# Convergence of the Forgetting Factor Algorithm for Identifying Time-Varying Stochastic Systems

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**Abstract**: In the paper [1], the authors established the convergence properties of the forgetting factor algorithm for time-varying systems. In this note, we: 1) Point out the errors in the proof of convergence; 2) Show the upper bound and lower bound of convariance matrix of recursive forgetting factor algorithm (RFFA); 3) Provide a correction of convergence properties of the algorithm presented in [1].

**Key words**: parameter estimation; forgetting factor; exponential convergence; persistent excitation

# 辨识时变随机系统遗忘因子算法的收敛性分析

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**摘要**: 文[1]给出了辨识时变随机系统遗忘因子算法的收敛性分析. 本文指出了文[1]中的一些证明和结论上的错误, 并给出了相应的改正和完善.

**关键词**: 参数估计; 遗忘因子; 指数收敛; 持续激励

## 1 Introduction

Parameter identification problems have been received great attention from many researchers for many years. As we know, the recursive least square (RLS) algorithm with forgetting factor is often used to identify time-varying systems because of its good robustness. In [1], the convergence properties of the RLS algorithm with forgetting factor were presented. However, there are some errors in that paper as following:

1) A key result is used to obtain Eq. (4) in [1]. That is

$$mI \leqslant \varphi(t)\varphi^{\mathrm{T}}(t) \leqslant MI. \tag{1}$$

According to algebra, for any vector $\varphi(t)$, we have

$$\mathrm{rank}(\varphi(t)\varphi^{\mathrm{T}}(t)) \leqslant \min\{\mathrm{rank}(\varphi(t)), \mathrm{rank}(\varphi^{\mathrm{T}}(t))\} \leqslant 1. \tag{2}$$

Clearly, Eq. (1) is wrong. This leads to the wrong results in that paper.

2) On pp. 637, "For the time invariant stochastic systems: i) PEE (parameter estimation error) given by RLS algorithm converges to zero under the mean square sense, and its convergence rate is of $\left(\frac{1}{\sqrt{t}}\right)$". The convergence rate should be $\left(\frac{1}{t}\right)$.

3) On pp. 637, "the mean square PEE given by RLS algorithm is unbounded". As will be pointed in Remark 2 of Section 3, the upper bound of the mean square PEE provided by the ordinary RLS algorithm tends to infinity, but it doesn't mean the PEE is unbounbded.

In this note, we first show the upper bound and lower bound of covariance matrix of forgetting factor algorithm. Then, we further provide the correction of convergence properties presented in [1] on RFFA for time-varying stochastic systems.

## 2 System model and the algorithm

We consider a single-input and single-output (SISO) system represented by:

$$y(t) = \varphi^{\mathrm{T}}(t)\theta(t) + v(t), \tag{3}$$

where $\varphi(t) = [-y(t-1), -y(t-2), \cdots, y(t-n_a), u(t-1), u(t-2), \cdots, u(t-n_b)]^{\mathrm{T}}$ is system input-output regression vector, $\theta(t) = [a_1(t), a_2(t), \cdots, a_{n_a}(t), b_1(t), b_2(t), \cdots, b_{n_b}(t)]^{\mathrm{T}}$, is system time-varying unknown parameter vector and to be identified, $y(t)$ and $u(t)$ are observable output and input

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sequence of the system, respectively. $v(t)$ is the system stochastic noise sequence.

The forgetting factor algorithm employed to identify $\theta(t)$ can be described by the following equations:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + P(t+1)\cdot \varphi(t)[y(t) - \varphi^{T}(t)\hat{\theta}(t)], \quad (4)$$

$$P^{-1}(t+1) = \lambda P^{-1}(t) + \varphi(t)\varphi^{T}(t), \quad (5)$$

where $\lambda \in (0,1)$ is the forgetting factor, $P(1) = \varepsilon I$ is a positive definite matrix, and $\hat{\theta}(1)$ is arbitrary. $\hat{\theta}(t)$ denotes the estimate of the parameter $\theta(t)$. When $\lambda = 1$ in the above equations, we have the ordinary RLS algorithm.

## 3 The convergence of the algorithm

As a preliminary to the main result, the definition of the persistent excitation is given. Then we show that if $\{\varphi(t)\}$ is persistently exciting, $P^{-1}(t+1)$ is bounded.

**Definition** The measurement vector sequence $\{\varphi(t)\}$ is said to be persistently exciting, if for some constant integer s and all $k$ there exist positive constants $m$ and $M$ such that

$$0 < mI \leqslant \sum_{j=k}^{k+s}\varphi(j)\varphi^{T}(j) \leqslant MI < \infty. \quad (6)$$

**Lemma 1** If the measurement vector of the system sequence $\{\varphi(t)\}$ is persistently exciting, then for all $n \geqslant 1$,

$$\lambda^{ns}P^{-1}(i) + \frac{\lambda^{s}(1-\lambda^{ns})}{1-\lambda^{s}}mI \leqslant P^{-1}(ns+i) \leqslant \lambda^{ns}P^{-1}(i) + \frac{1-\lambda^{ns}}{1-\lambda^{s}}MI \quad (7)$$

or

$$O(\lambda^{n}) + \frac{\lambda^{s}(1-\lambda^{ns})}{1-\lambda^{s}}mI \leqslant p^{-1}(ns+i) \leqslant O(\lambda^{n}) + \frac{1-\lambda^{ns}}{1-\lambda^{s}}MI, \quad (8)$$

where $1 \leqslant i \leqslant s-1$.
As $n \to \infty$

$$\frac{\lambda^{s}}{1-\lambda^{s}}mI \leqslant P^{-1}(ns+i) \leqslant \frac{1}{1-\lambda^{s}}MI \quad (9)$$

or

$$\frac{1-\lambda^{s}}{M}I \leqslant P(ns+i) \leqslant \frac{1-\lambda^{s}}{m\lambda^{s}}I. \quad (10)$$

Proof From (1), we obtain that

$$P^{-1}(ns+i) = \lambda^{s}P^{-1}[(n-1)s+i] + \sum_{j=1}^{s}\lambda^{s-j}\varphi[(n-1)s+i+j]\varphi^{T}\cdot [(n-1)s+i+j]. \quad (11)$$

Also, from (6) and $0 < \lambda < 1$

$$\lambda^{s}P^{-1}[(n-1)s+i] + \lambda^{s}mI \leqslant P^{-1}(ns+i) \leqslant \lambda^{s}P^{-1}[(n-1)s+i] + MI. \quad (12)$$

So for $n \geqslant 1$

$$P^{-1}(ns+i) \leqslant \lambda^{s}P^{-1}[(n-1)s+i] + MI \leqslant \lambda^{ns}P^{-1}(i) + \sum_{j=0}^{(n-1)s}\lambda^{js}MI = \lambda^{ns}P^{-1}(i) + \frac{1-\lambda^{ns}}{1-\lambda^{s}}MI,$$

$$P^{-1}(ns+i) \geqslant \lambda^{s}P^{-1}[(n-1)s+i] + \lambda^{s}mI \geqslant \lambda^{ns}P^{-1}(i) + \sum_{j=0}^{ns}\lambda^{js}mI = \lambda^{ns}P^{-1}(i) + \frac{\lambda^{s}(1-\lambda^{ns})}{1-\lambda^{s}}mI.$$

As $n \to \infty$, the result follows.

**Remark 1** According to [2], $\{\varphi(t)\}$ can be guaranteed to be persistently exciting if the system is controllable and observable.

**Lemma 2** If the measurement vector sequence of the system $\{\varphi(t)\}$ is persistently exciting, and $\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{t}\varphi(j)\varphi^{T}(j)$ exists, then

$$0 < \frac{m}{s}I \leqslant \lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{t}\varphi(j)\varphi^{T}(j) = E[\varphi(j)\varphi^{T}(j)] \leqslant \frac{M}{s}I < \infty. \quad (13)$$

Proof Let $t = ns + i$ $(0 \leqslant i \leqslant s-1)$, then

$$\frac{1}{t}\sum_{j=1}^{t}\varphi(j)\varphi^{T}(j) = \frac{1}{ns+i}\sum_{j=1}^{ns+i}\varphi(j)\varphi^{T}(j) \leqslant \frac{1}{ns+i}(n+1)MI,$$

$$\frac{1}{t}\sum_{j=1}^{t}\varphi(j)\varphi^{T}(j) = \frac{1}{ns+i}\sum_{j=1}^{ns+i}\varphi(j)\varphi^{T}(j) \geqslant \frac{1}{ns+i}nmI.$$

As $t \to \infty$, i.e. $n \to \infty$, we can obtain

$$\frac{m}{s}I \leqslant \lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{t}\varphi(j)\varphi^{T}(j) = E[\varphi(j)\varphi^{T}(j)] \leqslant \frac{M}{s}I, \quad (14)$$

it proves Lemma 2.

Based on Lemma 1 and 2, we now present our main results.

**Theorem 1** For time-varying system, if the following conditions hold

1) The measurement vector sequence $\{\varphi(t)\}$ is persistently exciting, and $\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{t}\varphi(j)\varphi^{\mathrm{T}}(j)$ exists;

2) The noise sequences $v(t)$ is an independent random variable with zero mean and square bounded, i.e.

$$E[\varphi(l)\varphi(k)] = \delta_{kl}\sigma^2 < \infty,$$

$$\delta_{kl} = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases}$$

3) The parameter change rate $\alpha(t) = \theta(t) - \theta(t-1)$ is bounded, $\alpha(t)$ and $v(t)$ are independent, i.e.

$$E\|\alpha(t)\|^2 \leqslant M_1 < \infty,$$

$$E[\alpha(k)v(l)] = 0,$$

then under mean square sense, the PEE given by RLS algorithm with forgetting factor converges exponentially to a region, i.e. as $t \to \infty$

$$\frac{m\sigma^2(n_a+n_b)(1-\lambda^s)^2}{sM^2(1-\lambda)^2} \leqslant E\|\tilde{\theta}(t+1)\|^2 \leqslant$$

$$\frac{M\sigma^2(n_a+n_b)(1-\lambda^s)^2}{sm^2\lambda^{2s}(1-\lambda)^2} + \frac{M^2M_1}{m^2\lambda^{2s}(1-\lambda)^2}. \quad (15)$$

**Theorem 2** For time invariant system, if Condition 1 and 2 hold, then under mean square sense, the PEE provided by RLS algorithm with forgetting factor converges exponentially to a region, i.e. as $t \to \infty$

$$\frac{m\sigma^2(n_a+n_b)(1-\lambda^s)^2}{sM^2(1-\lambda)^2} \leqslant$$

$$E\|\tilde{\theta}(t+1)\|^2 \leqslant \frac{M\sigma^2(n_a+n_b)(1-\lambda^s)^2}{sm^2\lambda^{2s}(1-\lambda)^2}. \quad (16)$$

As the further results of the above theorems, we can obtain the following corollaries.

**Corollary 1** For time-varying system, if $\lim_{t\to\infty} v(t) = 0$ and the Condition 1 and 3 hold, then under mean square sense, the RLS algorithm with forgetting factor converges exponentially to a region, i.e. as $t\to\infty$

$$E\|\tilde{\theta}(t+1)\|^2 \leqslant \frac{M^2M_1}{m^2\lambda^{2s}(1-\lambda)^2}. \quad (17)$$

**Corollary 2** For time-invariant system, if $\lim_{t\to\infty} v(t) = 0$ and Condition 1 and 2 hold, then under mean square sense, parameter estimate provided by the forgetting factor algorithm converges exponentially to the true value of the system parameter.

**Corollary 3** For time invariant stochastic system, if Condition 1 and 2 hold, then the parameter estimate provided by the recursive least square algorithm converges to a region at rate of $\left(\frac{1}{t}\right)$, i.e. as $t \to \infty$

$$\frac{ms\sigma^2(n_a+n_b)}{M^2} \leqslant$$

$$E\|\tilde{\theta}(t+1)\|^2 \leqslant \frac{Ms\sigma^2(n_a+n_b)}{m^2}. \quad (18)$$

**Corollary 4** For time invariant stochastic system, if $\lim_{t\to\infty} v(t) = 0$ and Condition 1 and 2 hold, then the parameter estimate provided by the recursive least square algorithm converges to one point $\theta$ (the true value of the system parameter) at rate of $\left(\frac{1}{t}\right)$.

The proof of the theorems and corollaries are similar as those in [1], and are omitted here.

**Remark 2** For time-varying systems, let $\lambda \to 1$, from (15), we can find the upper bound of the parameter error caused by parameter variation tends to infinity. This shows that the upper bound of the parameter error provided by the ordinary recursive least square algorithm can not be given, but it doesn't mean the PEE is unbounded.

## 4 Conclusion

The main results in this note show the upper bound and lower bound of covariance matrix of RFFA, and demonstrate that the parameter estimate obtained by RFFA for both time-varying and time invariant stochastic systems converges exponentially to a region provided that $\{\varphi(t)\}$ is persistently exciting. Furthermore, it is also proved that the parameter estimate given by the ordinary RLS algorithm for the time invariant systems in presence of noise converges to a region at rate of $\left(\frac{1}{t}\right)$. Thus, these give a correction of the results presented in [1].

### References

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