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# Noninteracting Decomposition of Linear Systems

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**Abstract**

In this paper, the input-output block decomposition problem is discussed. We first prove a theorem for compatible $(A, B)$-invariant subspaces. Using it, necessary and sufficient conditions for noninteracting decomposition problem have been obtained. Then, an algorithm follows. Finally we show that what we obtained is a canonical form.

## 1. Introduction

This paper tackles the problem of noninteracting decomposition of linear systems. The problem can be described as follows: suppose a linear dynamic system has been given in advance. The problem is whether there exists a state feedback control such that one block of inputs controls and affects only one block of outputs.

The noninteracting control problem has been studied for more than twenty years. Some of the earliest work in this problem via the transfer functions can be found in [1,2]. Recently, [3] made a new contribution by using this approach.

Another way for studying this problem is the geometric approach. Using the concept of controllability subspaces, some progress has been done. For more detailed accounts of the long history of noninteracting control of linear systems, we refer to [4,5]. A nice formulation and some more results about noninteraction problem can be found in [6], where it is called Restricted Decoupling Problem.

This paper discusses the solvability, algorithm and canonical form of the noninteracting decomposition problem. The paper is organized as follows. In section 2, we discuss the compatible $(A,B)$-invariance. Section

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3 presents necessary and sufficient conditions for the solvability of noninteracting decomposition problem. The algorithm is obtained in section 4. It has been also shown in this section that the obtained structure of the decomposed system is a canonical form.

## 2. Compatible (A,B) - invariant Subspaces

Consider a linear system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^r$.

We say that a subspace $V$ is $(A,B)$ - invariant if $AV \subset V + G$, where $G = S_p\{B\}$. It is well known [6] that $V$ is $(A,B)$ - invariant if and only if there exists a feedback law $F$, such that

$$(A+BF)V \subset V \tag{2}$$

In decoupling problems the common $F$ for several $(A,B)$ - invariant subspaces plays very important role. Thus we give the following definition.

**Definition 1** $k$ subspaces $V_1, \cdots, V_k$ are called compatible $(A,B)$ - invariant if there exists a $F$ such that (2) holds for every $V_i$.

In general, it is difficult to find common $F$ for all $V_i's$. But the following result is enough for the noninteracting decomposition problem.

**Theorem 2** Let $V_1, \cdots, V_k$ be $k$ subspaces of $R^n$. If i) $V_1 + \cdots + V_k = R^n$, ii) each $V_i$ is $(A,B)$ - invariant, iii) $G = G \cap V_1 + \cdots + G \cap V_k$, then $V_i + \sum_{i=1}^{k} V_i \cap (\sum_{j \neq i} V_j)$, $i = 1, \cdots, k$ are compatible $(A,B)$ - invariant subspaces.

A key lemma to prove the theorem is the following

**Lemma 3** If $D_0, D_1, \cdots, D_k$ are linearly independent subspaces. $\sum_{i=0}^{k} D_i = R^n$. Let $V_i = D_0 + D_i$, $i = 1, \cdots, k$ be $k$ $(A,B)$ - invariances. If $G = G \cap V_1 + G \cap V_2 + \cdots + G \cap V_k$, then $V_1, V_2, \cdots, V_k$ are compatible $(A,B)$ - invariant subspaces.

**Proof.** Choose coordinates such that

$$D_i = S_p\{(0, \cdots, 0, I_{n_i}, 0, \cdots, 0)^T\}, i = 0, 1, \cdots, k. \tag{3}$$

where $n_i = \dim(D_i)$.

Since $G = G \cap V_1 + \cdots + G \cap V_k$ and $V_i = D_0 + D_i$, we can find an $m \times m$ nonsingular matrix $Q$ such that

$$BQ=\left(\begin{array}{c|cccc} * & & * & & \\ \hline & L_1 & & 0 & \\ 0 & & L_2 & & \\ & & & \ddots & \\ & 0 & & & L_h \end{array}\right) \tag{4}$$

where $L_i$ is an $n_i \times t_i$ matrix with $\mathrm{rank}(L_i)=t_i \leqslant \dim(G \cap V_i)$.

Decompose A to be

$$A=(A_{ij}),\ i=0,\ \cdots,\ k,\ j=0,\ \cdots,\ k,$$

where $A_{ij}$ is an $n_i \times n_j$ matrix. Since each $D_0+D_i$ is $(A,B)$-invariant, it follows from (3) and (4) that

$$A\begin{pmatrix} I_{n_0} & 0 \\ 0 & I_{n_i} \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} A_{00} & A_{0i} \\ A_{10} & A_{1i} \\ \vdots & \vdots \\ A_{k0} & A_{ki} \end{pmatrix} \in D_0+D_i+S_p\begin{pmatrix} 0 & 0 \\ L_1 & 0 \\ \vdots & \ddots \vdots \\ 0 & L_h \end{pmatrix} \tag{5}$$

Thus there exist $t_j \times n_i$ matrices $F_{ji}$, such that

$$A_{ji}=L_jF_{ji},\ i=0,\ \cdots,\ k,\ j=1,\ \cdots,\ k,\ j \neq i$$

Set

$$F=-Q\left(\begin{array}{c|cccc} 0 & & 0 & & \\ \hline F_{10} & 0 & F_{12} & & F_{1k} \\ & F_{21} & 0 & \cdots & \\ \vdots & & F_{32} & & \vdots \\ & & \vdots & & F_{k-1,k} \\ F_{k0} & F_{k1} & F_{k2} & \cdots & 0 \end{array}\right) \tag{6}$$

it follows that

$$A+BF=\left(\begin{array}{c|cccc} * & & * & & \\ \hline & A_{11} & & & \\ 0 & & A_{22} & & \\ & & & \ddots & \\ & 0 & & & A_{kk} \end{array}\right) \tag{7}$$

A straightforward computation shows that

$$(A+BF)V_i \subset V_i,\ i=1,\ \cdots,\ k.$$

## 3. Noninteracting Decomposition Problem

Recall the system (1). Let $y=(y^1,\ \cdots,\ y^k)$ be a decomposition of the output y. The noninteracting decompsition problem is defined precisely as

**Definition 4** The noninteracting decomposition problem (NDP) is that of finding a feedback control $u=F+Qv$, where $F$ is an $m\times n$ matrix and $Q$ is an $m\times m$ nonsingular matrix, such that there exists a decomposition $(v^1, \cdots, v^k)$ of $v$, which satisfies the following two conditions:

i) $v^i$ does not affect $y^j$, $j\neq i$,

ii) $v^i$ controls $y^i$ completely.

Denote $K_i=\bigcap\limits_{j\neq i}\ker(C^j)$, $i=1, \cdots, k$. Let $R_i$ and $V_i$ be the largest controllability subspace contained in $K_i$ respectively.

Now we state the following theorem which solves the NDP.

**Theorem 5** Assume system (1) is completely controllable. Then NDP is solvable if and only if, either one of the following two conditions is satisfied

i) $G=G\cap R_1+\cdots+G\cap R_k$;

ii) $G=G\cap V_1+\cdots+G\cap V_k$.

## 4. Noninteracting Decomposition Algorithm

It is well known that the calculation of the largest $(A,B)$-invariant subspace $V_i$ contained in $K_i$ is much easier than that of $R_i$, the largest controllability subspace in $K_i$. Therefore we set up the algorithm with respect to $V_1,\cdots,V_K$. But one may see that the algorithm is also available for $R_1, \cdots, R_k$. The following algorithm is proved by the above constructive proof.

**Algorithm 6**

Step 1. Compute the largest $(A,B)$-invariant subspaces $V_i\subset\sum\limits_{j\neq i}\ker(C^j)$, $i=1, \cdots, k$.

Step 2. Check $G=G\cap V_1+\cdots+G\cap V_k$.

Step 3. Choose $w_j^i, j=1, \cdots, n_i$ as a basis of $(\bigcap\limits_{j\neq i}V_j)^\perp$; Choose $w_j^0$, $j=1, \cdots, n_0$, such that $\sum\limits_{i=0}^{k}n_i=n$ and $w_j^i$, $i=0, \cdots, k$, $j=1, \cdots, n_i$ be a basis of $R^n$. Set new coordinate frame z by

$$z_j^i=(w_j^i)^Tx,\ i=0, 1, \cdots, k,\ j=1\cdots, n_i,$$

or brievely $z=Wx$.

Step 4. Choose $Q$ such that WBQ has the form of (4).

Step 5. Decompose $WAW^{-1}$ as $WAW^{-1}=(A_{ij})$, $i, j=0, 1,\cdots,k$, where $A_{ij}$ is an $n_i\times n_j$ matrix. Compute

$$F_{ji} = (L_j^T L_j)^{-1} L_j^T A_{ji}, \quad i = 0, 1, \cdots, k, \; j = 1, \cdots, k, \; j \neq i$$

Using equation (6) to construct F.

Step 6 Construct the control u as

$$u = FWx + Qv. \tag{8}$$

To show that the control (8) obtained in the Algorithm 6 solves the NDP, we give the following theorem.

**Theorem 7** Assume system (1) is completely controllable, then the NDP is solvable (with respect to the given partition of output), if and only if, Algorithm 6 is executable. Moreover under the coordinate frame $z = Wx$ we have the following decomposed form

$$\begin{aligned} \dot{z}^0 &= A_0 z + B_0 v \\ \dot{z}^1 &= A_{11} z^1 + L_1 v^1 \\ &\cdots\cdots \\ \dot{z}^k &= A_{kk} z^k + L_k v^k \\ y^i &= C^i z^i, \quad i = i, \cdots, k. \end{aligned} \tag{9}$$

Next, we show that the input-output decomposed form (9) is a canonical form in the following sense.

**Theorem 8** Let x be a canonical coordinate frame. If there exists control $u = F'x + Q'v$, such that under x one has another input-output decomposed form

$$\begin{aligned} \dot{x}^0 &= A_0' x + B_0' v \\ \dot{x}^1 &= A_{11}' x^1 + L_1' v'^1 \\ &\cdots\cdots \\ \dot{x}^k &= A_{kk}' x^k + L_k' v'^k \\ y^i &= C'^i x^i, \quad i = 1, \cdots, k, \end{aligned}$$

where each $L_i'$ has full rank. Then for each decomposed subsystem

$$\begin{aligned} \dot{x}^i &= A_{ii}' x^i + L_i' v'^i \\ y^i &= C'^i x^i, \quad i = 1, \cdots, k, \end{aligned}$$

there exists linear transformations $T_i$: $z^i \to x^i$ as $x^i = T_i z^i$, $i = 0, \cdots, k$, $n_i \times t_i$ matrix $F_i$ and $t_i \times t_i$ nonsingular matrix $Q_i$, such that

$$T_i (A_{ii} + L_i F_i) T_i^{-1} = A_{ii}' \tag{10a}$$

$$T_i L_i Q_i = L_i', \quad i = 1, \cdots, k. \tag{10b}$$

## 5. Conclusion

The noninteracting decomposition problem for arbitray given partition of outputs has been solved by the necessary and sufficient condition

$$G = G\cap V_1 + \cdots + G\cap V_k,$$

where $V_i$ is the largest $(A,B)$-invariant subspace in the kernel of corresponding output blocks.

An algorithm has been presented, which yields the input-output decomposed form (9). It has also been shown that the decomposed form (9) is a canonical form in the sense that the decomposed subsystems contained in (9) are unique module subsystem substate linear transformations and subsystem substate feedbacks.

All the results obtained in this paper has been extended to affine nonlinear systems[7].

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# 线性系统的不相关解耦*

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**摘　要**

本文讨论线性系统的输入－输出块解耦问题．首先证明一个关于相容（$A$，$B$）－不变子空间族的定理，再利用它证明不相关解耦的充要条件．然后，给出解耦算法．最后证明，由算法得到的解耦形式是一种标准形式．

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