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Design of State Feedback Law Possessing Integrity Against Actuator Failures

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Abstract

This paper is concerned with the design of state feedback law which possesses integrity against actuator failures. Some sufficient conditions for the existence of such state feedback law and a numerical example are given in the paper.

Key words——Integrity; State feedback lows; Actuator; Failures

1. Introduction

For some fields requiring high reliable control systems, the feedback control law should make close loop system stable even in the presence of some actuator failures. If a control law satisfy this requirement, then it is said to possess integrity.

If a plant is stable, then there exist state feedback control laws that can preserve the stability of close loop system in the presence of arbitrary actuator failures [1]. Joshi[2] studied the integrity problem for the case in which the plants may have unstable dynamics. But the conditions for the existence of control law possessing integrity given by him are rigorous, so these conditions have limited uses.

This paper investigates the application of Bass' algorithm of designing state feedback law to the design of control law possessing integrity against actuator failures. It is proved that Bass' algorithm can design a state feedback law that have infinite gain margin for each input channel. Based on the property, the state feedback law with integrity can be obtained by adjusting the gain of each input

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channel properly.

2. State Feedback Law Possessing Integrity

Consider the following linear time-invariant multi-input plant

$$\dot{x} = Ax + Bu \tag{1}$$

where, the pair (A, B) is controllable and $x \in R^n$, $u \in R^r$.

Take the state feedback law as

$$u = Kx = -B'P^{-1}x \tag{2}$$

where, P is the symmetric positive definite solution of the following Lyapunov equation

$$P(A+\beta I)' + (A+\beta I)P = 2BB' \tag{3}$$

and β in (3) is an arbitrary real number greater than $\|A\|$ which will make $-(A+\beta I)$ asymptotically stable.

Theorem 1 (Bass' algorithm[3]) If we apply the state feedback law (2) to the plant (1), then the close loop system

$$\dot{x} = (A+BK)x \tag{4}$$

is asymptotically stable.

Proof Since the pair (A, B) is controllable, it is easy to see that $(A+\beta I, B)$ is also controllable. So the equation (3) has a unique symmetric positive definite solution P provided that $-(A+\beta I)$ is asymptotically stable, which is guaranteed by the choice of β. Rewriting (3) as

$$(A-BB'P^{-1})P + P(A-BB'P^{-1})' = -2\beta P \tag{5}$$

we know from the Lyapunov theorem that $(A-BB'P^{-1})$ is asymptotically stable.

Now let

$$u = K'x = \text{diag}(\alpha_1, \cdots, \alpha_r)Kx = -\text{diag}(\alpha_1, \cdots, \alpha_r)B'P^{-1}x \tag{6}$$

then the variation range of α_i that will preserve the stability of the following system

$$x = (A+BK')x \tag{7}$$

is defined as the gain margin of the ith input channel of (4) [4].

Theorem 2 If we apply the state feedback law (2) to the plant (1), then the close loop system (4) has a gain margin $[1, \infty)$ for each input channel.

Proof By using (5), we have

$$
\begin{aligned}
&P(A-B\mathrm{diag}(\alpha_1,\cdots,\alpha_r)B'P^{-1})'+(A-B\mathrm{diag}(\alpha_1,\cdots,\alpha_r)B'P^{-1})P\\
&=P(A-BB'P^{-1}+B\mathrm{diag}(1-\alpha_1,\cdots,1-\alpha_r)B'P^{-1})'+(A-BB'P^{-1}\\
&\quad+B\mathrm{diag}(1-\alpha_1,\cdots,1-\alpha_r)B'P^{-1})P\\
&=P(A-BB'P^{-1})'+(A-BB'P^{-1})P+2B\mathrm{diag}(1-\alpha_1,\cdots,1-\alpha_r)B'\\
&=-2\beta P+2B\mathrm{diag}(1-\alpha_1,\cdots,1-\alpha_r)B'
\end{aligned}
\tag{8}
$$

Since P is positive definite, the right hand side of (8) will be negative definite when $\alpha_i\geqslant 1$ $(i=1,\cdots,r)$. Hence, $(A-B\mathrm{diag}(\alpha_1,\cdots,\alpha_r)B'P^{-1})$ is asymptotically stable provided that $\alpha_i\geqslant 1$ $(i=1,\cdots,r)$ from the Lyapunov Theorem. That is, the system (4) has at least $[1,\infty)$ gain margin for each input channel.

If we take (6) as state feedback law, which is equivalent to inserting a constant gain in each input channel of (4), then α_i can be taken as the parameters to be adjusted for integrity of close loop system. Joshi first explained this for LQG state feedback law in [2]. In the following, we will explain that this is also the case for taking (6) as state feedback law.

Only two states are distinguished for each actuator in the following: normal state and failed state. so, there are at most 2^r-1 failure states for a system having r actuators. In order to represent the kth actuator failure state, we define

$$D_K=\mathrm{diag}(d_{K1},\cdots,d_{Kr})$$

where,

$$
d_{Ki}=\begin{cases}0 & \text{the actuator } i \text{ fails in the kth failure state;}\\ 1 & \text{the actuator } i \text{ is normal in the kth failure state.}\end{cases}
$$

Thus, when the close loop system (7) is in the kth failure state, the real input going into the plant (1) can be expressed as

$$u=-D_k\mathrm{diag}(\alpha_1,\cdots,\alpha_r)B'P^{-1}x=-\mathrm{diag}(d_{K1}\alpha_1,\cdots,d_{kr}\alpha_r)B'P^{-1}x$$

Theorem 3 Take (6) as state feedback law, then it has integrity with respect to the kth failure state if

$$-\beta P+B\mathrm{diag}(1-d_{k1}\alpha_1,\cdots,1-d_{kr}\alpha_r)B'<0 \tag{9}$$

Proof when the system (7) change from the normal state to the kth failure state, its state equation becomes

$$\dot{x}=(A-B\mathrm{diag}(d_{k1}\alpha_1,\cdots,d_{kr}\alpha_r)B'P^{-1})x \tag{10}$$

Since

$$
\begin{aligned}
&P(A-B\mathrm{diag}(d_{K1}\alpha_1,\cdots,d_{Kr}\alpha_r)B'P^{-1})'+(A-B\mathrm{diag}(d_{k1}\alpha_1,\cdots,d_{kr}\alpha_r)B'P^{-1})P\\
&=-2\beta P+2B\mathrm{diag}(1-d_{k1}\alpha_1,\cdots,1-d_{kr}\alpha_r)B'
\end{aligned}
$$

(10) is asymptotically stable provided that the right hand side of the above equation is negative definite, i. e., (9) holds.

If the pair (A,B) is not completely controllable. then the solution P of (3) may not be positive definite. Hence the inverse of P may not exist.

Assume that the pair (A,B) is stabilizable. So, there exist a nonsingular matrix T which can transform (1) to the following form by the transformation $x=Tz$

$$\begin{vmatrix} \dot{z}_1 \\ \dot{z}_2 \end{vmatrix} = \begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} + \begin{vmatrix} B_1 \\ B_2 \end{vmatrix} u \tag{11}$$

where, A_1 includes all of the unstable modes of (1) and A_2 all of the stable modes. Thus, the pair (A_1, B_1) is controllable from the stabilizability of (A, B). Let B' be an arbitrary real number greater than $\|A_1\|$ which will make $-(A_1+\beta' I)$ asymptotically stable. Hence, the following Lyapunov equation

$$P_1(A_1+\beta' I)' + (A_1+\beta' I)P_1 = 2B_1B_1' \tag{12}$$

will have a unique positive definite solution P_1.

If we take

$$u = [K_1 \quad 0] \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \tag{13}$$

as state feedback law, then (11) becomes

$$\begin{vmatrix} \dot{z}_1 \\ \dot{z}_2 \end{vmatrix} = \begin{vmatrix} A_1+B_1K_1 & 0 \\ B_2K_1 & A_2 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \tag{14}$$

Therefore, the stability of (14) is guaranteed provided that the subsystem

$$\dot{z} = (A_1+B_1K_1)z \tag{15}$$

is asymptotically stable. So, we have the following corollary.

Corollary 1: If we take (6) as state feedback law for plant (1) except that $K=-B'P^{-1}$ is substituted by $K=[-B_1'P_1^{-1} \quad 0]T^{-1}$, then the close loop system (7) has integrity with respect to the kth failure state if:

$$-\beta' P_1 + B_1 \mathrm{diag}(1-d_{k1}\alpha_1, \cdots, 1-d_{kr}\alpha_r)B_1' < 0 \tag{16}$$

Proof If the condition (16) is satisfied for the kth failure state, then the system (15) with $K_1 = -\mathrm{diag}(\alpha_1,\cdots,\alpha_r)B_1{}'P_1^{-1}$ will be stable in this failure state from the theorem 3. So, the system (7) with $K=[-B_1{}'P_1^{-1}\ \ 0]T^{-1}$ has integrity against the kth failure state.

Assume that the indices of the normal and the failed actuators are $i_1,\cdots,i_{q(k)}$ and $j_1,\cdots,j_{(r-q(k))}$ respectively and the condition

$$R(b_{j1},\cdots,b_{j(r-q(k))})\subseteq R(b_{i1},\cdots,b_{iq(k)}) \tag{17}$$

is satisfied in the kth failure state, where $R(\cdot)$ denotes the range space of a matrix, b_i represents the ith column of B and $q(k)$ the number of failed actuator in the kth failure state. Then there exist a matrix S_k with $q(k)$ rows and $(r-q(k))$ columns, such that $(b_{j1},\cdots,b_{j(r-q(k))})$ can be expressed as

$$(b_{j1},\cdots,b_{j(r-q(k))}) = (b_{i1},\cdots,b_{iq(k)}S_k)$$

In this case, we have

$$\begin{aligned} &B\ \mathrm{diag}(1-d_{k1}\alpha_1,\cdots,1-d_{kr}\alpha_r)B' \\ &= (b_{i1},\cdots,b_{iq(k)})\mathrm{diag}(1-\alpha_{i1},\cdots,1-\alpha_{iq(k)})(b_{i1},\cdots,b_{iq(k)})' \\ &\quad + (b_{j1},\cdots,b_{j(r-q(k))})(b_{j1},\cdots,b_{j(r-q(k))})' \\ &= (b_{i1},\cdots,b_{iq(k)})\{\mathrm{diag}(1-\alpha_{i1},\cdots,1-\alpha_{iq(k)})+S_kS_k'\}(b_{i1},\cdots,b_{iq(k)})'\leqslant 0 \end{aligned} \tag{18}$$

provided that $\alpha_i\geqslant 1+\lambda_{\max}(S_kS_k')(i=i_1,\cdots,i_{q(k)})$, where, $\lambda_{\max}(\cdot)$ represents the largest eigenvalue of a matrix. Thus, we have the following theorem.

Theorem 4 If the condition (17) is satisfied for the kth failure state, then the state feedback law (6) has integrity against this failure state provided that α_i $(i=i_1,\cdots,i_{q(k)})$ is sufficiently large.

Proof If the condition (17) is satisfied for the kth failure state and α_i $(i=i_1,\cdots,i_{q(k)})$ are sufficiently large, then (18) will hold. So, the feedback law (6) has integrity against this failure state from the theorem 3.

Generally speaking, it is difficult for a failure state to satisfy the condition (17). But using the corollary (1) we can get a condition easy to satisfy in the following corollary.

Corollary 2: Suppose the condition

$$R(b_{1j1},\cdots,b_{1j(r-q(k))})\subseteq R(b_{1i1},\cdots,b_{1iq(k)}) \tag{19}$$

is satified, where, b_{1i} denotes the ith column of B_1. Then the state feedback law (6) with $K=[-B_1{}'P_1^{-1}\ \ 0]T^{-1}$ possess integrity with respect to the kth failure state provided that α_i $(i=i_1,\cdots,i_{q(k)})$ is sufficiently large.

Proof If the condition (19) is satisfied, then from the theorem 4 we have

$$-\beta' P_1 + B_1 \mathrm{diag}(1-d_{k1}\alpha_1,\cdots,1-d_{kr}\alpha_r)B_1'<0$$

provided that α_i $(i=i_1,\cdots,i_{q(k)})$ in (6) are sufficiently large. So, (6) possess integrity with respect to the kth failure state from the corollary 1.

Generally, the number of the unstable modes of a plant is very small compared with the order of the plant in many practical situations. So, the dimension of b_{1i} is much smaller than that of b_i. Thus, the condition (19) may be satisfied even though (17) does not hold for the kth failure state.

From the proving process of the corollary 2, we can see that if we let every α_i be sufficiently large regardless of the state of failures, then the state feedback law (6) with $K=[-B_1'P_1^{-1}\ 0]T^{-1}$ will possess integrity against all of the failure states that satisfy (19). Thus, we have the following theorem.

Theorem 5 There exist state feedback laws that have integrity against arbitrary failure states satisfying (19).

3. Example

Consider a double effect pilot plant evaporator [5]. Assume that the dynamics of the evaporator are represented by a fifthorder system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{bmatrix}=\begin{bmatrix}0.0 & 0.0 & -0.0034 & 0.0 & 0.0\\ 0.0 & -0.041 & 0.0013 & 0.0 & 0.0\\ 0.0 & 0.0 & -1.1471 & 0.0 & 0.0\\ 0.0 & 0.0 & -0.0036 & 0.0 & 0.0\\ 0.0 & 0.094 & 0.0057 & 0.0 & -0.051\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}$$

$$+\begin{bmatrix}-1.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.948\\ 0.916 & -1.0 & 0.0\\ -0.6 & 0.0 & 0.0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} \tag{20}$$

where, the variable $x_1, \cdots, x_5$ are all measurable. The open loop poles are$\{0.0, 0.0, -1.146, -0.041, -0.051\}$. So, the open loop system is not asymptotically stable. Examining the B in (20), we can see that the condition (17) can not be satisfied for any failure state. Now, we transform (20) into the following form:

$$\begin{vmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{vmatrix} = \begin{vmatrix} 0.0 & & & & \\ & 0.0 & & & \\ & & -1.146 & & \\ & & & -0.041 & \\ & & & & -0.51 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{vmatrix} + \begin{vmatrix} -1.0 & 0.0 & -0.0028 \\ 0.92 & -1.0 & -0.0029 \\ 0.0 & 0.0 & 0.9417 \\ 0.0 & 0.0 & 0.0011 \\ -0.6 & 0.0 & -0.0057 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} \tag{21}$$

So, we have

$$A_1 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \qquad B_1 = \begin{vmatrix} -1.0 & 0.0 & -0.0028 \\ 0.92 & -1.0 & -0.0029 \end{vmatrix}$$

It is easy to see that (19) is satisfied for any single actuator failures. So, there exist state feedback laws which have integrity against any single actuator failures by the theorem 5. This is also the best result we can expect since the plant (20) is not stabilizable under any other actuator failure states.

Now, we take $\beta' = 0.1$. Solving (12) and using (13), we get

$$K_1 = \begin{vmatrix} 9.999784e-2 & -1.519918e-6 \\ 9.199572e-2 & 9.999701e-2 \\ 7.837625e-4 & 5.475794e-4 \end{vmatrix}$$

When we take (6) with $K = [K_1 \ \ 0]T^{-1}$ as state feedback law for (20), the real part of the close loop poles in normal and failure conditions are all negative (see Table 1.). So, we have obtained a state feedback law possessing integrity against any single actuator failures without adjusting α_i.

Next, we make some variation for α_i: $\alpha_1 = 100, \alpha_2 = 100, \alpha_3 = 10000$, the corresponding real part of the close loop poles are listed in Table 2.

Table 1 Real part of close loop poles

normal state	$-1.000000e-1$	$-9.999999e-2$	-1.146	-0.041	-0.51
actuator 1 fails	$-7.897615e-7$	$-9.999999e-2$	-1.146	-0.041	-0.51
actuator 2 fails	$-1.000000e-1$	$-2.987593e-6$	-1.146	-0.041	-0.51
actuator 3 fails	$-9.999623e-2$	$-1.000000e-1$	-1.146	-0.041	-0.51

Table 2 Real part of close loop poles

normal state	-10.0374500	-10.000000	-1.146	-0.041	-0.51
actuator 1 fails	$-7.816315e-3$	-10.0297100	-1.146	-0.041	-0.51
actuator 2 fails	-10.00779	$-2.996212e-2$	-1.146	-0.041	-0.51
actuator 3 fails	-9.999623	-10.000000	-1.146	-0.041	-0.51

From Table 2, we can see that the (6) still has integrity when we enlarge α_i. This is fully pretictable from the theorem 2 and 5. It can also be seen that the degree of stability of the close loop system has been improved.

From the example, we can see that the degree of stability of the close loop system (7) can be manipulated by α_i. So, we can adjust α_i for the degree of stability of close loop system and for integrity.

4. Conclusion

A simple method for designing state feedback laws that have integrity against actuator failures is proposed in this paper. The main computational work this method involved is solving the Lyapunov equation. So, the method is a computationally simple one. In the method, the gain to be determined has a close relation with α_i or β, therefore, trade-off between gain and degree of stability or integrity of close loop system can be made easily. The example given in the paper illustrated the effectiveness of the method proposed.

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对执行器失效具有完整性的状态反馈律设计

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摘 要

本文考虑对执行器失效具有完整性的状态反馈律的设计问题．文中给出了这样的状态反馈律存在的一些充分条件和一个数值例子．

关键词：完整性；状态反馈律；执行器；失效