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Identification of the Fission Cross Section of a Nuclear Reactor

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Abstract

It is important to determine the fission cross sectiin in the theory and applications of neclear reacter. In this paper we prove uniqueness and existence of the optimal fission cross section, and give the optimality condition.

1. The setting of the problem

In the theory and applications of nuclear reactor a very important problem is to determine the fission cross section. In this paper we use single energy static diffusion approximation. Consider the eigenvalue problem of the elliptic operator of second order

$$-\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial\varphi(x)}{\partial x_j}+\sigma_a(x)\varphi(x)=\frac{\nu}{\lambda}\sigma_f(x)\varphi(x),\quad x\in\Omega \tag{1}$$

$$\varphi(\mathbf{x})=0,\mathbf{x}\in\Gamma$$

where the domain Ω occupied by the reactor is a bounded convex open set in R^3, enclosed by a sufficiently smooth unilateral surface Γ; $a_{ij}(x)=a_{ji}(x)$; $\varphi(x)$ is the neutron flux distribution; $\sigma_a(x)$ is the macro-absorption cross section; $\sigma_f(x)$ is the macro-fission cross section; $\nu>0$ is the multipticalion fator; λ is the eigenvalue.

We assume that there is a constant $c_0>0$ such that

$$\sum_{i,j=1}^{3}a_{ij}(x)\xi_i\xi_j\geq c_0\sum_{i=1}^{3}\xi_i^2,\quad\forall(\xi_i)\in R^3,\ x\in\Omega;$$

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that $\sigma_a(x) \in L^\infty(\Omega)$, $\sigma_a(x)) \geq 0$, a.e. $x \in \Omega$ and $\sigma_f(x) \in L^\infty(\Omega)$, $\sigma_f(x) > 0$, a.e $x \in \Omega$. We now define in $L^2(\Omega)$

$$L\varphi = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial \varphi(x)}{\partial x_j},$$

$$D(L) = \{v | \in L^2(\Omega),\ Lv \in L^2(\Omega), v/\Gamma = 0\}$$

and the bounded linear operators

$$\Sigma_a \varphi = \sigma_a(x)\varphi(x)$$

$$\Sigma_f \varphi = \sigma_f(x)\varphi(x)$$

It is known that under the above assumptions the operator $T = \nu(L + \Sigma_a)^{-1}\Sigma_f$ is compact in $L^2(\Omega)$, the largest eigenvalue $\lambda(\sigma_f)$ of T, which is algebraically simple, is equal to the spectrum radius and the eigenfunction $\varphi(\sigma_f)$, belonging to $\lambda(\sigma_f)$, andν may be chosen to be positive almost everywhere. (See [1]) For simplicity, we assume $\nu = 1$.

The problem is to determine the fission cross section as exactly as possible by measuring the critical flue distribution. We regard it as the identifiction problem of a distributed parameter system.

From the physical consideration, the admissible set of fission cross sections is

$$U = \{\sigma_f | \in L^\infty(\Omega),\ c_1 \leq \sigma_f(x) \leq c_2\}$$

where c_1, c_2 are positive constants.

Note that, while the critical flue distribution is being measured, the reactor is in a critical state, which means that the largest eigenvalue of the system (1) is equal to 1 and the corresponding eigenfunction, namely the critical flue distribu tion, is positive almost everywhere. To determine φ uniquely, a reasonable requirement is that the total amount of neutrons corresponding to φ, is equal to that being measured, that is

$$\int_\Omega \varphi(\sigma_f)(x)dx = \int_\Omega \varphi_d(x)dx \qquad (2)$$

Let $g = 1$, (2) can be rewritten with the scalar product in $L^2(\Omega)$:

$$\langle \varphi(\sigma_f), g \rangle = \langle \varphi_d, g \rangle \qquad (3)$$

Thus we will consider

$$(L + \Sigma_a)\varphi = \sigma\varphi \qquad (4)$$

$$\langle \varphi g, \rangle = p \qquad (5)$$

where $p = \langle \varphi_d, g \rangle$. Using a quadratic cost

$$J(\varphi)=\frac{1}{2}\int_{\Omega}|\varphi(x)-\varphi_d(x)|^2dx$$

we set an optimal problem of determining φ_0 such that

$$J(\varphi_0)=\inf_{\varphi\in V} J(\varphi) \tag{6}$$

$$V=\{\varphi|\varphi\geq 0,\ \langle\varphi,g\rangle=p, \exists\sigma\in \mathrm{U}\ \mathrm{s{\cdot}t}\ (4)\ \mathrm{holds}\}$$

We will show the existence and uniqueness of the solution of (6) and give the sufficient and necessary conditions of optimality.

2. Existence and uniqueness of the optimal solution

Lemma 1 V is a closed convex set in $L^2(\Omega)$

Proof 1) Convexity. Suppose that $\varphi_1,\varphi_2\in V$, $t\in(0,1)$, $\varphi_t=t\varphi_1+(1-t)\varphi_2$, then $\varphi\geq 0$, $\langle\varphi,g\rangle=p$. We have $\sigma_1,\sigma_2\in \mathrm{U}$ such that $(L+\Sigma_a)\varphi_i=\sigma_i\varphi_i$, $i=1,2$.

Define

$$\sigma=\frac{t\sigma_1\varphi_1+(1-t)\sigma_2\varphi_2}{t\varphi_1+(1-t)\varphi_2}$$

then $\sigma\in \mathrm{U}$, $(L+\Sigma_a)\varphi=\sigma\varphi$.

2) Closeness. Consider a convergent sequence $\{\varphi_n\}_1^\infty\subset V$, $\varphi_n\to\varphi$ in L^2.

It is obvious that $\langle\varphi,g\rangle=p$. For any $f\in C^\infty(\overline{\Omega})$ with $f\geq 0$, $\int_\Omega\varphi f dx=\lim_{n\to\infty}\int_\Omega\varphi_n f dx\geq 0$, thus $\varphi\geq 0$, a.e $x\in\Omega$. Suppose that $\sigma_i\in u$, $(L+\Sigma_a)\varphi_i=\sigma_i\varphi_i$, U is a bounded set in $L^\infty(\Omega)$, so that it is sequentially weak compact. We may assume that $\{\sigma_i\}$ converges to $\sigma\in \mathrm{U}$. For any $f\in C_0^\infty(\overline{\Omega})$, $\langle\varphi_i,(L+\Sigma_a)f\rangle=\langle\varphi_i,\sigma_i f\rangle$. Let $i\to\infty$, we obtain $\langle\varphi,(L+\Sigma_a)f\rangle=\langle\varphi,\sigma f\rangle,\forall f\in C_0^\infty(\overline{\Omega})$, which implies that $\varphi\in V$.

Lemma 2 Let λ^* be the first eigenvalue of the operator $L+\Sigma_a$, and $\varphi^*\geq 0$ be the corresponding eigenfunction: $(L+\Sigma_a)\varphi^*=\lambda^*\varphi^*$. Then the set V is nonempty if

$$c_1\leq\lambda^*\leq c_2$$

Proof The part of "if" is evident. Suppose that $\varphi\in V$, $\sigma\in \mathrm{U}$. $(L+\Sigma_a)\varphi=\sigma\varphi$. By making scalar production with φ^*, we get

$$\begin{aligned}\langle\sigma\varphi,\ \varphi^*\rangle&=\langle(L+\Sigma_a)\varphi,\ \varphi^*\rangle\\&=\langle\varphi,\ (L+\Sigma_a)\varphi^*\rangle=\lambda^*\langle\varphi,\ \varphi^*\rangle\end{aligned}$$

Since $c_1\leq\sigma\leq c_2$,

$$c_1\langle\varphi_1, \varphi^*\rangle\leq\langle\sigma\varphi, \varphi^*\rangle\leq c_2\langle\varphi, \varphi^*\rangle$$

thus $c_1\leq\lambda^*\leq c_2$, for $\langle\varphi, \varphi^*\rangle\neq 0$.

We now state the main result of this paper.

Theorem 3 Suppose that $c_1\leq\lambda^*\leq c_2$, then (6) has an unique optimal solution φ_0, satisfying

$$\langle\varphi_0-\varphi_d, \varphi-\varphi_0\rangle\geq 0, \quad \forall\varphi\in V \tag{7}$$

Proof Since V is a non-empty closed convex set, the existence and uniqueness of φ_0 and the optimal condition (7) follow from the general result of convex analysis. We simply recall:

Theorem A[3]. Let V be a nonempty closed convex set in a Hilbert space H, φ_d be a fixed point in H, then there exists an unique point φ^0 in V such that

$$\|\varphi_0-\varphi_d\|=\inf_{\varphi\in V}\|\varphi-\varphi_d\|$$

Moreover φ_0 is the optimal solution if (7) holds.

Remark Strictly speaking, we should show the uniqueness of the optimal critical fission cross section σ, which can be done in the following way. Suppose that $(L+\Sigma_a)\varphi=\sigma_i\varphi$, $\varphi\in V$, $\sigma_i\in U$ $i=1,2$, then $\sigma_1\varphi=\sigma_2\varphi$. Since $\varphi>0$ for a.e $x\in\Omega$, $\sigma_1=\sigma_2$.

3. Optimality condition

Although (7) is really an optimal condition for φ_0, However it is not convenient for use, since the structure of V is not clear. In this section, we introduce a more analytical condition. Define, at first, the dual state ϕ satisfying the equation

$$(L+\Sigma_a)\phi=\sigma_0\phi+(\varphi_0-\varphi_d)+\alpha g \tag{8}$$

where the constant α is to be chosen. For (8) being solvable, we must have

$$\langle\varphi_0-\varphi_d+\alpha g, \varphi_0\rangle=0$$

or

$$\alpha=-\frac{1}{p}\langle\varphi_0-\varphi_d, \varphi_0\rangle$$

In this case, using a special solution (8), we can simplify (7) and obtain

$$\begin{aligned}\langle\varphi_0-\varphi_d, \varphi-\varphi_0\rangle&=\langle(L+\Sigma_a)\phi-\sigma_0\phi-\alpha g, \varphi-\varphi_0\rangle\\&=\langle(L+\Sigma_a)\phi-\sigma_0\phi, \varphi-\varphi_0\rangle=\langle\phi, (L+\Sigma_a-\sigma_0)(\varphi-\varphi_0)\rangle\\&=\langle\phi, (\sigma-\sigma_0)\varphi\rangle\end{aligned}$$

Thus the optimality condition (7) can be rewritten as

$$\langle(\sigma-\sigma_0)\varphi,\ \phi\rangle \geq 0,\quad \forall\varphi\in V \tag{9}$$

To simplify (9) further, we need some lemmas in [2].

Lemma 4 The solution mapping $\sigma_f \longmapsto (\varphi(\sigma_f),\ \lambda(\sigma_f))$ defined by (1) with the restriction (2) is a Frèchet differentiable mapping from $\sigma_f \in U$ (with $L^\infty(\Omega)$-topology) into $L^2(\Omega)\times R$

Lemma 5 For any $h\in L^\infty(\Omega)$ and $t\in R$ small enough, set $\sigma_t=\sigma_0+th$, $\varphi_t=\varphi(\sigma_t)$, $\lambda_t=\lambda(\sigma_t)$, then

$$\left.\frac{d}{dt}\lambda_t\right|_{t=0}=\frac{\lambda_0\langle h\varphi,\ \varphi_0\rangle}{\langle\sigma_0\varphi_0,\ \varphi_0\rangle}$$

Lemma 6 Suppose that $\varphi\in V$, $(L+\Sigma_a)\varphi=\sigma\varphi$, $h\in L^\infty(\Omega)$, $\sigma+h\in U$, $\int h\varphi^2=0$ and $|h|\geq\varepsilon>0$, $\forall$a·e $x\in\Omega$, then there exists $t_0>0$ and $\varphi_t\in V$ for $0\leq t<t_0$ such that

$$(L+\Sigma_a)\varphi_t=\sigma_t\varphi_t$$

where

$$\sigma_t=\sigma+t\,h+0(t)$$

Proof Let $\tilde{\sigma}_t=\sigma+t\,h$, $\varphi_t=\varphi(\tilde{\sigma}_t)$, $\lambda_t=\lambda(\tilde{\sigma}_t)$. By Lemma 5, $\lambda_t=\lambda(\sigma)+0(t)=1+0(t)$, thus

$$(L+\Sigma_a)\varphi_t=(1+0(t))\,\tilde{\sigma}_t\varphi_t$$

Denote $\sigma_t=(1+0(t))\,\tilde{\sigma}_t=\sigma+t\,h+0(t)$, we will show that $\sigma_t\in U$ as t is small enough. In fact, if $h<-\varepsilon$, then

$$\begin{aligned}\sigma_t&=\sigma+th+0(t)\\&\leq\sigma-t\varepsilon+0(t)\leq\sigma\leq c_2;\end{aligned}$$

if $h>\varepsilon$, we have

$$\begin{aligned}\sigma_t&=\sigma+\frac{3}{2}t\,h-\frac{1}{2}t\,h+0(t)\\&\leq\sigma+\frac{3}{2}t\,h\leq\sigma+h\leq c_2\end{aligned}$$

Similarly, $\sigma_t\geq c_1$ as t is small enough.

Theorem 7 φ_0 is the optimal solution of (6) if

$$\int_\Omega(\sigma(x)-\sigma_0(x))\phi(x)\varphi_0(x)dx\geq 0 \tag{10}$$

for any $\sigma(x)\in U$ satisfying

$$\int_\Omega\sigma(x)\,\varphi_0^2(x)dx=\int_\Omega\sigma_0(x)\,\varphi_0^2(x)dx \tag{11}$$

where $\phi(x)$ is defined as (8).

Proof "Only if". Set $h(x)=\sigma(x)-\sigma_0(x)$. First we assume that h satisfies the assumption of Lemma 6, that is, $|h(x)|\geq\varepsilon>0$. In this case, by Lemma 6, we have $\sigma_t=\sigma+th+0(t)$, $\varphi_t=\varphi(\sigma_t)\in V$ and

$$\int_\Omega(\sigma_t-\sigma_0)\phi\varphi_t\geq 0$$

Let $t\to 0^+$, we obtain (10). For a function $\sigma\in L^\infty(\Omega)$ in general, we argue by an approximation.

Lemma 8 Let $\sigma\in U$, $\sigma\not\equiv\sigma_0$, satisfying (11), then we can find $\sigma_\varepsilon\in U$ such that σ_ε satisfies (11), $|\sigma_\varepsilon-\sigma_0|\geq\varepsilon>0$ and $\sigma_\varepsilon\to\sigma$ in $L^\infty(\Omega)$ as $\varepsilon\to 0$.

Proof Denote $\Omega_a^b=\{x\in\Omega|a\leq\sigma(x)-\sigma_0(x)\leq b\}$. Since $\int_\Omega\sigma\varphi_0^2dx=\int_\Omega\sigma_0\varphi_0^2dx, \varphi_0>0, \sigma-\sigma_0\not\equiv 0$, there exist $\alpha,\delta>0$ s.t. mes $\Omega_\alpha^{+\infty}\geq\delta$, mes $\Omega_{-\infty}^{-\alpha}\geq\delta$. Define σ_ε as following: $\sigma_\varepsilon(x)=\pm\varepsilon+\sigma_0(x)$ in $\Omega_{-\varepsilon}^{\varepsilon}$, where we choose the sign of $\pm\varepsilon$ so that $c_1\leq\sigma_\varepsilon(x)\leq c_2$. This is always possible provided $c_1-c_2\geq 2\varepsilon$. If $\int_{\Omega_{-\varepsilon}^{\varepsilon}}\sigma_\varepsilon\varphi_0^2dx=\int_{\Omega_{-\varepsilon}^{\varepsilon}}\sigma_0\varphi_0^2dx$, we define simply $\sigma_\varepsilon=\sigma$ in $\Omega\setminus\Omega_{-\varepsilon}^{\varepsilon}$. Conversely, suppose, for example, $\int_{\Omega_{-\varepsilon}^{\varepsilon}}\sigma_\varepsilon\varphi_0^2\,dx>\int_{\Omega_{-\varepsilon}^{\varepsilon}}\sigma\varphi_0^2dx$, we define

$$\sigma_\varepsilon(x)=\begin{cases}\sigma(x)-d & \text{in } \Omega_\alpha^{+\infty}\\ \sigma(x) & \text{in } \Omega\setminus(\Omega_\alpha^{+\infty}\cup\Omega_{-\varepsilon}^{\varepsilon})\end{cases}$$

where d is chosen so that $\int_\Omega\sigma_\varepsilon\varphi_0^2dx=\int_\Omega\sigma\varphi_0^2dx$. Notice that $d=0(\varepsilon)$, so $c_1\leq\sigma_\varepsilon(x)\leq c_2$. Moreover

$$|\sigma_\varepsilon-\sigma_0|\geq|\sigma(x)-\sigma_0(x)|-|\sigma_\varepsilon(x)-\sigma(x)|\geq\alpha-d\geq\varepsilon,\quad\forall x\in\Omega_\alpha^{+\infty}$$

and

$$|\sigma_\varepsilon-\sigma|\leq 0(\varepsilon)$$

Now return to prove Theorem 7. Take σ_ε as the approximation sequence in Lemma 8, then

$$\int(\sigma_\varepsilon-\sigma_0)\phi\,\varphi_\varepsilon\geq 0$$

Let $\varepsilon \to 0^+$, we obtain again (10).

The "if" part of Theorem 7. We show that (9) follows from (10), (11). Set $\Sigma_\varepsilon = \{x \in \Omega | \varphi_0(x) \leq \varepsilon\} \cup \left\{x \in \Omega | \varphi(x) \geq \frac{1}{\varepsilon}\right\}$, then mes $\Sigma_\varepsilon \to 0$, as $\varepsilon \to 0$. If $\sigma = \sigma_0$ in Σ_ε for some $\varepsilon > 0$, let

$$h(x) = \begin{cases} 0 & x \in \Sigma_\varepsilon \\ \dfrac{\sigma(x) - \sigma_0(x)}{\varphi_0(x)} \varphi(x) & \text{otherwise.} \end{cases}$$

For t small enough, $\sigma_t = \sigma_0 + th \in \mathrm{U}$ and $\int_\Omega \sigma_t \varphi_0^2 dx = \int_\Omega \sigma_0 \varphi_0^2 dx$, since $\int_\Omega \sigma \sigma_0 (\varphi - \varphi_0) dx = 0$. From (10),

$$\int_\Omega (\sigma - \sigma_0) \phi \varphi dx \geq 0$$

For σ in general, we choose $l \in L^\infty$, such that $l = 0$ in Σ_ε for some $\varepsilon > 0$ and $\int_\Omega \sigma \sigma_0 (\varphi - \varphi_0) l = 0$. Set

$$h(x) = \begin{cases} \dfrac{\sigma(x) - \sigma_0(x)}{\varphi_0(x)} \varphi(x) l(x) & X \in \Omega \setminus \Sigma_\varepsilon \\ 0 & X \in \Sigma_\varepsilon \end{cases}$$

$$\sigma_t = \sigma_0 + th$$

then

$$\int_\Omega \sigma_t \varphi_0^2 dx = \int_\Omega \sigma_0 \varphi_0^2 \, dx + t \int_\Omega (\sigma - \sigma_0) \varphi \ \varphi_0 \ l \ dx = \int \sigma_0 \varphi_0^2 dx$$

and

$$\int_\Omega (\sigma - \sigma_0) l \ \phi \ \varphi \ dx \geq 0$$

Using an argument of approximation, we get

$$\int_\Omega (\sigma - \sigma_0) \phi \varphi \ dx \geq 0$$

Remark Condition (10) is of the bang-bang type. In fact, from (10) we have

$$\sigma_0(x) = \begin{cases} c_1 & \text{if } \phi(x) > 0 \\ c_2 & \text{if } \phi(x) < 0 \end{cases}$$

References

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核反应堆裂变截面的辨识

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摘　要

本文考虑反应堆裂变截面的辨识问题。反应堆处于临界状态，由二阶椭圆算子的本征问题刻划。给出了最优解的存在性、唯一性以及最优解的充分必要条件。