第1卷 第2期 控制理论与应用 Vol. 1, No. 2
1984年4月 CONTROL THEORY AND APPLICATIONS April, 1984

The Stability of a Class of Large Scale System

Wang Muqiu, Wang Lian

(Institute of Mathematics, Academia Sinica, Beijing)

Abstract

In this paper, We consider the stability of the equalibrium $x=0$ of composite system

$$\dot{x}=h(x,t) \tag{3}$$

with decomposition

$$\dot{x}_i=g_i(x_i,t)+f_i(x_1,\cdots,x_r,t)\ (i=1,\cdots,r) \tag{4}$$

where

$$x_i=\mathrm{col}(x_1^{(i)},\cdots,x_{n_i}^{(i)}),\ i=1,\cdots,r,\ n_1+\cdots+n_r=n,$$

$$x^T=(x_1^T,\cdots,x_r^T);\ g_i(x_i,t)=\mathrm{col}(g_1^{(i)}(x_1^{(i)},\cdots,x_{n_i}^{(i)},t),\cdots,g_{n_i}^{(i)}(x_1^{(i)},\cdots,x_{n_i}^{(i)},t)),$$

$$x_i\in R^{n_i},\ t\in J=[t_0,\infty),\ g_i:\ R^{n_i}\times J\longrightarrow R^{n_i};\ f_i(x_1,\cdots,x_r,t)$$

$$=\mathrm{col}(f_1^{(i)}(x_1^{(1)},\cdots,x_{n_1}^{(1)},\cdots,x_1^{(r)},\cdots,x_{n_r}^{(r)},t),\cdots,f_{n_i}^{(i)}(x_1^{(1)},\cdots,x_{n_1}^{(1)},\cdots,$$

$$x_1^{(r)},\cdots,x_{n_r}^{(r)},t))$$

$$f_i:\ R^{n_1}\times R^{n_2}\times\cdots\times R^{n_r}\times J\longrightarrow R^{n_i},\ i=1,\cdots,r.$$

under suitable assumptions for f_i and g_i, using the method of scalar Lyapunov function, we obtain the conditions under which the equilibrium $x=0$ of Eq (3) is asymptotically stable or is unstable respectively.

By making use of the method of vector Lyapunov functon, we have studied the asymptotical stable problems of the zero solution for the non-autonomous and nonlinear composite system

$$\dot{x}=g(x,t) \tag{1}$$

with decomposition

Manuscript received April 13, 1983; revised September 3, 1983.

$$\dot{x}_i = g_i(x_i, t) + \sum_{j=1,\, i\neq j}^{r} A_{ij}(t)\, x_j \quad (i=1,\cdots,r) \tag{2}$$

where

$$x_i = \mathrm{col}(x_1^{(i)},\cdots, x_{n_i}^{(i)}),\ g_i(x_i, t) = \mathrm{col}(g_1^{(i)}(x_1^{(i)},\cdots, x_{n_i}^{(i)}, t),\cdots,$$

$$g_{i}^{(i)}(x_1^{(i)},\cdots, x_{n_i}^{(i)}, t)),$$

$$i=1,\cdots, r,\ n_1+\cdots+n_r = n.\quad x^T=(x_1^T,\cdots, x_r^T),\ x_i\in R^{n_i},\ t\in J=[t_0, +\infty)$$

$$g_i:\ R^{n_i}\times J \longrightarrow R^{n_i}.$$

In what follows, we always assume that $g_i(x_i, t)=0$ for all $t_0\leq t<+\infty$ if and only if $x_i=0$. The term $A_{ij}(t)$ is a $n_i\times n_j$ matrix, and all elements of $A_{ij}(t)$ are continuous and bounded. If its element is $a_{ij}(t)$, then we have

$$|a_{ij}(t)|\leq B_{ij} \qquad (B_{ij}>0 \text{ is a constant}).$$

In this paper, we shall make use of the method of scalar Lyapunov function to investigate the stability and instability of the zero solution for the composite system

$$x = h(x, t) \tag{3}$$

with decomposition

$$\dot{x}_i = g_i(x_i, t) + f_i(x_1,\cdots, x_r, t) \quad (i=1,\cdots, r) \tag{4}$$

where

$$f_i(x_1,\cdots, x_r, t) = \mathrm{col}(f_1^{(i)}(x_1^{(1)},\cdots, x_{n_1}^{(1)},\cdots, x_1^{(r)},\cdots, x_{n_r}^{(r)}, t),\cdots,$$

$$f_n(x_1^{(1)},\cdots, x_{n_1}^{(1)},\cdots, x_1^{(r)},\cdots, x_{n_r}^{(r)}, t)),\ f_i: R^{n_1}\times R^{n_2}\times\cdots\times R^{n_r}\times J \rightarrow R^{n_i}$$

and $g_i(x_i, t)$ as above. Suppose that:

(1) there exists a domain D: $t\geq t_0$, $|x_s^{(k)}|\leq H$, $s=1,\cdots, n_k$; $k=1,\cdots, r$ such that the following inequality

$$|f_i^{(k)}(x_1,\cdots, x_r, t)|\leq A_k\{|x_1^{(1)}|+\cdots+|x_{n_1}^{(1)}|+\cdots+|x_1^{(r)}|+\cdots+|x_{n_r}^{(r)}|\},$$

$$i=1,\cdots, n_k \tag{5}$$

is satisfied in this domain, where $A_k\geq 0$ $(k=1,\cdots, r)$ being constants.

(2) In this domain, the function $f_i(x_1,\cdots, x_r, t)$ $(i=1, 2\cdots r)$ is

continuous and satisfies some conditions which guarantee the existence and the uniqueness of the solution of the equation (4) with any initial value in this one.

Therefore we obtain the following conclusions:

Theorem 1 Suppose that $g_i(x_i, t)$ is differentiable with respect to x_i, and the Jacobi matrix

$$g_{ix_i}(x_i, t)=\begin{pmatrix} \dfrac{\partial g_1^{(i)}}{\partial x_1^{(i)}} & \cdots & \dfrac{\partial g_1^{(i)}}{\partial x_{n_i}^{(i)}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial g_{n_i}^{(i)}}{\partial x_1^{(i)}} & \cdots & \dfrac{\partial g_{n_i}^{(i)}}{\partial x_{n_i}^{(i)}} \end{pmatrix}$$

is continuous. We denote by $\mu_i(x_i, t)$ the largest eigenvalue of matrix $g_{ix_i}(x_i, t)+g^T_{ix_i}(x_i, t)$, if there exists $\mu_i>0$, such that $\mu_i(x_i, t)\leq -2\mu_i<0$ for all $(x_i, t)\in R^{n_i}\times J$ $(i=1,\cdots,r)$ and satisfies the following inequality

$$nA_i+c<2\mu_i-\delta_i \qquad (i=1,\cdots,r)$$

where $c=\sum\limits_{i=1}^{r} n_iA_i$ and $\delta_i>0$ may be chosen arbitrarily small, then the zero solution of (3) is asymptotically stable.

Proof Consider r isolated subsystems

$$\dot{x}=g_i(x_i, t) \qquad (i=1,\cdots, r) \tag{6}$$

For the system (6), We construct Lyapunov function

$$v_i(x_i)=x_i^T x_i.$$

It is evident that $v_i(x_i)$ is a positive definite function, but

$$\left.\frac{dv_i(x_i)}{dt}\right|_{(6)}=\dot{x}_i^T x_i+x_i^T\dot{x}_i=g_i^T(x_i, t)x_i+x_i^Tg_i(x_i, t), \tag{7}$$

because the function $g_i(x_i, t)$ is differentiable with repsect to x_i, and the Jacobi matrix $g_{ix_i}(x_i, t)$ is continuous in the product space $R^{n_i}\times J$, therefore

$$\frac{d}{d\theta}g_i(\theta x_i,\ t)=g_{ix_i}(\theta x_i,\ t)x_i$$

besides, because $g_i(0,\ t)=0$, hence

$$g_i(x_i,t)=\left(\int_0^1 g_{ix_i}(\theta x_i,\ t)d\theta\right)x_i$$

substituting it into (7), we obtain

$$\left.\frac{dv_i(x_i)}{dt}\right|_{(6)}=x_i^T\left\{\int_0^1 g_{ix_i}(\theta x_i,\ t)d\theta+\left(\int_0^1 g_{ix_i}(\theta x_i,t)d\theta\right)^T\right\}x_i$$

$$=x_i^T\left(\int_0^1 M_i(\theta x_i,\ t)d\theta\right)x_i=\int_0^1(x_i^T M_i(\theta x_i,\ t)x_i)d\theta$$

$$=x_i^T M_i(\xi x_i,\ t)x_i\le\mu_i(y,\ t)x_i^T x_i\le-2\mu_i x_i^T x_i. \qquad (8)$$

where we introduce the symmetrical matrix $M_i(y,t)=g_{ix_i}(y,t)+g_{ix_i}^T(y,t)$.

The inequality (8) shows that $\left.\frac{dv_i(x_i)}{dt}\right|_{(6)}$ is a negative definite function, but owing to the function $v_i(x_i)$ in itself being positive definite, hence the zero solution of isolated subsystem (6) is asymptotically stable.

In regard to the system (4), we construct Lyapunov function as follows

$$v(x)=\sum_{i=1}^{r}v_i(x_i)=\sum_{i=1}^{r}x_i^T x_i.$$

It is evident that $v(x)$ is positive definite but

$$\left.\frac{dv(x)}{dt}\right|_{(3)}=\sum_{i=1}^{r}\left.\frac{dv_i(x_i)}{dt}\right|_{(3)}$$

$$=\sum_{i=1}^{r}(g_i^T(x_i,\ t)+f_i^T(x_1,\cdots,x_r,\ t))x_i+x_i^T(g_i(x_i,\ t)+f_i(x_1,\cdots,x_r,\ t))$$

$$= \sum_{i=1}^{r} (g_i^T(x_i, t)\, x_i + x_i^T g_i(x_i, t)) + \sum_{i=1}^{r} (f_i^T(x_1, \cdots, x_r, t)\, x_i$$

$$+ x_i^T f_i(x_1, \cdots, x_r, t)) \leq -2 \sum_{i=1}^{r} \mu_i x_i^T x_i + 2 \sum_{i=1}^{n_1} x_i^{(1)} f_i^{(1)}(x_1, \cdots, x_r, t)$$

$$+1 \sum_{i=1}^{n_2} x_i^{(2)} f_i^{(2)}(x_1, \cdots, x_r, t) + \cdots + 2 \sum_{i=1}^{n_r} x_i^{(r)} f_i^{(r)}(x_1, \cdots, x_r, t)$$

$$\leq -2 \sum_{i=1}^{r} \mu_i x_i^T x_i + 2 \sum_{i=1}^{n_1} |x_i^{(1)}|\,|f_i^{(1)}(x_1, \cdots, x_r, t)|$$

$$+2 \sum_{i=1}^{n_2} |x_i^{(2)}|\,|f_i^{(2)}(x_1, \cdots, x_r, t)| + \cdots + 2 \sum_{i=1}^{n_r} |x_i^{(r)}|\,|f_i^{(r)}(x_1, \cdots, x_r, t)|$$

$$\leq -2 \sum_{i=1}^{r} \mu_i x_i^T x_i + 2A_1 \sum_{i=1}^{r_1} |x_i^{(1)}| \left(\sum_{i=1}^{n_1} |x_i^{(1)}| \right.$$

$$\left. + \sum_{i=1}^{n_2} |x_i^{(2)}| + \cdots + \sum_{i=1}^{n_r} |x_i^{(r)}| \right) + 2A_2 \sum_{i=1}^{n_2} |x_i^{(2)}| \left(\sum_{i=1}^{n_1} |x_i^{(1)}| \right.$$

$$\left. + \sum_{i=1}^{n_2} |x_i^{(2)}| + \cdots + \sum_{i=1}^{n_r} |x_i^{(r)}| \right) + \cdots + 2A_r \sum_{i=1}^{n_r} |x_i^{(r)}| \left(\sum_{i=1}^{n_1} |x_i^{(1)}| \right.$$

$$\left. + \sum_{i=1}^{n_2} |x_i^{(2)}| + \cdots + \sum_{i=1}^{n_r} |x_i^{(r)}| \right) \leq -2 \sum_{i=1}^{r} \mu_i x_i^T x_i$$

$$+ A_1 \left[2n_1 \sum_{i=1}^{n_1} x_i^{(1)2} + n_2 \sum_{i=1}^{n_1} x_i^{(1)2} + n_1 \sum_{i=1}^{n_2} x_i^{(2)2} + \cdots + n_r \sum_{i=1}^{n_1} x_i^{(1)2} \right.$$

$$\left. + n_1 \sum_{i=1}^{n_r} x_i^{(r)2} \right] + A_2 \left[n_2 \sum_{i=1}^{n_1} x_i^{(1)2} + n_1 \sum_{i=1}^{n_2} x_i^{(2)2} + 2n_2 \sum_{i=1}^{n_2} x_i^{(2)2} + \right.$$

$$+\cdots+n_r\sum_{i=1}^{n_2}x_i^{(2)2}+n_2\sum_{i=1}^{n_r}x_i^{(r)2}\Bigg]+\cdots+A_r\Bigg[\,n_r\sum_{i=1}^{n_1}x_i^{(1)2}+n_1\sum_{i=1}^{n_r}x_i^{(r)2}$$

$$+n_r\sum_{i=1}^{n_2}x_i^{(2)2}+n_2\sum_{i=1}^{n_r}x_i^{(r)2}+\cdots+2n_r\sum_{i=1}^{n_r}x_i^{(r)2}\Bigg]$$

$$=-2\sum_{i=1}^{r}\mu_i x_i^{\mathrm{T}}x_i+A_1\Bigg[(n+n_1)\sum_{i=1}^{n_1}x_i^{(1)2}+n_1\sum_{i=1}^{n_2}x_i^{(2)2}+\cdots+n_1\sum_{i=1}^{n_r}x_i^{(r)2}\Bigg]$$

$$+A_2\Bigg[\,n_2\sum_{i=1}^{n_1}x_i^{(1)2}+(n+n_2)\sum_{i=1}^{n_2}x_i^{(2)2}+\cdots+n_2\sum_{i=1}^{n_r}x_i^{(r)2}\Bigg]+\cdots$$

$$+A_r\Bigg[\,n_r\sum_{i=1}^{n_1}x_i^{(1)2}+n_r\sum_{i=1}^{n_2}x_i^{(2)2}+\cdots+(n+n_r)\sum_{i=1}^{n_r}x_i^{(r)2}\Bigg]$$

$$=[-2\mu_1+(n+n_1)A_1+n_2A_2+\cdots+n_rA_r]\sum_{i=1}^{n_1}x_i^{(1)2}+[-2\mu_2+n_1A_1$$

$$+(n+n_2)A_2+\cdots+n_rA_r]\sum_{i=1}^{n_2}x_i^{(2)2}+\cdots+[-2\mu_r+n_1A_1+n_2A_2+\cdots$$

$$+(n+n_r)A_r]\sum_{i=1}^{n_r}x_i^{(r)2}$$

let $c=\sum_{i=1}^{r}n_iA_i$, then we have

$$\left.\frac{dv(x)}{dt}\right|_{(3)}\leq(-2\mu_1+nA_1+c)\sum_{i=1}^{n_1}x_i^{(1)2}+(-2\mu_2+nA_2+c)\sum_{i=1}^{n_2}x_i^{(2)2}+\cdots$$

$$+(-2\mu_r+nA_r+c)\sum_{i=1}^{n_r}x_i^{(r)2},$$

according to the assumptions of theorem, we know $nA_i+c<2\mu_i-\delta_i$ $(i=1,\cdots,r)$, We obtain

$$\left.\frac{dv(x)}{dt}\right|_{(3)}\leq-\delta_1\sum_{i=1}^{n_1}x_i^{(1)2}-\delta_2\sum_{i=1}^{n_2}x_i^{(2)2}+\cdots-\delta_r\sum_{i=1}^{n_r}x_i^{(r)2}.$$

because $\delta_i>0$ $(i=1,\cdots,r)$, therefore $\left.\frac{dv(x)}{dt}\right|_{(3)}$ is a negative definite function, but $V(x)$ in itself is positive definite, it follows from here that the zero solution of the system (4) is asymptotically stable.

Theorem 2 Suppose that $g_i(x_i,t)$ is differnetiable with respect to x_i, and the Jacobi matrix

$$g_{ix_i}(x_i,t)=\begin{pmatrix}\frac{\partial g_1^{(i)}}{\partial x_1^{(i)}} & \cdots & \frac{\partial g_1^{(i)}}{\partial x_{n_i}^{(i)}}\\ \cdots & \cdots & \cdots\\ \frac{\partial g_{n_i}^{(i)}}{\partial x_1^{(i)}} & \cdots & \frac{\partial g_{n_i}^{(i)}}{\partial x_{n_i}^{(i)}}\end{pmatrix}$$

is continuous. We denote by $\mu_i^*(x_i,t)$ the least eigenvalue of matrix

$$g_{ix_i}(x_i,t)+g_{ix_i}^T(x_i,t).$$

if there exists $\mu_i^*>0$, such that

$$\mu_i^*(x_i,t)\geq 2\mu_i^*>0 \qquad \text{for all } (x_i,t)\in R^{n_i}\times J\ (i=1,\cdots,r)$$

and satisfies the following inequality

$$nA_i+c<2\mu_i^*-\delta_i^* \qquad (i=1,\cdots,r)$$

where $c=\sum_{i=1}^{r}n_iA_i$, but $\delta_i^*>0$ may be chosen arbitrarily small, then the zero solution of the system (4) is unstable.

Proof Consider r isolated subsystems

$$\dot{x}_i=g_i(x_i,t) \qquad i=1,\cdots,r. \tag{6}$$

In regard to the system (6), we construct Lyapunov function

$$v_i(x_i) = x_i^T x_i.$$

It is evident that $v_i(x_i)$ is a positive definite function, but

$$\left.\frac{dv_i(x_i)}{dt}\right|_{(6)} = g_i^T(x_i, t)x_i + x_i^T g_i(x_i, t) \tag{7}$$

by the same means as theorem 1, we obtain

$$\left.\frac{dv_i(x_i)}{dt}\right|_{(6)} = x_i^T M(\xi x_i, t)x_i \geq \mu_i^*(y, t)x_i^T x_i$$

$$\geq 2\mu_i^* x_i^T x_i. \tag{9}$$

It has been shown that $\frac{dv_i(x_i)}{dt}$ is a positive definite function with respect to $x_1^{(i)}, \cdots, x_{n_i}^{(i)}$ but the function $v_i(x_i)$ in itself is a positive definite function, too. Therefore the zero solution of system (6) is unstable.

In regard to system (4), we construct Lyapunov function as follows

$$v(x) = \sum_{i=1}^{r} v_i(x_i) = \sum_{i=1}^{r} x_i^T x_i.$$

It is evident that $v(x)$ is a positive definite function with respect to x, but

$$\left.\frac{dv(x)}{dt}\right|_{(4)} = \sum_{i=1}^{r} \left.\frac{dv_i(x_i)}{dt}\right|_{(4)}$$

$$= \sum_{i=1}^{r}(g_i^T(x_i, t)x_i + x_i^T g_i(x_i, t)) + \sum_{i=1}^{r}(f_i^T(x_1, \cdots, x_r, t)x_i + x_i^T f_i(x_1, \cdots, x_r, t))$$

$$\geq 2\sum_{i=1}^{r} \mu_i^* x_i^T x_i + 2\sum_{i=1}^{n_1} x_i^{(1)} f_i^{(1)}(x_1, \cdots, x_r, t) + 2\sum_{i=1}^{n_2} x_i^{(2)} f_i^{(2)}(x_1, \cdots, x_r, t) + \cdots$$

$$+2\sum_{i=1}^{n_r} x_i^{(r)} f_i^{(r)}(x_1,\cdots,x_r,t) \geq 2\sum_{i=1}^{r} \mu_i^* x_i^T x_i - 2\sum_{i=1}^{n_1} |x_i^{(1)}|\,|f_i^{(1)}(x_1,\cdots,x_r,t)|$$

$$-2\sum_{i=1}^{n_2} |x_i^{(2)}|\,|f_i^{(2)}(x_1,\cdots,x_r,t)| + \cdots - 2\sum_{i=1}^{n_r} |x_i^{(r)}|\,|f_i^{r}(x_1,\cdots x_r,t)|$$

$$\geq 2\sum_{i=1}^{r} \mu_i^* x_i^T x_i - 2A_1 \sum_{i=1}^{n_1} |x_i^{(1)}| \left(\sum_{i=1}^{n_1} |x_i^{(1)}| + \sum_{i=1}^{n_2} |x_i^{(2)}| + \cdots + \sum_{i=1}^{n_r} |x_i^{(r)}| \right)$$

$$-2A_2 \sum_{i=1}^{n_2} |x_i^{(2)}| \left(\sum_{i=1}^{n_1} |x_i^{(1)}| + \sum_{i=1}^{n_2} |x_i^{(2)}| + \cdots + \sum_{i=1}^{n_r} |x_i^{(r)}| \right) + \cdots$$

$$-2A_r \sum_{i=1}^{n_r} |x_i^{(r)}| \left(\sum_{i=1}^{n_1} |x_i^{(1)}| + \sum_{i=1}^{n_2} |x_i^{(2)}| + \cdots + \sum_{i=1}^{n_r} |x_i^{(r)}| \right)$$

$$\geq 2\sum_{i=1}^{r} \mu_i^* x_i^T x_i - A_1 \left[(n+n_1) \sum_{i=1}^{n_1} x_i^{(1)2} + n_1 \sum_{i=1}^{n_2} x_i^{(2)2} + \cdots + n_1 \sum_{i=1}^{n_r} x_i^{(r)2} \right]$$

$$-A_2 \left[n_2 \sum_{i=1}^{n_1} x_i^{(1)2} + (n+n_2) \sum_{i=1}^{n_2} x_i^{(2)2} + \cdots + n_2 \sum_{i=1}^{n_r} x_i^{(r)2} \right] + \cdots$$

$$-A_r \left[n_r \sum_{i=1}^{n_1} x_i^{(1)2} + n_r \sum_{i=1}^{n_2} x_i^{(2)2} + \cdots + (n+n_r) \sum_{i=1}^{n_r} x_i^{(r)2} \right]$$

$$= (2\mu_1^* - (n+n_1)A_1 - n_2A_2 - \cdots - n_rA_r) \sum_{i=1}^{n_1} x_i^{(1)2}$$

$$+ (2\mu_2^* - n_1A_1 - (n+n_2)A_2 - \cdots - n_rA_r) \sum_{i=1}^{n_2} x_i^{(2)2} + \cdots$$

$$+ (2\mu_r^* - n_1A_1 - n_2A_2 - \cdots - (n+n_r)A_r \sum_{i=1}^{n_r} x_i^{(r)2}$$

let $c=\sum_{i=1}^{r} n_i A_i$, then We obtain

$$\left.\frac{dv(x)}{dt}\right|_{(4)} \geq (2\mu_1^* - nA_1 - c)\sum_{i=1}^{n_1} x_i^{(1)\,2} + (2\mu_2^* - nA_2 - c)\sum_{i=1}^{n_2} x_i^{(2)\,2}$$

$$+\cdots+(2\mu_r^* - nA_r - c)\sum_{i=1}^{n_r} x_i^{(r)\,2}.$$

by the assumption of theorem 2, we have $nA_i + c < 2\mu_i^* - \delta_i^*$, that is $2\mu_i^* - nA_i - c > \delta_i^*$ $(i=1, 2, \cdots, r)$, we obtain

$$\left.\frac{dv(x)}{dt}\right|_{(4)} \geq \delta_1^* \sum_{i=1}^{n_1} x_i^{(1)\,2} + \cdots + \delta_r^* \sum_{i=1}^{n_r} x_i^{(r)\,2}$$

because $\delta_i^* > 0$ $(i=1, \cdots, r)$, hence $\left.\frac{dv(x)}{dt}\right|_{(4)}$ is a positive definite function, but the $v(x)$ in itself is a positive definite function, too. It follows that the zero solution of the system(4) is unstable. This completes the proof of the theorem.

Reference

[1] Michel, A. N. and Miller, R. K., Qualitative analysis of large scale dynamical systems, Academic Press, New York, Sanfrancisco, London (1977).

[2] Wang Muqiu and Wang Lian, The stability of a class of non-linear autonomous system, Kexue Tongbao (to appear).