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Comparison Principle of the Discrete Large-scale Systems in the Theory of Stability and Its Applications*

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Abstract

In this paper, we establish the comparison principle of the discrete systems and study the decomposition problem in the theory of stability. At the same time the region of stability of a nonlinear time-invarant system is estimated as an example.

I. Introduction

On the stability analysis of the large-scale systems, there are two methods based on Lyapunov functions which are called "scalar Lyapunov function method" and "vector Lyapunov function method" respectively. The advantage of the former is that the regions of stability can be estimated very fast. Thus this method is a powerful tool for solving many technical and engineering problems. In this respect, Liu Yong-qing, Wang Mu-qiu, Wang Lian[1-5], A. N. Michel, R. K. Miller, D.D. Siljak[6][7] and others studied many cases, such as time-varying, time-invariant, delay and non-linear, continuous and discrete systems. Vector Lyapunov function method and comparison principle, in general, can give one a result that possess-es weeker conditions (i.e. a larger region of stability) than the sca-lar one. In 1966, F.N.Bailey[8] established the comparison principle of linear system. During the past years, A. N. Michel, R.K. Miller, D.D.Siljak studied the stability of continuous large-scale systems.

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But until 1982, the comparison principle and vector Lyapunov function method of discrete-time systems has not been established. After the works cited in the References, it is possible, as described in this paper, to obtain a series of results of comparison principle of discrete systems by applying the ideas of A.N.Michel, R.K. Miller and Siljak.

II. Scalar Case

We begin by considering a scalar difference equation of the form

$$y(m+1)=f[y(m),\ m],\quad m=\tau+1,\ \tau+2,\ \cdots, \tag{2.1}$$

where $y(m)\in R$, $m\in I\triangleq[\tau,\infty]$, $f(m):R_\rho\times I\to R$ ($R_\rho=\{x\,|\,x\in R;\ 0\leq\|x-x_0\|<\rho\}$, Assume that Eq.(2.1) possesses solution $y(m)=y(m;\ y_\tau,\tau)$ for every $y_\tau=y(\tau;y_\tau,\tau)$, which is unique. Also, we assume that $y(m)=0$ is an isolated equilibrium. For the sake of brevity, we frequently write $y(m)$ in place of $y(m;\ y_\tau,\ \tau)$ to denote solutions with $y(\tau)=y_\tau$.

Definition 1. A function $f\ [x(m),\ m]$ belongs to the class H_1 if for any fixed $m\in I$ and all $x'(m),\ x''(m)\in R$ such that $x'(m)\leq x''(m)$ the inequality $f[x'(m),\ m]\leq f[x''(m),\ m]$ is satisfied.

Theorem 1. If $x(m)\in R$ such that $x(\tau)\leq y(\tau)=y_\tau$ and if

$$x(m+1)\leq f[x(m),\ m]\quad \forall m\in I$$

where f: $R_\rho\times I\to R$, $f(m)\in H_1$ then inequality $x(m)\leq y(m)$ holds for all $m\in I$, where $y(m)$ is the solution of Eq. (2.1).

Proof. The proof is very simple using iteration method. When $m=\tau$, we have $x(\tau)\leq y(\tau)$. Because of $f(m)\in H_1$, then

$$x(\tau+1)\leq f[x(\tau),\ \tau]\leq f[y(\tau),\ \tau]=y(\tau+1)$$

Thus we get $x(m)\leq y(m)$ for all $m=\tau+1,\ \tau+2,\ \cdots$. The proof is complete.

Theorem 2. If $x(m)\in R$ such that $x(\tau)\geq y(\tau)$. Assume that $y(m)$ is the unique solution of Eq.(2.1) and if $x(m+1)\geq f[x(m,\ )m]\ \forall m\in I$ where $f(m)\in H_1$, then $x(m)\geq y(m)$ for all $m\in I$.

Proof We omit the proof here.

Now we study the stability of the system as follows:

$$x(m+1)=g[x(m),\ m] \tag{2.2}$$

where $g[\cdot,\ m]:R^n\times I\to R^n$, $\forall m\in I$, $x(m)\in R^n$. Assume that the solutions of Eq.(2.2) satisfy the unique existence condition.

Definition 2. A function $V[x(m),\ m]$ such that $V(0,\ m)=0$ is Positive Definite if there exists a positive definite function $\Phi[x(m)]$ in which variable m is an implicant, and $V[x(m), m] \geq \phi[x(m)]$ for all $m \in I$, where function $\phi[x(m)] = 0$ if and only if $x(m) = 0$.

Theorem 3. If $V[x(m), m]: R^n \times I \to R_+$ is a positive definite Lyapunov function such that

$$V[x(m+1), m+1]|_{(2.2)} = V[g(x(m), m), m+1]|_{(2.2)} \leq f[V(m), m]$$

where $f[V(m), m] \in H_1$. Then the following statements are true.

(i) If the trivial solution of Eq. (2.1) is stable, then the trivial solution of Eq. (2.2) is also stable.

(ii) If the trivial solution of Eq. (2.1) is uniformly asymptotically stable, then the trivial solution of the Eq.(2.2) is also uniformly asymptotically stable.

Proof: (i) For any $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that if $|x(0)| < \delta(\varepsilon)$, we have $|x(m)| < \varepsilon$ $(m \in I)$, where $\varepsilon > 0$ is a sufficiently small number. Suppose now that the result is not true. Then for $1/n$ (n is a natural number), when $|x(0)| < 1/n$, $\exists T(n) \in I$, we have $|x[T(n)]| \geq \varepsilon$. Since $V[x(m), m]$ is a positive definite function, there exists a positive number $\lambda > 0$ such that

$$V[x(T(n)), T] \geq \lambda \tag{*}$$

We notice that the trivial solution of system (2.1) is stable, that is, for $\lambda > 0$, $\exists \eta(\lambda) > 0$ when $|y(0)| < \eta(\lambda)$ we have $|y(m)| < \lambda$. Then by Theorem 1. we conclude that $|V[x(T(n), T]| < |y(T)| < \lambda$. It contradicts inequality (*) and Theorem 3. is established.

(ii) By the proof of result(i), it is evident that system (2.2) is stable. Since $\lim_{m\to\infty} y(m) = 0$, we get $\lim_{m\to\infty} V(m) = 0$. Because function $V[x(m), m]$ is positive definite, i. e. for $\forall m \in I$, $V[x(m), m] \geq \phi[x(m)]$ then $\lim_{m\to\infty} \phi[x(m)] = 0$ implies that $\lim_{m\to\infty} x(m) = 0$. Thus the proof is complete.

Theorem 4. If $V[x(m), m]: R^n \times I \to R_+$ is a positive definite function such that

$$V[x(m+1), m+1]|_{(2.2)} = V[g(xm), m)), m+1] \geq f[V(m), m] \forall m \in I,$$

where $f[V(m), m] \in H_1$. If the trivial solution of Eq.(2.1) is unstable, then the trivial solution of Eq.(2.2) is unstable.

Proof: For any $\forall \varepsilon > 0$, no matter how we choose $\delta(\varepsilon) > 0$, such that if $|x(0)| < \delta(\varepsilon)$, there always exists a time $T \in I$. When $m > T$ then $|x_s(m)| \geq \varepsilon$ $(\forall s = 1, 2, \cdots, n)$. Suppose now that this is not true, i. e.

we always have $|x_i(m)|<\varepsilon$. Since $V[x(m), m]$ is positive definite, then $|V[x(m), m]|<\lambda$ $(\lambda>0)$. Notice that system (2.1) is unstable, i. e. for $\lambda>0$, no matter how we take $\eta(\lambda)>0$, if $|y(0)|<\eta(\lambda)$, there exists a time $K\in I$, when $m\geq K$ such that $|y(m)|\geq\lambda$. Particuarly, we can take $y(0)>0$, then $y(m)>0$ for all $m\in I$. Now using Theorem 2, when $m\geq\max\{T, K\}$ we have

$$|V[x(m), m]|\geq|y(m)|\geq\lambda$$

Thus we obtain a contradicton and the proof is complete.

Ⅲ. The Case of N-Dimension Vector

We now consider the vector-valued comparison difference equation of the form

$$y(m+1)=f[y(m), m] \tag{3.1}$$

where $y(m)\in R^n$, $m\in I$, $f[y(m), m]:D\to R^n$ is a function definited on open set D in R^{n+1}. The elements in R^{n+1} are $(y(m), m)$. We assume that the solution of Eq.(3.1) satisfies unique existence condition and denote it by $y(m)=y(m; y_\tau, \tau)$ which passes through point (τ, y_τ). We assume that $y(m)=0$ is an isolated equilibrium.

Definition 3. A vector-valued function $f[x(m, m)=\{f_1[x(m), m],\cdots, f_n[x(m),m]\}^T$ belongs to class H_n (expressed as $f\in H_n$) if for any $(x'(m), m)$, $(x''(m), m)\in D$, such that $x'(m)\leq x''(m)$, the inequality

$$f_i[x'(m),m]\leq f_i[x''(m),m] \ \forall m\in I; \ i=1, 2, \cdots, n$$

is satisfied.

Theorem 5. For $x(m)\in R^n$ if $x(\tau)\leq y(\tau)=y_\tau$, and if

$$x(m+1)\leq f[x(m), m]$$

where $f[x(m), m]: R^n\times I\to R^n$, $f[x(m), m]\in H_n$, assume that $y(m)$ is the solution of Eq. (3.1), then inequality $x(m)\leq y(m)$ holds for all $m\in I$.

Theorem 6. For any $x(m)\in R^n$, if $x(\tau)\geq y(\tau)$ and if

$$x(m+1)\geq f[x(m), m]$$

where $f[x(m), m]\in H_n: R^n\times I\to \in R^n$. $\forall m\in I$, then inequality $x(m)\geq y(m)$ holds for all $m\in I$. We assume that $y(m)$ is the solution of Eq.(3.1).

The proofs of Theorem 5. and Theorem 6. are the same as Theorem 1. and Theorem 2. But we must notice that $x(m)$, $y(m)$ and $f(\cdot)$ are n-dimension vectors.

Now we consider system

$$x(m+1)=g[x(m), m] \tag{3.2}$$

where $x(m)\in R^n$, $m\in I$, $g[x(m), m]:R^n\times I\to R^n$. We assume that the solutions of Eq. (3.2) astisfy unique existence condition. $x(m)=0$ is its isolated equilibrium point.

Theorem 7. Assume that

$V[x(m), m]=\{V_1[x(m), m], \cdots, V_n[x(m), m]\}^T$

is an n-dimention vector Lyapunov function, $V[x(m), m]: R^n \times I \to R^n_+$, its every component $V_i[x(m), m]: R^n \times I \to R_+$ is a positive definite scalar Lyapunov function. On its domain of definition, if

$V[x(m+1), m+1]|_{(3.2)} = V[g(x(m), m), m+1] \leq f[V(m), m]$

where $f[x(m), m] \in H_n: R^n \times I \to R^n$, $m \in I$. Then the following statements are true.

(i) If the trivial solution of comparison equation (3.1) is stable, then the trivial solution of system (3.2) is also stable.

(ii) If the trivial solution of comparison equation(3.1)is uniformly asymptotically stable, then the trivial solution of system (3.2) is also uniformly asymtotically stable.

Proof: In fact, according to Theorem 5. we have $V(m) \leq y(m)$, $\forall m \in I$, *i.e.* fer every component of $V(m)$ and $y(m)$

$V_i(m) \leq y_i(m)$; $(i=1, 2, \cdots, m)$

By the proof of Theorem 3, Theorem 7. can be obtained immediately.

Theorem 8. Assume that $V[x(m), m)$, $V_i[X[m), m]$ $f[V(m), m]$ are the same as Theorem 7. If

$V[x(m+1), m+1]|_{(3.2)} = V[g(x(m), m), m+1] \geq f[V(m), m]$

then instability of the trivial solution of Eq. (3.1) implies instability of the trivial solution of Eq. (3.2).

Proof: the proof is similar to the proof of Theorem 7. We notice the direction of the sign of inequlity here, using Theorem 4. and Theorem 6. the statement is an evident fact.

IV. Applications of Comparison Principle and Decomposition of the Large-scale System

We now discuss stability of discrete large-scale system by means of the theorems that have been established above and are usually called "comparison principle". We assume that

(φ): $x(m+1) = g[x(m), m] + h[x(m), m] \triangleq q[x(m), m]$

(Σ_i): $z_i(m+1) = g_i[z_i(m), m] + h_i[z(m), m]$,

(φ_i): $z_i(m+1) = g_i[z_i(m), m]$

$i = 1, 2, \cdots, r$.

where (Σ_i) is decomposition of the composite large-scale or interconnected system, (φ), (φ_i) are isolated subsystems of (φ), $x(m)$

$\in R^n$, $z_i(m) \in R^{n_i}$, $m \in I$, $\Sigma_{i=1}^r n_i = n$.

The origin is the unique equilibrium point of (φ) and (φ_i).

Definition 4. An isolated sub-system (φ_i) possesses Property A if, on domain $|x_s(m)|<M$, $m \in I$, there exists a positive definite function $V_i[z_i(m), m]: R^{n_i} \times I \to R_+$ such that

$\Delta V[z_i(m), m]|_{(\varphi_i)} = V[z_i(m+1), m+1]|_{(\varphi_i)} - V[z_i(m), m] = V[g_i(z_i(m), m), m+1] - V[z_i(m), m] = \sigma_i \phi_i[z_i(m)]$

where σ_i is a constant, $\phi_i[z_i(m)]$ is a positive definite function in which m is not explicit.

In fact, we have assumed that there always exists a positive definite Lyapunov function of the subsystem (φ_i) here. Clearly, if $\sigma_i<0$ the equilibrium $z_i(m)=0$ of (φ_i) is stable (or asymptotically stable). If $\sigma_i>0$ the equilibrium of (φ_i) is unstable.

Theorem 9. For the large-scale or interconnected system (φ) with decomposition (Σ_i), if

(i) Every isolated subsystem (φ_i) possesses Property A.

(ii) We take a vector Lyapunov function

$V[x(m), m] = \{V_1[z_1(m), m], \cdots, V_r[z_r(m)m, m]\}^T$.

such that

$$V_i[z_i(m+1), m+1]|_{(\Sigma_i)} \leq f_i[V(z(m)m, m]$$

where $f_i[V(m), m] \in H_r$, $i=1,2,\cdots,r$, Then the uniformly asymptotical stability of the trivial solution of the vector comparison equation (3.1) implies the uniformly asymptotical stability of the trivial solution of the interconnected system(φ). In this Theorem we may assume $\sigma_i < 0$ $(i=1,2,\cdots,r)$. Thus we can get the stability of the interconnected system (φ) from the stability of the subsystem(φ_i) That is, the interconnected terms of (φ) is as a disturbance to composite system. As a particular form of Theorem 9, we have

Theorem 10. For the interconnected system (φ) with decomposition (Σ_i), if

(i) every isolated subsystem possesses Property A. and

(ii) its Lyapunov function $V_i[z_i(m), m]$: $R^{n_i} \times I \to R_+$ satisfies inequality

$$V_i[z_i(m+1), m+1]\Big|_{(\Sigma_i)} \leq \sum_{j=1}^{r} G_{ij} V_j[z_j(m), m],$$

$$i = 1, 2, \cdots, r.$$

or $\quad V(m+1) \leq G\cdot V(m)$

where G is an $r\times r$ constant matrix which elements $G_{ij} \geq 0$ $(i, j = 1, 2, \cdots, r)$ and $f[V(m), m] = G\cdot V(m) \in H_r$;

(iii) all the eigenvalues of $r\times r$ matrix G satisfy $|\mu_i| < 1$ $(i = 1, 2, \cdots, r)$. Then the trivial solution of composite system (φ) is uniformly asymptotically stable.

Using the principle of stability in the first approximation and Theorem 10. we immediately obtain the following result.

Theorem 11. For interconnected system (φ) with decomposttion (Σ_i), if every subsystem (φ_i) possesses Property A and $V(m)$ is a vector Lyapunov function, such that

$$V(m+1)\Big|_{(\Sigma_i)} \leq G\cdot V(m) + f^*[V(m), m] \qquad (*)$$

where $\{GV(m) + f^*[V(m), m]\} \in H_r$, $f^*[V(m), m]$ is assumed to consist of second or higher order terms i. e.

$$\lim_{|V(m)|\to 0} |f^*[V(m), m]|/|V(m)| = 0,$$

uniformly in $m \in I$. Then the following are true.

(i) If the absolute-value of all eigenvalues μ_i $(i = 1, 2, \cdots, r)$ of matrix G satify $|\mu_i| < 1$, the trivial solution of (φ) is uniformly asymtotically stable.

(ii) If the inequality $(*)$ is reversed and if G has at least one eigenvalues which absolute-value is greater than unity, the trivial solution of interconnected system (φ) is unstable.

Example. Consider the 2-dimensional system

$$\begin{cases} x(m+1) = \dfrac{ay(m)}{1+x^2(m)} + y^2(m) \\ \\ y(m+1) = \dfrac{bx(m)}{1+y^2(m)} + x^2(m) \end{cases} \tag{4.1}$$

$$m = 0, 1, 2, \cdots.$$

where a and b are real constants, $x(m), y(m) \in R$,

We take $V[x(m), y(m)] = x^2(m) + y^2(m)$ and notice that inequality

$(a_1^s + a_2^s)^{1/s} \leq (a_1^t + a_2^t)^{1/t}$ is true for $a_1, a_2 \geq 0$ and $0 < t < s$. Then we have

$$V[x(m+1), y(m+1)]\Big|_{(5\cdot5)} = \left[\frac{ay(m)}{1+x^2(m)} + y^2(m)\right]^2$$

$$+\left[\frac{bx(m)}{1+y^2(m)} + x^2(m)\right]^2 \leq \frac{a^2y^2(m)}{[1+x^2(m)]^2} + \frac{b^2x^2(m)}{[1+y^2(m)]}$$

$$+ x^4(m) + y^4(m) + \frac{2|a|\cdot|y(m)|^3}{1+x^2(m)} + \frac{2|b|\cdot|x(m)|^3}{1+y^2(m)} \leq M^2[x^2(m)$$

$$+y^2(m)] + x^4(m) + y^4(m) + 2x^2(m)y^2(m) + 2M[x^2(m)+y^2(m)]^{3/2}$$

$$= M^2V[x(m), y(m)] + V^2[x(m), y(m)] + 2MV^{3/2}[x(m], y(m)]$$

where $M = \max\{|a|, |b|\}$. Its comparison system is

$$r(m+1) = M^2 r(m) + r^2(m) + 2Mr^{3/2}(m) \qquad (4.2)$$

The origin is a unique equilibrium point of equation (4.2) in set $S = \{r | r \in R;\ 0 \leq r \leq (1-M)^2\}$ and

$$\lim_{r(m)\to 0} [r^2(m) + 2Mr^{3/2}(m)]/r(m) = 0$$

By Theorem 11, the trivial solution of Eq. (4.1) is asymptotically stable if condition $M^2 < 1$ is satisfied.

V. Conclusions

We can use the comparison principle and vector Lyapunov function to get the stability properties of large-scale discrete-time systems from its subsystems. Generally speeking, we can always ddecompose the n-order system into r-order $(1 \leq r \leq n)$. But the step may bring about some conditions which are not neccssary.

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