown in Fig. 4.

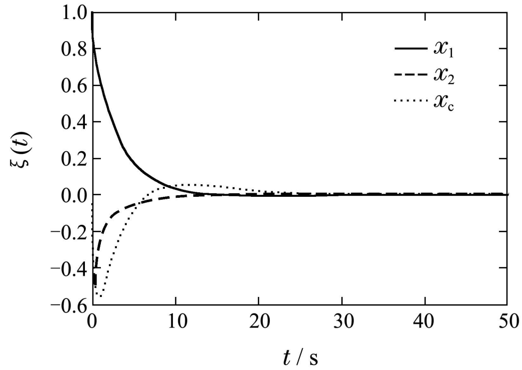

Fig. 4 Curves of state vectors by dynamic compensator $n_{\rm c} = 1$

• The controller with second order, that is $n_{\rm c} = 2$. In light of Algorithm 1 and MATLAB LMI Toolbox, we can obtain the optimal dynamic compensator with the gain:

$$A_{\rm c} = \begin{bmatrix} -3.1448 & 1.5383 \\ -2.5226 & -10.0037 \end{bmatrix},$$
$$B_{\rm c} = \begin{bmatrix} -2.1264 \\ -0.2312 \end{bmatrix},$$
$$C_{\rm c} = [-9.7856\ \ 0.2039],$$
$$D_{\rm c} = -4.9153,\ J^* = 0.9250.$$

The trajectories of the state vectors are shown in Fig. 5.

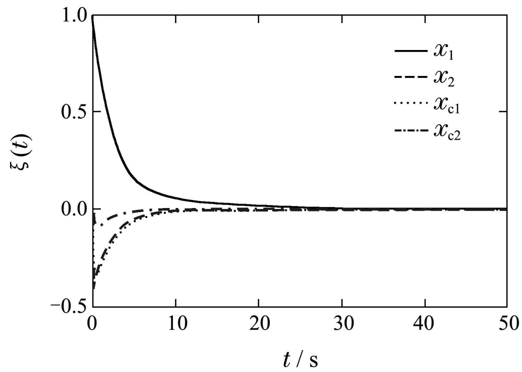

Fig. 5 Curves of state vectors by dynamic compensator $n_{\rm c} = 2$

By observing these figures and results, we can conclude that the closed-loop system based on dynamic compensator can achieve better trajectory performance (the overshoot of the state $x_2(t)$ decreases). Moreover, we can see that the performance of the closed-loop system can become better with the increase of the dynamic order $n_{\rm c}$.

ii) Set the exogenous disturbance system (2) with the the following given parameters

$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ F = [1\ \ 0].$$

Now $v(t)$ represents disturbance signal with sinusoidal specification. Choose the performance index (6) and the weight matrices are the same as above.

Using the controller (24)–(26) and dynamic compensator (8) with $n_{\rm c} = 1$ and $n_{\rm c} = 2$, respectively, the state curves are shown in Fig.6 and Fig.7.

By observing these figures and results, we can find that the amplitude of the state variables decreases based on dynamic compensator. Moreover, the amplitude reduces to less than 0.05 with the increase of the dynamic order n_c.

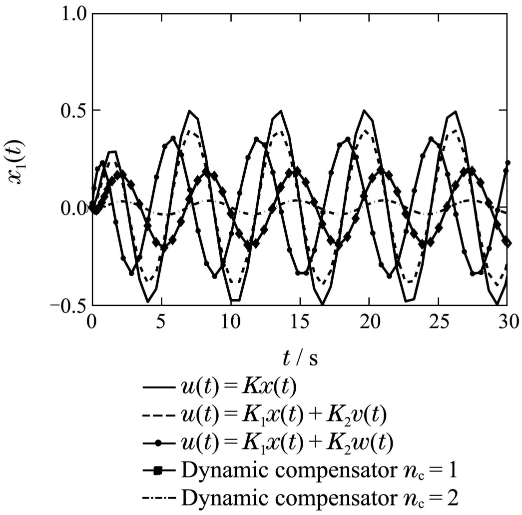

Fig. 6 Comparison curves of $x_1(t)$

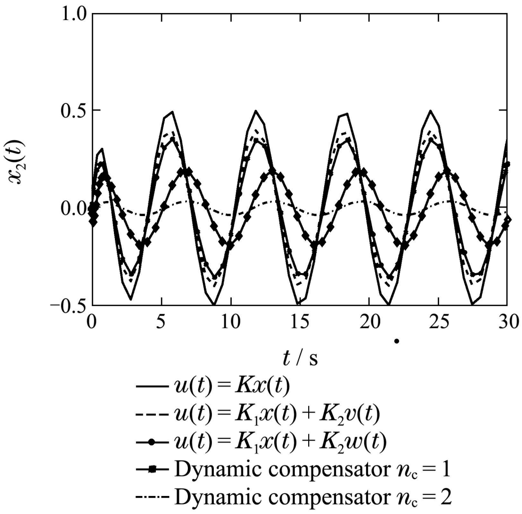

Fig. 7 Comparison curves of $x_2(t)$

The performance indices are shown in Table 1.

Table 1 Performance index comparison.

Controller form	Index
(24)	3.2578
(25)	2.6716
(26)	1.7663
(8) with $n_c = 1$	0.9269
(8) with $n_c = 2$	0.6481

Above all, the numerical examples show that the performance is much better with respect to dynamic compensator than that of the traditional feedback optimal control.

5 Conclusion

In this paper we have considered the LQ optimal control for linear time-invariant system with exogenous disturbance signal with known dynamic characteristics based on dynamic compensation. It is shown that if there exists a dynamic compensator with proper dynamic order such that the closed-loop system is asymptotically stable, then a Lyapunov equation has a symmetric positive-definite solution and the given quadratic performance index can be expressed as a simple form. By transforming the Lyapunov equation into a BMI, an algorithm has further been proposed in terms of path-following algorithm, then the optimal dynamic compensator and the minimum value of the quadratic performance index can be obtained by MATLAB LMI Toolbox. Finally, one numerical example is given to show that the proposed method can make the system achieve better performance with respect to additive two kinds of persistent disturbances than the traditional state feedback, and more, the dynamic compensators with higher order can achieve better performance by comparison of the quadratic performance indices.

References:

[1] ANDERSON B D O, MOORE J B. *Linear Quadratic Methods Optimal Control* [M]. New York: Dover Publications, 1990.

[2] HU Shousong, WANG Zhiquan, HU Weili. *Optimal Control Theory and Systems* [M]. Beijing: Science Press, 1994.
(胡寿松, 王执铨, 胡维礼. 最优控制理论与系统 [M]. 北京: 科学出版社, 1994.)

[3] LEVINE W S, ATHANS M. On the determination of the optimal constant output feedback gains for linear multivariable systems [J]. *IEEE Transactions on Automatic Control*, 1970, 15(1): 44 – 48.

[4] HUANG L, LI Z. Fundamental theorem for optimal output feedback problem with quadratic performance criterion [J]. *International Journal of Control*, 1989, 50(6): 2341 – 2347.

[5] GU G X. On the existence of linear optimal control with output feedback [J]. *SIAM Journal of Control and Optimization*, 1990, 28(3): 711 – 719.

[6] GAO F, LIU W Q, SREERAM V, et al. Characterization and selection of global optimal output feedback gains for linear time-invariant systems [J]. *Optimal Control Applications and Methods: Optimal Control Application Method*, 2000, 21(5): 195 – 209.

[7] JACOB E, ARIE W. A result on output feedback linear quadratic control [J]. *Automatica*, 2008, 44(1): 265 – 271.

[8] LEVINE W S, JOHNSON T L, ATHANS M. Optimal limited controllers state variable feedback for linear systems [J]. *IEEE Transactions on Automatic Control*, 1971, 16(6): 785 – 793.

[9] HU Shousong, GAO Lixin. A design method of optimal regular using dynamic output feedback [J]. *Journal of Nanjing Aeronautical Institute*, 1988, 20(4): 61 – 68.
(胡寿松, 高莉新. 动态输出反馈最优调节器设计方法 [J]. 南京航空学院学报, 1988, 20(4): 61 – 68.)

[10] BLANCHINI F, SZNAIER M. Persistent disturbance rejection via static-state feedback [J]. *IEEE Transactions on Automatic Control*, 1995, 40(6): 1127 – 1131.

[11] LU W W, BALAS G J, LEE E B. Linear quadratic performance with worst case disturbance rejection [J]. *International Journal of Control*, 2000, 73(16): 1516 – 1524.

[12] CADOTTE P, MANNOR S, MICHALSKA H, et al. Design of L1-optimal controllers with flexible disturbance rejection level [J]. *IEEE Transactions on Automatic Control*, 2006, 51(5): 868 – 873.

[13] DI LOTETO M, LOISEAU J J, LAFAY J F. Disturbance attenuation by dynamic output feedback for input-delay systems [J]. *Automatica*, 2008, 44(3): 2202 – 2206.

[14] JIANG Y, LIU S. Global disturbance rejection of a class of nonlinear systems with unknown exosystems [J]. *Journal of Control Theory and Applications*, 2009, 7(4): 379 – 383.

[15] TANG G Y. Feedforward and feedback optimal control for linear systems with sinusoidal disturbances [J]. *High Technology Letter*, 2001, 7(4): 16 – 19.

[16] TANG Gongyou, ZHAO Chong. Optimal disturbance rejection via dynamic output feedback for linear systems [J]. *Information and Control*, 2008, 37(1): 119 – 128.
(唐功友, 赵翀. 线性系统的动态输出反馈最优扰动抑制 [J]. 信息与控制, 2008, 37(1): 119 – 128.)

[17] HASSIBI A, HOW J, BOYD S. A path-following method for solving BMI problems in control [C] //*Proceedings of American Control Conference*. California: IEEE, 1999: 1385 – 1389.

[18] ZHANG G S, LIU L. Linear quadratic optimal control based on dynamic compensation for rectangular descriptor systems [J]. *Acta Automatica Sinica*, 2010, 36(12): 1752 – 1757.

[19] SYROMS V L, ABDALLAH C T, DORATO P, et al. Static output feedback — a survey [J]. *Automatica*, 1997, 33(3): 125 – 137.

作者简介:

刘　蕾　(1984–), 女, 博士, 现于北京大学力学系做博士后研究工作, 目前研究方向为奇异摄动系统、最优控制, E-mail: liulei1223@pku.edu.cn;

张国山　(1961–), 男, 博士, 博士生导师, 目前研究方向为广义系统、鲁棒控制、最优控制, E-mail: zhanggs@tju.edu.cn.