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On the Stability of Linear Time-variant Large-scale Systems*

Liu Jinxian

(Department of Mathematic, Hebei Chemical Engineering Institute, Shijiazhuang)

Liu Yongqing

(Department of Automation, South China University of Technology, Guangzhou)

Abstract

In this paper, the stability test of high-order, time-variant and large-scale system is simplified to that of a lower-order time-invariant one by using comparison principle and Lyapunov function, whose derivative is noncontinuous. In the meantime, stability conditions of subsystems are omitted and those of time variant large-scale system are widened.

1. Background

The stability property of systems whose coefficients are variant with time t is significant for both ordinary differential equation theory and engineering technology. The study of stability of time-variant systems is rather a complicated problem, so far the sufficient and necessary condition for stability of linear one is not yet found. Hacker [1] proposed the "coefficient-freezing" method in 1970, however there is not only difference but some opposite case between the frozen system and the original on their stability properties. Rosenbrock [2] proposed that the derivative of coefficients should be limitted when using the "coefficient-freezing" method, but he studed only a specifical class of systems.

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Qin Yuan-xun, Wang Mu-qiu and Wang Lian [3] showed the stability property of systems with slowly-changing coefficients by constructing Lyapunov function. Wang Mu-qiu, Tian Xiu-gong [4] investigated the stability of a class of time-variant, large-scale systems (supposed the coefficient matrixes of subsystems are symmetric and real, and their eigenvalues all are negatve) by using comparison principle and vector Lyapunov function. Zhang Ze-mian and Zhang Hong-liang [5] also studied the stability of this class of large scale systems by using scalar-sum Lyapunov function.

In present paper, the stability test of high-order, timevariant, large-scale system is simplified to that of lower-order, time-invariant auxiliary one by using comparison principle combined with Lyapunov function, whose derivative is noncontinuous. In the meantime the stability conditions of subsystems are omitted and those of time-variant whole system are widened.

Definition [6]. Functional $\varphi_i(x_i, \dot{x}_i)$ is defined as follow;

$$\varphi_i(x_i, \dot{x}_i) = \begin{cases} 1, \text{ if } x_i>0; \text{ or if } x_i=0 \text{ and } \dot{x}_i>0 \\ 0, \text{ if } x_i=0 \text{ and } \dot{x}_i=0 \\ -1, \text{ if } x_i<0; \text{ or if } x_i=0 \text{ and } \dot{x}_i<0 \end{cases}$$

Lemma [6]. B is a matrix whose diagonal elements are negative while the others nonnegative. (BX also belongs to the class H). If $X(t,X_0,t_0)$ and $Y(t,Y_0,t_0)$ are solutions of

$$D^+X \leqslant BX$$

$$\dot{Y} = BY$$

resp., and if $X_0 \leqslant Y_0$ then $X(t, X_0, t_0) \leqslant Y(t, r_0, t_0)$ is true for all $t_0 \leqslant t < +\infty$.

2. Stability test of time-variant systems

Consider n first-order, linear time-variant systems

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j \qquad (i=1,2,\cdots,n) \tag{1}$$

Suppose their coefficients $a_{ij}(t)$ all are continuous and bounded and

$$a_{ii}(t) \leqslant -\alpha < 0 (i=1,2,\cdots,n) \tag{2}$$

$$\text{Denote}\begin{cases} \overline{a}_{ii}=\sup\limits_{t_0\leqslant t<+\infty}\{a_{ii}(t)\} \quad (i=1,2,\cdots,n) \\ \overline{a}_{ij}=\sup\limits_{t_0\leqslant t<+\infty}\{|a_{ij}(t)|\} \quad (i\neq j,\ i,j=1,2,\cdots,n) \end{cases} \tag{3}$$

Letting $$V_i=|x_i| \quad (i=1,2,\cdots,n)$$

then
$$\begin{aligned} D^+V_{1(1)} &= \varphi_1\dot{x}_1 = \varphi_1 a_{11}(t)x_1+\varphi_1 a_{12}(t)x_2+\cdots+\varphi_1 a_{1n}(t)x_n \\ &\leqslant a_{11}(t)|x_1|+|a_{12}(t)||x_2|+\cdots+|a_{1n}(t)||x_n| \\ &\leqslant \overline{a}_{11}V_1+\overline{a}_{12}V_2+\cdots+\overline{a}_{1n}V_n \end{aligned}$$

Similarly
$$D^+V_{2(1)}\leqslant \overline{a}_{21}V_1+\overline{a}_{22}V_2+\cdots+\overline{a}_{2n}V_n$$
$$\cdots\cdots\cdots\cdots$$
$$D^+V_{n(1)}\leqslant \overline{a}_{n1}V_1+\overline{a}_{n2}V_2+\cdots+\overline{a}_{nn}V_n$$

Consider the auxiliary set of equations

$$\begin{cases} \dfrac{dV_1^*}{dt}=\overline{a}_{11}V_1^*+\overline{a}_{12}V_2^*+\cdots+\overline{a}_{1n}V_n^* \\ \dfrac{dV_2^*}{dt}=\overline{a}_{21}V_1^*+\overline{a}_{22}V_2^*+\cdots+\overline{a}_{2n}V_n^* \\ \cdots\cdots\cdots\cdots \\ \dfrac{dV_n^*}{dt}=\overline{a}_{n1}V_1^*+\overline{a}_{n2}V_2^*+\cdots+\overline{a}_{nn}V_n^* \end{cases} \tag{4}$$

The zero-solution for (4) is asymptotically stable if eigenvalues of its coefficient matrix all have negative real parts.

Under the lemma it follows that

$$V_i(t,V_i^0,t_0)\leq V_i^*(t,V_i^0,t_0) \quad (i=1,2,\cdots,n)$$

Hence $$\lim_{t\to\infty} V_i(t,V_i^0,t_0)=0 \quad (i=1,2,\cdots,n)$$

Therefore $$\lim_{t\to\infty} x_i(t)=0 \quad (i=1,2,\cdots,n)$$

i. e. the zero solution for system (1) is also asymptotically stable. Thus the following theorem is obtained.

Theorem 1 The zero-solution for system (1) is asymptotically stable if their coefficients are continuous and bounded and satisfy the condition $a_{ii}(t)\leq -a<0 (i=1,2,\cdots,n;\ \forall t\geq t_0)$, and if eigenvalues of auxiliary equation (4) all have negative real parts.

3. Order-reducing for Time-variant Large-scale Systems in the Stability Theory

First take a third-order system as an example.

$$\begin{cases} \dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 \\ \dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3 \\ \dot{x}_3 = a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 \end{cases} \tag{5}$$

Suppose its coefficients $a_{ij}(t)$ all are continuous and bounded and suppose $a_{ii}(t) \leq -\alpha < 0$ $(i = 1, 2, \cdots, n;\ \forall t \geq t_0)$. Matrix A is divided into 2×2 blocks

$$A(t) = \left(\begin{array}{cc|c} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ \hline a_{31}(t) & a_{32}(t) & a_{33}(t) \end{array} \right) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \tag{6}$$

Let

$$\begin{aligned} b_{11} &= \sup_{t_0 \leq t < +\infty} \{ a_{11}(t) + |a_{21}(t)|, a_{22}(t) + |a_{12}(t)| \} \\ b_{12} &= \sup_{t_0 \leq t < +\infty} \{ |a_{13}(t)|, |a_{23}(t)| \} \\ b_{21} &= \sup_{t_0 \leq t < +\infty} \{ |a_{31}(t)|, |a_{32}(t)| \} \\ b_{22} &= \sup_{t_0 \leq t < +\infty} \{ a_{33}(t) \} \end{aligned} \tag{7}$$

Choosing that $V_1 = (|x_1| + |x_2|)$, $V_2 = |x_3|$

then we have that

$$\begin{aligned} D^+V_{1(5)} &= \varphi_1 \dot{x}_1 + \varphi_2 \dot{x}_2 = \sum_{j=1}^{3} \varphi_1 a_{1j}(t)x_j + \sum_{j=1}^{3} \varphi_2 a_{2j}(t)x_j \\ &\leq (a_{11}(t) + |a_{21}(t)|)|x_1| + (a_{22}(t) + |a_{12}(t)|)|x_2| \\ &\qquad + (|a_{13}(t)| + |a_{23}(t)|)|x_3| \\ &\leq b_{11}(|x_1| + |x_2|) + b_{12}|x_3| = b_{11}V_1 + b_{12}V_2 \end{aligned}$$

$$\begin{aligned} D^+V_{2(5)} &= \varphi_3 \dot{x}_3 = \varphi_3 a_{31}(t)x_1 + \varphi_3 a_{32}(t)x_2 + \varphi_3 a_{33}(t)x_3 \\ &\leq |a_{31}(t)||x_1| + |a_{32}(t)||x_2| + a_{33}(t)|x_3| \\ &\leq b_{21}(|x_1| + |x_2|) + b_{22}|x_3| = b_{21}V_1 + b_{22}V_2 \end{aligned}$$

Consider the auxiliary set of equations

$$\begin{cases} \dfrac{dV_1^*}{dt} = b_{11} V_1^* + b_{12} V_2^* \\ \dfrac{dV_2^*}{dt} = b_{21} V_1^* + b_{22} V_2^* \end{cases} \tag{8}$$

If coefficients of the auxiliary equations satisfy

i) $b_{11} \leqslant -\beta < 0$, $b_{22} \leqslant -\beta < 0$, β is a positive constant; and

ii) eigenvalues both have negative real parts;

then the zero-solution of system (8) is asymptotically stable. According to the lemma we have inequalities

$$V_1(t, V_1^0, t_0) \leqslant V_1^*(t, V_1^0, t_0), \quad V_2(t, V_2^0, t_0) \leqslant V_2 (t, V_2^0, t_0)$$

Hence $\lim\limits_{t\to\infty} V_1(t, V_1^0, t_0) = \lim\limits_{t\to\infty} V_2(t, V_2^0, t_0) = 0$

while $V_1 = |x_1| + |x_2|$, $V_2 = |x_3|$

therefore we have $\lim\limits_{t\to\infty} x_i(t) = 0 \quad (i = 1, 2, 3)$.

i. e. the system (5) is asymptotically stable.

Here the asymptotical stability of time-variant third-order system (5) is deduced from that of time-invariant second-order system (8), and the order-reducing is accomplished of systems with variant coefficients.

Let's now concern ourselves with the order-reducing of nth-order large-scale system in the stability.

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t) x_j \quad (i = 1, 2, \cdots, n) \tag{9}$$

Also rewrite it in the form $\dot{x} = A(t)x$ (9)'

Suppose that $a_{ij}(t)$ are all continuous and bounded, and that $a_{ii}(t) \leqslant -\alpha < 0$ $(i = 1, 2 \cdots, n;\ \forall t \geqslant t_0)$. The coefficient matrix $A(t)$ is divided into $m \times m$ $(m < n)$ blocks

$$A(t) = \begin{pmatrix} A_{11}(t) \cdots A_{1m}(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ A_{m1}(t) \cdots A_{mm}(t) \end{pmatrix} \tag{10}$$

Where $A_{rs}(t)$ is $n_r \times n_s$ submatrix, $(r, s = 1, \cdots, m)$, $n_1 + \cdots + n_m = n$.

Let
$$\left\{\begin{array}{l} b_{11}=\sup\limits_{\substack{t_0\leqslant t<+\infty\\ j=1,\cdots,n_1}}\left\{a_{jj}(t)+\sum\limits_{i=1,i\neq j}^{n_1}|a_{ij}(t)|\right\} \\ b_{12}=\sup\limits_{\substack{t_0\leqslant t<+\infty\\ j=n_1+1,\cdots,n_1+n_2}}\left\{\sum\limits_{i=1}^{n_1}|a_{ij}(t)|\right\} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{1m}=\sup\limits_{\substack{t_0\leqslant t<+\infty\\ j=n-n_m+1,\cdots,n}}\left\{\sum\limits_{i=1}^{n_1}|a_{ij}(t)|\right\} \end{array}\right. \tag{11}$$

$$\left\{\begin{array}{l} b_{r1}=\sup\limits_{\substack{t_0\leqslant t<+\infty\\ j=1,\cdots,n_1}}\left\{\sum\limits_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r}|a_{ij}(t)|\right\} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{rr}=\sup\limits_{\substack{t_0\leqslant t<+\infty\\ j=n_1+\cdots+n_{r-1}+1,\cdots,n_1+\cdots+n_r}} \\ \qquad\left\{a_{jj}(t)+\sum\limits_{\substack{i=n_1+\cdots+n_{r-1}+1\\ i\neq j}}^{n_1+\cdots+n_r}|a_{ij}(t)|\right\}\quad (r=2,3,\cdots m) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b_{rm}=\sup\limits_{\substack{t_0\leqslant t<+\infty\\ j=n-n_m+1,\cdots,n}}\left\{\sum\limits_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r}|a_{ij}(t)|\right\}\quad (r=2,3,\cdots,m-1) \end{array}\right. \tag{12}$$

Letting $V_1=\sum\limits_{i=1}^{n_1}|x_i|,\cdots,V_m=\sum\limits_{i=n-n_m+1}^{n}|x_i|$

then
$$D^+V_{1(9)}=\sum_{i=1}^{n_1}\varphi_i\dot{x}_{i(9)}=\sum_{i=1}^{n_1}\left[\varphi_i\sum_{j=1}^{n}a_{ij}(t)x_j\right]$$
$$=\sum_{i=1}^{n_1}\left[\sum_{j=1}^{n_1}\varphi_i a_{ij}(t)x_j+\sum_{j=n_1+1}^{n_1+n_2}\varphi_i a_{ij}(t)x_j+\cdots+\sum_{j=n-n_m+1}^{n}\varphi_i a_{ij}(t)x_j\right]$$
$$\leqslant\sum_{j=1}^{n_1}\left[a_{jj}(t)|x_j|+\sum_{i=1,i\neq j}^{n_1}|a_{ij}(t)||x_j|\right]+\sum_{j=n_1+1}^{n_1+n_2}\sum_{i=1}^{n_1}|a_{ij}(t)||x_j|$$
$$+\cdots+\sum_{j=n-n_m+1}^{n}\sum_{i=1}^{n_1}|a_{ij}(t)||x_j|$$
$$\leqslant b_{11}V_1+b_{12}V_2+\cdots+b_{1m}V_m$$

$$D^+V_{r(9)}=\sum_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r}\varphi_i\dot{x}_i=\sum_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r}\left[\varphi_i\sum_{j=1}^{n}a_{ij}(t)x_i\right]$$

$$= \sum_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r} \left[\sum_{j=1}^{n_1} \varphi_i a_{ij}(t) x_j + \sum_{j=n_1+1}^{n_1+n_2} \varphi_i a_{ij}(t) x_j + \cdots + \sum_{j=n-n_m+1}^{n} \varphi_i a_{ij}(t) x_j \right]$$

$$\leqslant \sum_{j=1}^{n_1} \sum_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r} |a_{ij}(t)||x_j| + \cdots + \sum_{j=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r} \left[a_{jj}(t) + \sum_{\substack{i=n_1+\cdots+n_{r-1}+1 \\ i\neq j}}^{n_1+\cdots+n_r} |a_{ij}(t)| \right] |x_j|$$

$$+ \cdots + \sum_{j=n-n_m+1}^{n} \sum_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r} |a_{ij}(t)||x_j|$$

$$\leqslant b_{r1}V_1 + b_{r2}V_2 + \cdots + b_{rr}V_r + \cdots + b_{rm}V_m \quad (r = 2, \cdots, m-1)$$

$$D^+V_{m\,(9)} = \sum_{i=n-n_m+1}^{n} \varphi_i \dot{x}_i \leqslant \sum_{j=1}^{n_1} \sum_{i=n-n_m+1}^{n} |a_{ij}(t)||x_j| + \cdots + \sum_{j=n-n_m+1}^{n} \left[a_{jj}(t) + \sum_{\substack{i=n-n_m+1 \\ i\neq j}}^{n} |a_{ij}(t)| \right] |x_j|$$

$$\leqslant b_{m1}V_1 + b_{m2}V_2 + \cdots + b_{mm}V_m$$

Consider the auxiliary set of equations

$$\begin{cases} \dfrac{dV_1^*}{dt} = b_{11}V_1^* + b_{12}V_2^* + \cdots + b_{1m}V_m^* \\ \dfrac{dV_2^*}{dt} = b_{21}V_1^* + b_{22}V_2^* + \cdots + b_{2m}V_m^* \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{dV_m^*}{dt} = b_{m1}V_1^* + b_{m2}V_2^* + \cdots + b_{mm}V_m^* \end{cases} \tag{13}$$

If the coefficient matrix of the auxiliary equations (13) satisfies

i) $b_{ii} \leqslant -\beta < 0, i = 1, 2, \cdots, m, \beta$ is a positive constant; and

ii) all the eigenvalues have negative real parts, then the zero-solution for (13) is asymptotically stable.

According the lemma it is known that

$$V_r(t, V_r^0, t_0) \leqslant V_r^*(t, V_r^0, t_0) \quad (r = 1, \cdots, m)$$

Hence $\lim\limits_{t\to\infty} V_r(t,V_r^0\ ,t_0)=0 \quad (r=1,\cdots,m)$

while $V_r=\sum\limits_{i=n_1+\cdots+n_{r-1}+1}^{n_1+\cdots+n_r} |x_i| \quad (r=1,\cdots,m)$

therefor $\lim\limits_{t\to\infty} x_i(t)=0 \quad (i=1,\cdots,n)$

i. e. the zero-solution of system (9) is asymptotically stable. Here the stability study of nth-order, linear, time-variant system (9) is simplified to that of mth-order, time-invariant, auxiliary system (13), and the asymptotical stability property of the former is derived from that of the later, so the order-reducing for large-scale system is accomplished. Thus it follows that:

Theorem 2. If coefficients of linear system (9) with variant coefficients are continuous and bounded, and if $a_{ii}\ (t)\leqslant-\alpha<0\ \ (i=1,\cdots,n;\ \forall t\geqslant t_0)$, and if the coefficients of its auxiliary equations (13) satisfy

i) the diagonal elements are negative; and

ii) its eigenvalues all have negative real parts, then the zero-solution for (9) is asymptotically stable.

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线性时变连续大系统的稳定性

刘金宪 刘永清

（河北化工学院数学系，石家庄） （华南理工大学自动化系，广州）

摘 要

本文应用导数不连续的李雅普诺夫函数结合比较原理，把时变高维大系统稳定性判定，简化为低维定常系统稳定性判定，省略了子系统稳定性判定条件，减弱了时变大系统稳定性条件。