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具有时延和不确定拓扑的二阶多智能体系统的平均一致性

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摘要: 首先, 通过模型变换的方法将原系统分解为多个简单的子系统; 然后, 利用Lyapunov-Krasovskii的方法, 以线性矩阵不等式的形式给出了带有多时变时滞的二阶多智能体系统达到平均一致性的充分条件, 且只要求其通讯拓扑结构是联合连通的. 最后, 用仿真实验验证了本理论的正确性和有效性.

关键词: 平均一致性; 多时变时滞; 不确定性拓扑; 联合连通; 线性矩阵不等式
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Average consensus of second-order multi-agent systems with time-delays and uncertain topologies

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Abstract: First, by employing variable substitution method, the original system is decomposed into several subsystems, and then based on the Lyapunov-Krasovskii approach, some sufficient conditions in terms of linear matrix inequalities (LIMs) are given for average consensus of the second-order multi-agent systems with time-varying delays. The communication topologies between the agents only need to be jointly-connected. Finally, simulation results are provided to prove the effectiveness of the theoretical results.

Key words: average consensus; multiple time-varying delays; uncertain topologies; jointly-connected topologies; LMIs

1 Introduction

The cooperative control of multi-agent systems has attracted a considerable attention in recent years due to its broad applications within physics and control community. In the area of cooperative control of multi-agent systems, an important issue is to design distributed communication and control protocols based on local information to make the group of dynamic agents reach an agreement on certain quantities of interest in presence of limited and unreliable information exchange, dynamically changing interaction topologies as well as communication delays.

Recently, consensus problems have been investigated by many scholars from various perspectives[1–8]. In [3], consensus problems of first-order multi-agents are considered under three cases: 1) directed networks with fixed topology; 2) directed networks with switching topologies; and 3) undirected networks with communication time-delays and fixed topology. In [4], the author considered the problem of information consensus among agents in the presence of limited and unreliable information exchange with dynamically changing interaction topologies. In [5], the distributed reduced-order observer-based consensus protocols are proposed to solve the consensus problems for both continuous- and discrete-time linear multi-agent systems. By Lyapunov-based approach and related space decomposition technique, the work in [7] studied a coordination problem of a multi-agent system with jointly connected interconnection topologies.

In practice application, time-delays often arise naturally due to the finite transmission speed, the congestion of the communication channels and so on. It is well-known that the time-delays often affect the stability of systems. Moreover, due to the moving of the agents, the topologies between individual agents may change over time and the system may also exist parameter uncertainties. In [9], the consensus problem of networked Euler-Lagrange systems with unknown parameters was addressed, but it required the communication topology was fixed and had a spinning tree. In [10], the average consensus of first-order networks systems with time-delay and switching topologies was discussed. In [11], the average consensus problem in networks of dynamic agents with uncertain topologies and time-varying delays was addressed. In [12], the leader-follower consensus problem for a class of high-order multi-agent systems with existing external disturbances, model errors and time-delay was considered, here the communication topology was also supposed to be fixed and connected. In the study of variable topologies, joint connection is an important condition because it does not require the connection of the time-varying interconnection topology at any moment. The average consensus of

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first-order networks systems with time-delay and joint-connected topologies was considered in [13], and the consensus problems of second-order multi-agent system with constant time-delay and jointly topologies were investigated in [14].

In this paper, we consider the second-order networks systems with time-varying delays as well as jointly-connected topologies. By employing variable substitution method and Lyapunov-Krasovskii approach, some sufficient conditions in terms of linear matrix inequalities (LMIs) are given for average consensus of the networks systems. The conditions are composed of some decoupled parts corresponding to each possible connected component of the communication topology. In contrast to [10–11], the topologies in this paper only require joint-connected, so when the number of agents is large, the corresponding calculation is much smaller than [10–11]. In contrast to [13–14], we consider the second-order networks systems in presence of both uncertainties and time-varying delays.

Notation The following notation will be used throughout this paper. $\mathbb{R}^m$ denotes the set of all m dimensional real column vectors; I_m denotes the m dimensional unit matrix; $\otimes$ denotes the Kronecker product; $\mathbf{1}_n$ represents $[1\ \ 1\ \cdots\ 1]^{\mathrm{T}}$ with dimension n; 0 denotes a zero value or a zero matrix with appropriate dimensions; $\|\cdot\|$ refers to the standard Euclidean norm of vectors; the symbol $*$ denotes the symmetric term of a symmetric matrix.

2 Preliminaries and problem formulation

2.1 Graph theory

It is natural to model information exchange among agents by means of directed or undirected graphs. Let $G(\mathcal{V},\mathcal{E},\mathcal{A})$ be an undirected graph of order n, where $\mathcal{V}=\{v_1,\cdots,v_n\}$ is the node set representing agents, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of edges of the graph, and $\mathcal{A}=[a_{ij}]$ is the weighted adjacency matrix. The node indices belong to a finite index set $I=\{1,2,\cdots,n\}$. An edge of G is denoted by $e_{ij}=(v_i,v_j)$ representing that agents v_i and v_j can exchange information with each other. Since the graph is undirected, it means that the edges $e_{ij}\in\mathcal{E}$ and $e_{ji}\in\mathcal{E}$ are considered to be the same. The weighted adjacency matrix is defined as $a_{ii}=0$ and $a_{ij}\geqslant 0$, where $a_{ij}=a_{ji}>0$ if and only if $e_{ij}\in\mathcal{E}$. Obviously, $\mathcal{A}$ is a symmetric nonnegative matrix. The set of neighbors of node v_i is denoted by $\mathcal{N}_i=\{v_j\in\mathcal{V}:(v_i,v_j)\in\mathcal{E}\}$. The in-degree and out-degree of node v_i are defined as

$$d_{\text{in}}(v_i)=\sum_{j=1}^{n}a_{ji}=d_{\text{out}}(v_i)=\sum_{j=1}^{n}a_{ij}.$$

The Laplacian matrix corresponding to the undirected graph G is defined as $L=[l_{ij}]$, where $l_{ii}=d_{\text{out}}(v_i)$ and $l_{ij}=-a_{ij}$, $i\neq j$. It's easy to see that the Laplacian matrix L is also symmetric. A path is a sequence of connected edges of the form $(v_i,v_{i_1}),(v_{i_1},v_{i_2}),\cdots$, where $i_j\in\ell$ and $v_{i_j}\in\mathcal{V}$. A graph is connected if there is a path between every pair of vertices. The union of a collection of graphs $\bar{G}_1,\cdots,\bar{G}_m$, with the same node set $\mathcal{V}$, is defined as the graph $\bar{G}_{1-m}$ with the node set $\mathcal{V}$ and edge set equaling to the union of the edge sets of all of the graphs in the collection. Moreover, these graphs $\bar{G}_1,\cdots,\bar{G}_m$ are jointly-connected if their union graph $\bar{G}_{1-m}$ is connected.

Lemma 1[15] If the undirected graph G is connected, then its Laplacian L has the following properties:

1) zero is a simple eigenvalue of L with an associated eigenvector $\mathbf{1}_n$ satisfies $L\mathbf{1}_n=0$;

2) all its other $n-1$ eigenvalues are positive and real.

2.2 Problem statement

Suppose the ith agent has the dynamics as follows:

$$\begin{cases}\dot{x}_i(t)=v_i(t),\\ \dot{v}_i(t)=u_i(t),\end{cases}\tag{1}$$

where $x_i(t)\in\mathbb{R}$ is the position state, $v_i(t)\in\mathbb{R}$ is the velocity state, and $u_i(t)\in\mathbb{R}$ is the protocol.

We use the following linear consensus protocol:

$$\begin{aligned}u_i=&-k_1v_i(t)+\sum_{j\in N_i(t)}[a_{ij}(t)+\Delta a_{ij}(t)]\cdot\\&[x_j(t-\tau_{ij})-x_i(t-\tau_{ij})].\end{aligned}\tag{2}$$

To simplify the analysis, we make a model transformation. Let

$$\begin{aligned}&\bar{v}_i(t)=2v_i(t)/k_1+x_i(t),\\&\xi(t)=[x_1(t)\ \ \bar{v}_1(t)\ \cdots\ x_n(t)\ \ \bar{v}_n(t)],\\&A=\begin{bmatrix}-k_1/2 & k_1/2\\ k_1/2 & -k_1/2\end{bmatrix},\ B=\begin{bmatrix}0 & 0\\ 2/k_1 & 0\end{bmatrix}.\end{aligned}\tag{3}$$

Suppose that there are r different time-delays, denoted by $\tau_k(t)\in\{\tau_{ij},i,j\in I\}(k=1,2,\cdots,r)$. Under the protocol (2), the closed-loop network dynamics is

$$\begin{aligned}\dot{\xi}(t)=&(I_n\otimes A)\xi(t)-\sum_{k=1}^{r}[(L_{\sigma k}+\\&\Delta L_{\sigma k})\otimes B]\xi(t-\tau_k),\end{aligned}\tag{4}$$

where $L_{\sigma k}=[l_{kij}]$ is the matrix defined by

$$l_{kij}=\begin{cases}-a_{ij}, & j\neq i,\tau_k(\cdot)=\tau_{ij}(\cdot),\\ 0, & j\neq i,\tau_k(\cdot)\neq\tau_{ij}(\cdot),\\ \sum\limits_{j=1}^{n}l_{kij}, & j=i.\end{cases}$$

and $\Delta L_{\sigma k}=[\Delta l_{kij}]$ is the uncertain matrix defined by

$$\Delta l_{kij}=\begin{cases}-\Delta a_{ij}, & j\neq i,\tau_k(\cdot)=\tau_{ij}(\cdot),\\ 0, & j\neq i,\tau_k(\cdot)\neq\tau_{ij}(\cdot),\\ \sum\limits_{j=1}^{n}\Delta l_{kij}, & j=i.\end{cases}$$

Since the graph G_σ is undirected, it is easy to see $L_{\sigma k}$, $\Delta L_{\sigma k}$ are symmetric matrices, and $L_{\sigma k}\mathbf{1}=\mathbf{0}$, $\Delta L_{\sigma k}\mathbf{1}=\mathbf{0}$. Thus, $\mathbf{1}^{\mathrm{T}}\{(I_n\otimes A)\xi(t)-\sum\limits_{k=1}^{r}[(L_{\sigma k}+\Delta L_{\sigma k})\otimes B]\xi(t-\tau_k)\}=0$, i.e., $\sum\limits_{i=1}^{n}(\dot{x}_i(t)+\dot{\bar{v}}_i(t))=0$, which means $\alpha=(1/2n)\sum\limits_{i=1}^{n}(x_i(t)+\bar{v}_i(t))$ is an invariant quantity. Let $\delta(t)=\xi(t)-\alpha\mathbf{1}$. Then system (4) can be transformed into

$$\begin{aligned}\dot{\delta}(t)=&(I_n\otimes A)\delta(t)-\sum_{k=1}^{r}[(L_{\sigma k}+\\&\Delta L_{\sigma k})\otimes B]\delta(t-\tau_k).\end{aligned}\tag{5}$$

Obviously, $\sum\limits_{i=1}^{2n} = \delta_i(t) = 0$. Also, if $\lim\limits_{t\to\infty}\delta(t) = 0$, then $\lim\limits_{t\to\infty}\xi(t) = \alpha\mathbf{1}$, i.e., $\lim\limits_{t\to\infty}x_i(t) = \lim\limits_{t\to\infty}\bar{v}_i(t) = \alpha$, which implies $\lim\limits_{t\to\infty}x_i(t) = (1/n)\sum\limits_{i=1}^{n}(x_i(0) + v_i(0)/k_1) = (1/n)\sum\limits_{i=1}^{n}x_i(0)$ and $\lim\limits_{t\to\infty}v_i(t) = 0$ for any $i \in I$. That means average consensus is achieved.

In the following, we assume that time-varying delays in Eq.(5) satisfy

A1) $0 \leqslant \tau_k(t) \leqslant h_k$, $\dot{\tau}_k(t) \leqslant d_k$ for $t \geqslant 0$ and $k = 1, 2, \cdots, r$, where $h_k > 0$ and $d_k \geqslant 0$, or

A2) $0 \leqslant \tau_k(t) \leqslant h_k$, for $t \geqslant 0$ and $k = 1, 2, \cdots, r$, where $h_k > 0$. That is, nothing has been known about the derivative of τ_k.

Consider an infinite sequence of nonempty, bounded and contiguous time-intervals $[t_s, t_{s+1}], s = 0, 1, \cdots$, with $t_0 = 0$ and $t_{s+1} - t_s \leqslant T_1$ for some constant $T_1 > 0$. In each interval $[t_s, t_{s+1})$, there is a sequence of subintervals $[t_{s_0}, t_{s_1}), [t_{s_1}, t_{s_2}), \cdots, [t_{s_{m_r-1}}, t_{s_{m_r}})$ with $t_{s_0} = t_s$ and $t_{s_{m_r}} = t_{s+1}$ satisfying $t_{s_{j+1}} - t_{s_j} \geqslant T_2$, $0 \leqslant j \leqslant m_r - 1$ for some integer $m_r \geqslant 0$ and given constant $T_2 > 0$ such that the communication topology G_σ switches at t_{s_j} and it does not change during each subinterval $[t_{s_j}, t_{s_{j+1}})$. Evidently, there are at most $s_* = \lfloor T_1/T_2 \rfloor$ subintervals in each interval $[t_s, t_{s+1})$, where $\lfloor T_1/T_2 \rfloor$ denotes the maximum integer no larger than T_1/T_2.

Suppose that the (time-invariant) communication graph $G(t)$ on subinterval $[t_{s_j}, t_{s_{j+1}})$ has $l_\sigma \geqslant 1$ connected components with the corresponding sets of nodes denoted by $\varphi^1_{s_j}, \varphi^2_{s_j}, \cdots, \varphi^{l_\sigma}_{s_j}$. Then there exists a permutation matrix E_σ such that

$$E^{\mathrm{T}}_\sigma L_\sigma E_\sigma = \mathrm{diag}\{L^1_\sigma, L^2_\sigma, \cdots, L^{l_\sigma}_\sigma\},$$
$$E^{\mathrm{T}}_\sigma \Delta L_\sigma E_\sigma = \mathrm{diag}\{\Delta L^1_\sigma, \Delta L^2_\sigma, \cdots, \Delta L^{l_\sigma}_\sigma\},$$

and

$$\delta^{\mathrm{T}}(t)E_\sigma = [\delta^{1\mathrm{T}}_\sigma(t)\ \ \delta^{2\mathrm{T}}_\sigma(t)\ \cdots\ \delta^{l_\sigma\mathrm{T}}_\sigma(t)], \tag{6}$$

where each block matrix $L^i_\sigma \in \mathbb{R}^{d^i_\sigma \times d^i_\sigma}$ is a Laplacian matix of the corresponding connected component with d^i_σ denoting the number of nodes in $\varphi^i_{s_j}$. Then in each subinterval $[t_{s_j}, t_{s_{j+1}})$ system (5) can be decomposed into the following l_σ subsystems:

$$\dot{\delta}^i_\sigma(t) = (I_{d^i_\sigma} \otimes A)\delta^i_\sigma(t) - \sum_{k=1}^{r}(\widetilde{L}^i_{\sigma k} \otimes B)\cdot \delta^i_\sigma(t - \tau_k), \tag{7}$$

where $\delta^i_\sigma(t) = [\delta^{i\mathrm{T}}_{\sigma 1}(t)\ \cdots\ \delta^{i\mathrm{T}}_{\sigma d^i_\sigma}(t)]^{\mathrm{T}} \in \mathbb{R}^{d^i_\sigma}$ and $\widetilde{L}^i_{\sigma k} = L^i_{\sigma k} + \Delta L^i_{\sigma k}$ is the coefficient matrix associated with the variable $\delta^i_\sigma(t - t_k)$. It is easy to see that $L^i_{\sigma k}\mathbf{1} = 0$, $\Delta L^i_{\sigma k}\mathbf{1} = 0$, $(L^i_{\sigma k})^{\mathrm{T}} = L^i_{\sigma k}$, $(\Delta L^i_{\sigma k})^{\mathrm{T}} = \Delta L^i_{\sigma k}$, $L^i_\sigma = \sum\limits_{k=1}^{r} L^i_{\sigma k}$ and $\Delta L^i_\sigma = \sum\limits_{k=1}^{r} \Delta L^i_{\sigma k}$.

The following lemmas play an important role in the proof of the main results.

Lemma 2 Let $\Psi^i_\sigma = (I_{d^i_\sigma} \otimes A) - L^i_\sigma \otimes B$ with A and B as defined in Eq.(3) and satisfy $k^2_1 \geqslant 4d_{\max}$, where $d_{\max}$ denotes the largest diagonal entry of all possible L_σ, then, we can get that $-\Psi^i_\sigma$ is a Laplacian matrix with the same structure as $L^i_\sigma = [l_{ij}]$, i.e., if the graph of the Laplacian matrix L^i_σ is connected, then, $\mathrm{rank}(-\Psi^i_\sigma) = 2d^i_\sigma - 1$.

Proof Through directly calculate, we can get that

$$-\Psi^i_\sigma = \begin{bmatrix} \dfrac{k_1}{2} & -\dfrac{k_1}{2} \cdots & 0 & 0 \\ -\dfrac{k_1}{2} + \dfrac{2}{k_1}l_{11} & \dfrac{k_1}{2} & \cdots & \dfrac{2}{k_1}l_{1d^i_\sigma} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{k_1}{2} & -\dfrac{k_1}{2} \\ \dfrac{2}{k_1}l_{d^i_\sigma 1} & 0 & \cdots & -\dfrac{k_1}{2} + \dfrac{2}{k_1}l_{d^i_\sigma d^i_\sigma} & \dfrac{k_1}{2} \end{bmatrix}.$$

Obviously, $\Psi^i_\sigma\mathbf{1} = \mathbf{1}^{\mathrm{T}}\Psi^i_\sigma = 0$, and $-\dfrac{k_1}{2} + \dfrac{2}{k_1}l_{ii} \leqslant 0, i = 1, \cdots, d^i_\sigma$ since $k^2_1 \geqslant 4d_{\max}$. This means the off-diagonal entries of matrix $-\Psi^i_\sigma$ are nonpositive. Then, we can say that $-\Psi^i_\sigma$ is a Laplacian matrix. It is not hard to see that the graph of the Laplacian $-\Psi^i_\sigma$ is the extension of graph of the Laplacian L^i_σ by the following method: add a new node to each node of the graph of the Laplacian L^i_σ and connect them. Then, the new graph has the same structure as the original graph.

Lemma 3[10] Assume that G with the Laplacian matrix L is a strongly connected and balanced digraph, then we have

$$E^{\mathrm{T}}(L + L^{\mathrm{T}})E > 0,$$

where

$$E = \begin{bmatrix} I_{n-1} \\ E_0 \end{bmatrix},\ E_0 = (-1, -1, \cdots, -1).$$

Lemma 4 (Schur complement)[16] Let $S = S^{\mathrm{T}} = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$ be a given symmetric matrix, then the following three conditions are equivalent:

1) $S < 0$;
2) $S_{11} < 0$, $S_{22} - S^{\mathrm{T}}_{12}S^{-1}_{11}S_{12} < 0$;
3) $S_{22} < 0$, $S_{11} - S_{12}S^{-1}_{22}S^{\mathrm{T}}_{12} < 0$.

Lemma 5[11] For any real differentiable vector function $x(t) \in \mathbb{R}$ and any constant matrix $0 < W = W^t \in \mathbb{R}^{n\times n}$ and $0 \leqslant \tau_k(t) \leqslant h_k$, we have the following inequality:

$$h^{-1}_k[x(t) - x(t - \tau_k(t))]^{\mathrm{T}}W[x(t) - x(t - \tau_k(t))] \leqslant \int_{t-\tau_k(t)}^{t} \dot{x}^{\mathrm{T}}(s)W\dot{x}(s)\mathrm{d}s,\ t \geqslant 0. \tag{8}$$

Lemma 6[17] Let $Q = Q^{\mathrm{T}}$, R, H, $F(t)$ be a matrices with appropriate dimensions, and matrix $F(t)$ satisfied $F^{\mathrm{T}}(t)F(t) \leqslant I$, then

$$Q + HF(t)R + H^{\mathrm{T}}F^{\mathrm{T}}(t)R^{\mathrm{T}} < 0,$$

satisfied if and only if there exists a positive constant $\varepsilon > 0$ satisfied

$$Q + \varepsilon^{-1}HH^{\mathrm{T}} + \varepsilon R^{\mathrm{T}}R < 0.$$

3 Main result

Theorem 1 Consider a network of agents with time-delay satisfies A1), and the communication topologies graph is jointly-connected in each interval $[t_s, t_{s+1})$ and satisfies $(\Delta L^i_\sigma \otimes B)^{\mathrm{T}}(\Delta L^i_\sigma \otimes B) \leqslant a^2 I$ and $(\Delta L^i_{\sigma k} \otimes$

$B)^{\mathrm{T}}(\Delta L_{\sigma k}^i \otimes B) \leqslant a^2 I,\ i=1,\cdots,l_\sigma,\ k=1,\cdots,r$. If there exist some constants $\alpha,\ \beta,\ \gamma,\ \varepsilon$ such that the following LMIs is satisfied for each subinterval $[t_{s_j}, t_{s_{j+1}})$:

$$\Phi_\sigma^i = \begin{bmatrix} \Phi_{11}+\Delta_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ * & \Phi_{22}+\Delta_{22} & \Phi_{23} & 0 \\ * & * & \Phi_{33} & \Phi_{34} \\ * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{aligned} &\Phi_{11} = E^{\mathrm{T}}(\alpha(\Psi_\sigma^i+\Psi_\sigma^{i\mathrm{T}})+\gamma\sum_{k=1}^r d_k I)E,\\ &\Delta_{11} = \varepsilon a^2 E^{\mathrm{T}}E,\\ &\Psi_\sigma^i = (I_{d_\sigma^i}\otimes A) - L_\sigma^i \otimes B,\\ &\Phi_{12} = [E^{\mathrm{T}}(\alpha(L_{\sigma 1}^i\otimes B)+\beta(1-d_1)I)E\ \cdots\\ &\qquad E^{\mathrm{T}}(\alpha(L_{\sigma r}^i\otimes B)+\beta(1-d_r)I)E],\\ &\Phi_{13} = E^{\mathrm{T}}\Psi_\sigma^{i\mathrm{T}},\\ &\Phi_{14} = [E^{\mathrm{T}}\ \cdots\ E^{\mathrm{T}}\ \ 0],\\ &\Phi_{22} = \mathrm{diag}\{-E^{\mathrm{T}}(\frac{\gamma}{h_1}I+\beta(1-d_1)I)E,\cdots,\\ &\qquad -E^{\mathrm{T}}(\frac{\gamma}{h_r}I+\beta(1-d_r)I)E\},\\ &\Delta_{22} = \mathrm{diag}\{\varepsilon a^2 E^{\mathrm{T}}E,\cdots,\varepsilon a^2 E^{\mathrm{T}}E\},\\ &\Phi_{23} = \left[(L_{\sigma 1}^i\otimes B)E\ \cdots\ (L_{\sigma r}^i\otimes B)E\right]^{\mathrm{T}},\\ &\Phi_{33} = -(\gamma\sum_{k=1}^r h_k)^{-1},\\ &\Phi_{34} = [I\ \cdots\ I\ \ 0]. \end{aligned}$$

Then the protocol (2) solves the average consensus problem.

Proof We first prove that Eq.(9) is always feasible for any $0 \leqslant d_k < 1$ under the assumption of Theorem 1. Rewriting Eq.(9) by using Lemma 4, we obtain that

$$\begin{bmatrix} E^{\mathrm{T}}\alpha\bar{\Psi}_\sigma^i E & \alpha E^{\mathrm{T}}(L_{\sigma 1}^i\otimes B)E & \cdots & \alpha E^{\mathrm{T}}(L_{\sigma r}^i\otimes B)E \\ * & (\varepsilon a^2-\frac{\gamma}{h_1})E^{\mathrm{T}}E & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & (\varepsilon a^2-\frac{\gamma}{h_r})E^{\mathrm{T}}E \end{bmatrix} +$$

$$\beta\begin{bmatrix} \sum_{k=1}^r d_k E^{\mathrm{T}}E & (1-d_1)E^{\mathrm{T}}E & \cdots & (1-d_r)E^{\mathrm{T}}E & 0 \\ * & (d_1-1)E^{\mathrm{T}}E & 0 & 0 & 0 \\ * & * & \ddots & 0 & \vdots \\ * & * & * & (d_r-1)E^{\mathrm{T}}E & 0 \\ * & * & * & * & 0 \end{bmatrix}\cdot$$

$$\gamma(\sum_{k=1}^r h_k)R^{\mathrm{T}}R+\varepsilon^{-1}HH^{\mathrm{T}} < 0, \tag{10}$$

where $R = [\Psi_\sigma^i E\ \ (L_{\sigma 1}^i\otimes B)E\ \cdots\ (L_{\sigma r}^i\otimes B)E]^{\mathrm{T}}$, $\bar{\Psi}_\sigma^i = \alpha(\Psi_\sigma^i+\Psi_\sigma^{i\mathrm{T}})+\varepsilon a^2 I$ and

$$H = \begin{bmatrix} E^{\mathrm{T}} & E^{\mathrm{T}} & \cdots & E^{\mathrm{T}} & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ I & I & \cdots & I & 0 \end{bmatrix}.$$

Choosing appropriate $a,\ \varepsilon,\ \alpha,\ \beta,\ \gamma$ and h_m, it is easy to see that the inequality inequality (10) holds if $E^{\mathrm{T}}(\Psi_\sigma^{i\mathrm{T}}+\Psi_\sigma^i)E < 0$. It is apparent satisfied by Lemma 2 and Lemma 3. Hence, inequality (10) is always feasible for any $0 \leqslant d_k < 1$ and appropriate $a,\ \varepsilon,\ \alpha,\ \beta,\ \gamma$ and h_m.

Next, we prove that system (4) achieves average consensus under assumption A1). Define a common Lyapunov function for system (5) as follows:

$$\begin{aligned} V(t) = {}& \alpha\delta^{\mathrm{T}}(t)\delta(t)+\beta\sum_{k=1}^r\int_{t-\tau_k(t)}^t \delta^{\mathrm{T}}(s)\delta(s)\mathrm{d}s + \\ &\gamma\sum_{k=1}^r\int_{-\tau_k}^0\int_{t+\theta}^t \dot{\delta}^{\mathrm{T}}(s)\dot{\delta}(s)\mathrm{d}s\mathrm{d}\theta. \end{aligned}$$

From Eq.(6), $V(t)$ can be rewritten as

$$\begin{aligned} &V(t) = \\ &\sum_{i=1}^{l_\sigma}\{\alpha\delta_\sigma^{i\mathrm{T}}(t)\delta_\sigma^i(t)+\beta\sum_{k=1}^r\int_{t-\tau_k(t)}^t \delta_\sigma^{i\mathrm{T}}(s)\delta_\sigma^i(s)\mathrm{d}s + \\ &\gamma\sum_{k=1}^r\int_{-\tau_k}^0\int_{t+\theta}^t \dot{\delta}_\sigma^{i\mathrm{T}}(s)\dot{\delta}_\sigma^i(s)\mathrm{d}s\mathrm{d}\theta\}. \end{aligned}$$

Let $\eta_{\sigma k}^i(t) = \delta_\sigma^i(t)-\delta_\sigma^i(t-\tau_k)$. Rewrite system (7) as the following equivalent form:

$$\begin{aligned} \dot{\delta}_\sigma^i(t) = {}& \left[(I_{d_\sigma^i}\otimes A)-(L_\sigma^i+\Delta L_\sigma^i)\otimes B\right]\delta_\sigma^i(t) + \\ &\sum_{k=1}^r\left[(L_{\sigma k}^i+\Delta L_{\sigma k}^i)\otimes B\right]\eta_{\sigma k}^i(t) = \\ &\widetilde{\Psi}_\sigma^i\delta_\sigma^i(t)+\sum_{k=1}^r(\widetilde{L}_{\sigma k}^i\otimes B)\eta_{\sigma k}^i(t), \end{aligned} \tag{11}$$

where $\widetilde{\Psi}_\sigma^i = (I_{d_\sigma^i}\otimes A)-(L_\sigma^i+\Delta L_\sigma^i)\otimes B$, $\widetilde{L}_{\sigma k}^i = L_{\sigma k}^i+\Delta L_{\sigma k}^i$. Calculating $\dot{V}(t)$ along the trajectories of Eq.(11), we get

$$\begin{aligned} \dot{V}(t) = {}& \sum_{i=1}^{l_\sigma}\{\alpha\delta_\sigma^{i\mathrm{T}}(t)(\widetilde{\Psi}_\sigma^i+\widetilde{\Psi}_\sigma^{i\mathrm{T}})\delta_\sigma^i(t) + \\ &\beta\sum_{k=1}^r\delta_\sigma^{i\mathrm{T}}(t)\delta_\sigma^i(t)+\gamma\sum_{k=1}^r\tau_k\dot{\delta}_\sigma^{i\mathrm{T}}(s)\dot{\delta}_\sigma^i(s) + \\ &2\alpha\delta_\sigma^{i\mathrm{T}}(t)\sum_{k=1}^r(\widetilde{L}_{\sigma k}^i\otimes B)\eta_{\sigma k}^i(t) + \\ &\beta\sum_{k=1}^r(d_k-1)\delta_\sigma^{i\mathrm{T}}(t-\tau_k)\delta_\sigma^i(t-\tau_k) - \\ &\gamma\sum_{k=1}^r\int_{t-\tau_k}^t\dot{\delta}_\sigma^{i\mathrm{T}}(s)\dot{\delta}_\sigma^i(s)\mathrm{d}s\} \leqslant \\ &\sum_{i=1}^{l_\sigma}\{\delta_\sigma^{i\mathrm{T}}(t)[\alpha(\widetilde{\Psi}_\sigma^i+\widetilde{\Psi}_\sigma^{i\mathrm{T}})+\beta\sum_{k=1}^r d_k I]\delta_\sigma^i(t) + \\ &2\delta_\sigma^{i\mathrm{T}}(t)\sum_{k=1}^r[\alpha(\widetilde{L}_{\sigma k}^i\otimes B)-\beta(d_k-1)I]\eta_{\sigma k}^i(t) + \\ &\gamma\sum_{k=1}^r h_k[\widetilde{\Psi}_\sigma^i\delta_\sigma^i(t)+\sum_{k=1}^r(\widetilde{L}_{\sigma k}^i\otimes B)\eta_{\sigma k}^i(t)]^{\mathrm{T}}\cdot \\ &[\widetilde{\Psi}_\sigma^i\delta_\sigma^i(t)+\sum_{k=1}^r(\widetilde{L}_{\sigma k}^i\otimes B)\eta_{\sigma k}^i(t)] - \\ &\sum_{k=1}^r(\gamma/h_k+\beta(1-d_k))\eta_{\sigma k}^{i\mathrm{T}}(t)\eta_{\sigma k}^i(t)\} = \\ &\sum_{i=1}^{l_\sigma}y_i^{\mathrm{T}}(t)\widetilde{\Phi}_\sigma^i y_i(t), \end{aligned}$$

where $y_i^{\mathrm{T}}(t) = [\delta_\sigma^{i\mathrm{T}}(t)\ \ \eta_{\sigma 1}^i(t)\ \cdots\ \eta_{\sigma r}^i(t)]$, and

$$\widetilde{\varPhi}_\sigma^i = \begin{bmatrix} \widetilde{\varPhi}_{11} & \widetilde{\varPhi}_{12} \\ \widetilde{\varPhi}_{12}^{\mathrm{T}} & \widetilde{\varPhi}_{22} \end{bmatrix} + \gamma(\sum_{k=1}^{r} h_k) R^{\mathrm{T}} R, \tag{12}$$

$$\widetilde{\varPhi}_{11} = \alpha(\widetilde{\varPsi}_\sigma^i + \widetilde{\varPsi}_\sigma^{i\mathrm{T}}) + \beta \sum_{k=1}^{r} d_k I,$$

$$\widetilde{\varPhi}_{12} = [\alpha(\widetilde{L}_{\sigma 1}^i \otimes B) + \beta(1-d_1)I \ \cdots \ \alpha(\widetilde{L}_{\sigma r}^i \otimes B) + \beta(1-d_r)I],$$

$$\widetilde{\varPhi}_{22} = \mathrm{diag}\{-\frac{\gamma}{h_1}I - \beta(1-d_1)I, \cdots, -\frac{\gamma}{h_r}I - \beta(1-d_r)I\},$$

$$\widetilde{R} = [\varPsi_\sigma^i \ \widetilde{L}_{\sigma 1}^i \otimes B \ \cdots \ \widetilde{L}_{\sigma r}^i \otimes B].$$

Noting that $\mathbf{1}^{\mathrm{T}}\delta_\sigma^i \equiv 0$, we can rewrite $\delta_\sigma^i = E\widetilde{\delta}_\sigma^i$, and E is defined as in Lemma 3 with $2d_\sigma^i$ dimensional. Thus, by Eq.(12), we have

$$\dot{V} \leqslant \sum_{i=1}^{l_\sigma} \widetilde{y}_i^{\mathrm{T}}(t) W^{\mathrm{T}} \widetilde{\varPhi}_\sigma^i W \widetilde{y}_i(t), \tag{13}$$

where $y_i^{\mathrm{T}}(t) = [\widetilde{\delta}_\sigma^{i\mathrm{T}}(t) \ \widetilde{\eta}_{\sigma 1}^i \ \cdots \ \widetilde{\eta}_{\sigma r}^i]$, $\widetilde{\eta}_{\sigma k}^i(t) = \widetilde{\delta}_\sigma^i(t) - \widetilde{\delta}_\sigma^i(t-\tau_k)$ for $k = 1, \cdots, r$, and $W = \mathrm{diag}\{E, \cdots, E\} \in \mathbb{R}^{2d_\sigma^i(1+r)}$.

By straightforward computation and Lemma 4, we get that $W^{\mathrm{T}}\widetilde{\varPhi}_\sigma^i W < 0$ if and only if

$$\bar{\varPhi}_\sigma^i + aHF(t)G + aG^{\mathrm{T}}F^{\mathrm{T}}(t)H^{\mathrm{T}} < 0, \tag{14}$$

where $F(t) = a^{-1}\mathrm{diag}\{\Delta L_\sigma^i \otimes B^{\mathrm{T}}, \Delta L_{\sigma 1}^i \otimes B^{\mathrm{T}}, \cdots, \Delta L_{\sigma r}^i \otimes B^{\mathrm{T}}, \Delta L_\sigma^i \otimes B^{\mathrm{T}}\}$ satisfying $F^{\mathrm{T}}(t)F(t) \leqslant I$ for $t \geqslant 0$, H is defined in inequality (10) and

$$\bar{\varPhi}_\sigma^i = \begin{bmatrix} \varPhi_{11} & \varPhi_{12} & \varPhi_{13} \\ * & \varPhi_{22} & \varPhi_{23} \\ * & * & \varPhi_{33} \end{bmatrix},$$

$$G = \mathrm{diag}\{-E, E, \cdots, E, 0\}.$$

Then, by Lemma 6, we get that Eq.(14) satisfied if and only if there exits a positive $\varepsilon > 0$ satisfied

$$\bar{\varPhi}_\sigma^i + \varepsilon^{-1}HH^{\mathrm{T}} + \varepsilon a^2 G^{\mathrm{T}} G < 0. \tag{15}$$

By Lemma 4 again, we get that Eq.(9) is equivalent to Eq.(15), therefore, $\dot{V}(t) < 0$,

Therefore, there exist some positive constants $\mu_i, i = 1, \cdots, l_\sigma^i, \mu = \min\{\mu_i\}$ such that

$$\dot{V} \leqslant -\sum_{i=1}^{l_\sigma} \mu_i \|\widetilde{\delta}_\sigma^i(t)\| \leqslant -\mu\|\widetilde{\delta}_\sigma(t)\| \leqslant -(\mu/n)\|\delta_\sigma(t)\|.$$

This implies that the zero solution of system (7) is asymptotically stable by Theorem 2.1 in [18] Chapter 5, i.e.,

$$\lim_{t\to\infty} \delta_\sigma^i(t) = \lim_{t\to\infty} \|\xi_\sigma^i(t) - (1/d_\sigma^i)\mathbf{1}_{d_\sigma^i}^{\mathrm{T}} \sum_{i=1}^{d_\sigma^i} \xi_{\sigma k}^i\| = 0,$$

here $\xi_\sigma^i(t)$ corresponding to $\delta_\sigma^i(t)$. It is easy to get that $(1/d_\sigma^i)\mathbf{1}_{d_\sigma^i}^{\mathrm{T}} \sum_{i=1}^{d_\sigma^i} \xi_{\sigma k}^i$ is an invariant quantity in subinterval $[t_{s_j}, t_{s_{j+1}})$. Due to the solution of the system (7) is absolutely continuous, the states $\xi_\sigma^i(t)$ are continuous. Thus, the states of the agents tend to be equal to the average of theirs initial state as $t_{s_j} \to \infty$ if the topologies are jointly-connected in every two contiguous subintervals $[t_{s_j}, t_{s_{j+1}})$ and $[t_{s_{j+1}}, t_{s_{j+2}})$. Then by induction, the state of all agents tend to be equal to the average of their initial states. as $t \to \infty$, since the topologies are jointly-connected in each interval $[t_s, t_{s+1})$. This completes the proof.

When $d_k \geqslant 1$ or nothing has been known about the $\dot{\tau}_k$, we can obtain the following result.

Theorem 2 Consider a network of agents with time-delay satisfies A2), and the communication topologies graph is jointly-connected in each interval $[t_s, t_{s+1})$ and satisfies $\Delta L_\sigma^{i\mathrm{T}} \Delta L_\sigma^i \leqslant a^2 I$ and $\Delta L_{\sigma k}^{i\mathrm{T}} \Delta L_{\sigma k}^i \leqslant a^2 I$, $i = 1, \cdots, l_\sigma$, $k = 1, \cdots, r$. If there exist some constants α, β, γ, ε such that the following LMIs is satisfied for each subinterval $[t_{s_j}, t_{s_{j+1}})$:

$$\varOmega_\sigma^i = \begin{bmatrix} \varOmega_{11}+\varDelta_{11} & \varOmega_{12} & \varOmega_{13} & \varOmega_{14} \\ * & \varOmega_{22}+\varDelta_{22} & \varOmega_{23} & 0 \\ * & * & \varOmega_{33} & \varOmega_{34} \\ * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{16}$$

where

$$\varOmega_{11} = \alpha E^{\mathrm{T}}(\varPsi_\sigma^i + \varPsi_\sigma^{i\mathrm{T}})E,$$

$$\varOmega_{12} = [\alpha E^{\mathrm{T}}(L_{\sigma 1}^i \otimes B)E \ \cdots \ \alpha E^{\mathrm{T}}(L_{\sigma r}^i \otimes B)E],$$

$$\varOmega_{13} = \varPhi_{13}, \ \varOmega_{14} = \varPhi_{14},$$

$$\varOmega_{22} = \mathrm{diag}\{-\frac{\gamma}{h_1}E^{\mathrm{T}}E, \cdots, -\frac{\gamma}{h_r}E^{\mathrm{T}}E\},$$

$$\varOmega_{23} = \varPhi_{23}, \ \varOmega_{33} = \varPhi_{33}, \ \varOmega_{34} = \varPhi_{34}.$$

Then the protocol (2) solves the average consensus problem.

Proof we construct the Lyapnunov function as

$$V(t) = \alpha\delta^{\mathrm{T}}(t)\delta(t) + \gamma\sum_{k=1}^{r}\int_{-\tau_k}^{0}\int_{t+\theta}^{t}\dot{\delta}^{\mathrm{T}}(s)\dot{\delta}(s)\mathrm{d}s\mathrm{d}\theta.$$

The rest part of proof is similar to the analysis in Theorem 1.

4 Simulation

Numerical simulations will be given to demonstrate the effectiveness of the theoretical results. Consider a multi-agent systems consisting of six agents and the communication topologies are given in Fig.1. All graphs in this figure are not connected and the weight of each edge is 1 and $\Delta L(t) = 0.1\sin t \times L$. The communication topologies switches every 0.1s in the sequence of $G_1 \to G_2 \to G_3 \to G_4 \to G_1 \cdots$ and the initial position vector is $(-2, -5, 2, -1, 3, -3)$. We assume that the time-delays corresponding to the edges $(1,2)$, $(1,6)$, $(2,3)$, $(3,4)$, $(4,5)$ and $(5,6)$ are 0.3 s, 0.3 s, 0.2 s, 0.3 s, 0.2 s, 0.2 s, respectively. We choosing $k_1 = 2$, it is solved that one solution for Eq.(9) is

$$\alpha = 1.0133, \ \beta = 0.5778, \ \gamma = 1, \ \varepsilon = 8.7067.$$

Fig.2 shows the corresponding position trajectories of all agents. All agents asymptotically achieved to the average value of their initial positions -1.

Fig.3 shows the corresponding velocity trajectories of all agents. It is clear that average consensus can be asymptotically achieved, although the velocity trajectories are not perfectly smooth due to the switching of the network topology.

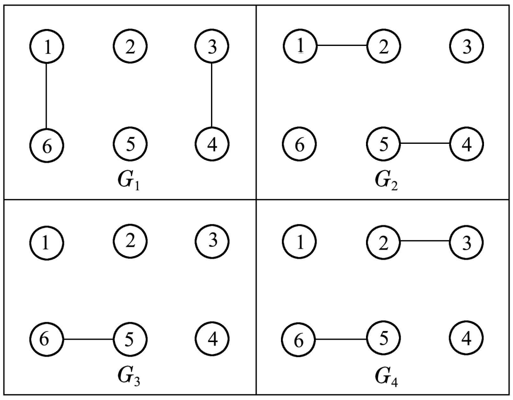

Fig. 1 Possible interaction topologies for six agents

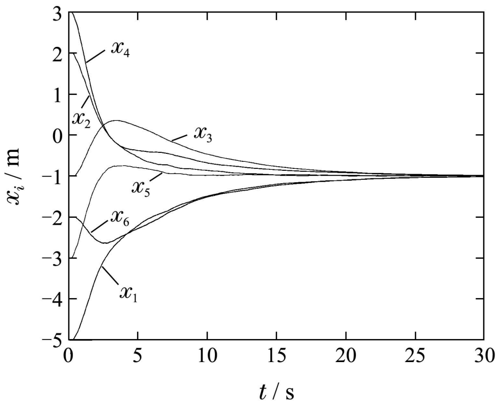

Fig. 2 Position trajectories of all agents

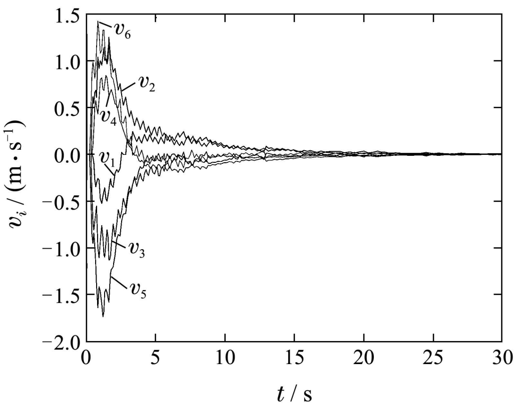

Fig. 3 Velocity trajectories of all agents

5 Conclusion

This paper has investigated the average consensus problem in networks of dynamic agents with multiple time-varying delays as well as uncertain topologies. By using Lyapunov-Krasovskii theory and decomposition technique, some sufficient conditions in terms of linear matrix inequalities (LMIs) have been given for the average consensus of the networks systems, where the communication topologies are only required jointly-connected. Finally, simulation results have been provided to demonstrate the effectiveness of our theoretical results.

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