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# Passivity, Stability and Optimality

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**Abstract**: In this paper, we consider passivity, stability and optimality and the relationship between them of affine nonlinear control systems. First, we study relations between the passivity of nonlinear systems and one of their linear approximate systems in order to analyse stability. Then we show that for affine nonlinear systems passivity, stability and optimality are equivalent via feedback in some sense. Finally, we go further to discuss the systems with measured outputs.

**Key words**: passivity; stability; optimality; affine nonlinear system

## 1 Introduction

Passivity, which was derived from network theory and other branches of physics, has become one of the powerful concepts to study control systems. In different cases, it changes a little into other similar concepts, for example, positive realness and dissipativity. With its deep physical meaning and its close relation with Lyapunov function, it has been applied widely, such as in analysis of stabilization of nonlinear systems[9], in control systems design[11], and in robot control[12]. Therefore, there are many research activities, including [2,3,7] and [4], related to passivity or passivity via feedback. In particular, after a summary of these works, [3] shows that it plays a key role in the stability or stabilizability of nonlinear systems.

Before discussing the problems concerned with passivity, we introduce its definition at first.

Consider an affine nonlinear system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathrm{R}^n, \tag{1.1a}$$

$$y = h(x), \quad u, y \in \mathrm{R}^m \tag{1.1b}$$

where $f$, $g$, $h$ are smooth with $f(0)=0$, $h(0)=0$, $g(0)\neq 0$.

We review some of basic concepts, which will be used in the following sections (referred to [3] for details).

**Definition 1.1** The system (1.1) is called to be passive if there exists a continuous nonnegative function $V: \mathrm{R}^n \to \mathrm{R}$, called the storage function, such that for all $u \in \mathrm{R}^m$, $x^0 \in \mathrm{R}^n$, $t \geqslant 0$, the following inequality holds:

$$V(x) - V(x^0) \leqslant \int_0^t y^{\mathrm{T}}(s)u(s)\mathrm{d}s \tag{1.2}$$

and $V(0)=0$.

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If $V$ is a $C^1$ nonnegative function, then by [3], (1.1) is passive if and only if

$$L_f V(x) \leqslant 0, \tag{1.3a}$$

$$L_g V(x) = h^{\mathrm{T}}(x) \tag{1.3b}$$

for each $x \in \mathrm{R}^n$.

**Definition 1.2** The system (1.1) is locally zero-state detectable if there exists a neighborhood U of 0 such that for all $x \in \mathrm{U}$

$$h(\Phi(t, x, 0)) = 0 \text{ for all } t \geqslant 0$$

implies

$$\lim_{t \to \infty} \Phi(t, x, 0) = 0$$

where $\Phi(t, x, 0)$ is the trajectory of (1.1) with initial value $x$ in $t=0$ without control. In addition, if U is the whole space $R^n$, then the system is zero-state detectable.

**Definition 1.3** A passive system with storage function $V$ is said to be strictly passive if there exists a positive definite function $S: \mathrm{R}^n \to \mathrm{R}$ such that for all $u \in \mathrm{R}^m$, $t \geqslant 0$, it holds:

$$V(x) - V(x^0) = \int_0^t y^{\mathrm{T}}(s) u(s) \mathrm{d}s - \int_0^{\mathrm{T}} S(x(s)) \mathrm{d}s \tag{1.4}$$

**Definition 1.4** The system (1.1) is passive via (state) feedback if there exists a control law

$$u = \alpha(x) + \beta(x) v$$

where $\alpha(x)$ and $\beta(x)$ are smooth vector-valued and matrix-valued function respectively, and $\beta(x)$ is invertible for all $x$, such that the closed-loop system

$$\begin{aligned} \dot{x} &= [f(x) + g(x)\alpha(x)] + g(x)\beta(x) v, \\ y &= h(x) \end{aligned} \tag{1.5}$$

is a passive system.

In the 1960's, Kalman first solved the linear quadric inverse optimal control problem[1]. Since then, there have been many papers related to stably optimal control problem and its inverse optimal problem. [5] considered the relationship between positive realness and optimality in the linear cases, while [6] involved that of control Lyapunov function and optimization.

In this paper the relationship between passivity, stability and optimality is discovered. The paper is organized as follows: First, we introduce linearization of nonlinear systems in order to discuss how to form storage functions for passive systems. Then we study the relationship between passivity, stability and optimality under the state feedback. Finally, we consider this problem for the systems with measured outputs.

## 2 Linear Approximation

Difficulties to find storage functions for systems prevent us from applying passivity widely. If we can linearize exactly an affine nonlinear system, then we will have a way to check if this system is passive (or passive via feedback) and to construct a storage function of the system. Unfortunately, by [8], there are few affine nonlinear systems which can be linearized exactly, and linearization procedure is quite hard to put into practice. On account of the fact, it is necessary to consider linear approximation in the nonlinear case.

Consider the system (1. 1) and its linear approximation system:

$$\dot{x} = Ax + Bu,$$
$$y = Cx \tag{2. 1}$$

where $A=\left.\frac{\partial f}{\partial x}\right|_{x=0}$, $B=g(0)$, $C=\left.\frac{\partial h}{\partial x}\right|_{x=0}$.

It is easy to get the following:

**Theorem 2. 1** If (1. 1) is passive (or passive via feedback), then so is (2. 1).

However, its inverse proposition is quite subtle. In general, we can only get some local results.

**Theorem 2. 2** If (1. 1) is strictly passive via feedback, then (1. 1) is locally strictly passive around $x=0$ via feedback.

The proof is based on the results of [3]. Notice (1. 1) is still locally minimum phase and relative degree $\{1, \cdots, 1\}$ at $x=0$ if (2. 1) is so.

**Remark 2. 3** For global stability, it is meanful and important to get (globally) strict passivity [3,9], but, generally, it is impossible because of the nonlinear complexity. However, we can analyse a special class of nonlinear systems. For example, we consider

$$\dot{x} = f(x) + Bu, \quad x \in \mathrm{R}^n, \tag{2. 2a}$$
$$y = Cx, \quad u,y \in \mathrm{R}^m \tag{2. 2b}$$

where $f(x)$ is a homogeneous polynomial vector field of odd degree. Based on [10], we know that if there exists a real $n\times n$ symmetric positive-definite matrix $P$, such that

$$PB = C^{\mathrm{T}}$$

and

$$\ker C = \{x \in \mathrm{R}^n : x^{\mathrm{T}}Pf(x) < 0\} \cup \{0\}.$$

Then (2. 2) will be strictly passsive via feedback.

**Theorem 2. 4** Consider (1. 1) and (2. 1) in the SISO case, if (2. 1) is strictly passive, then (1. 1) is locally strictly passive.

**Proof** If (2. 1) is strictly passive, then we can find a symmetric positive-definite matrices $P$, $Q$ such that

$$V_0 = x^{\mathrm{T}}Px,$$
$$L_{Ax}V_0 = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x = - x^{\mathrm{T}}Qx < 0,$$
$$(L_BV_0)^{\mathrm{T}} = B^{\mathrm{T}}Px = Cx.$$

To find a suitable V to satisfy

$$L_fV < 0, \quad L_gV = h^{\mathrm{T}}. \tag{2. 3}$$

Assume $V=V_0+V_1$, $V_1=o(x^2)$.

Rewriting (2. 3) in detail, we have

$$(L_BV_0)^{\mathrm{T}} + (L_{o(1)}V_0)^{\mathrm{T}} + (L_gV_1)^{\mathrm{T}} = Cx + o(x)$$

or equivalently

$$(L_gV_1)^{\mathrm{T}} = l(x) \tag{2. 4}$$

where $l(x)=o(x)$ is an infinitesimal function with higher than first power of $x$.

Notice that (2. 4) is a partially differential equation with one order and $g(0)\neq 0$. By virtue of theory of differential equation, (2. 4) has a solution $V_1(x)$ with initial value $V_1(0)=0$. Therefore, we get $V(x)=V_0(x)+V_1(x)$ which satisfies:

$$(L_gV_1)^{\mathrm{T}} = h(x)$$

and

$$L_fV = -x^{\mathrm{T}}Qx + o(x^2). \tag{2. 5}$$

Hence, there exists a neighborhood U of $x=0$ such that $L_fV$ is still negative definite in U. Then the result follows.

## 3 Optimal and Inverse Optimal Problem

This problem has been solved by Kalman and studied further in [5] in the linear case. Here, we will get corresponding conclusions in the nonlinear case.

**Theorem 3. 1** Consider an affine nonlinear system

$$\dot{x} = f(x) + g(x)u, \tag{3. 1}$$

where $x=0$ is the equibibrium as $u=0$ and a performance index

$$J(u) = \int_0^\infty [2S(x(t)) + u^{\mathrm{T}}(t)u(t)]\mathrm{d}t \tag{3. 2}$$

where $f$, $g$ are smooth, and $S(x)$ is a given positive-definite function. Then we have

1) The optimal control $u^*(x)$ can be written as

$$u^*(x) = -\frac{1}{2}\left(\frac{\partial J^*}{\partial x}g(x)\right)^{\mathrm{T}} \tag{3. 3}$$

where $J^*(x)$ is the optimal performance index value of (3. 1) (3. 2) with the initial point $x$.

2) $u^*(x)$ makes the optimal closed-loop system

$$\dot{x} = f(x) - \frac{1}{2}g(x)\left(\frac{\partial J^*}{\partial x}g(x)\right)^{\mathrm{T}}$$

be asymptotically stable.

3) If we take the output $y$ as follows:

$$y = h(x) \triangleq \frac{1}{2}\left(\frac{\partial J^*}{\partial x}g(x)\right)^{\mathrm{T}}.$$

Then (3. 1) with $y=h(x)$ as its output is passive via feedback.

Proof The proof of 1) 2) is regular, we only consider 3). We take a storage function $V$ as

$$V = \frac{1}{2}J^*(x).$$

It is easy to know that $V$ is nonegative and $L_gV=h^{\mathrm{T}}(x)$. We claim that $L_{f+gu^*}V<0$. In fact, it is the same that $L_{f+gu^*}J^*(x)<0$.

Notice that $J^*(x)$ and $u^*(x)$ in (3. 3) satisfy the following equality:

$$L_fJ^* + L_gJ^*u^* + 2S(x) + u^{*\mathrm{T}}u^* = 0. \tag{3. 4}$$

Hence

$$L_{f+gu^*}J^*(x) = -2S(x) - u^{*T}u^* < 0,$$

therefore, the result follows.

Conversely, we have to add some conditions to get the inverse results in the similar form.

**Theorem 3.2** If the system

$$\begin{aligned} \dot{x} &= [f(x) - \frac{1}{2}g(x)h(x)] + g(x)v, \\ y &= h(x) \triangleq -u^*(x) \end{aligned} \tag{3.5}$$

is passive. Then there exists an performance index in the form of (3.2) and $u^*(x)$ is just optimal control of the system (3.1) with this performance index.

Proof By passivity of (3.5), we have the storage function $V(x) \geqslant 0$, such that

$$L_{[f(x)-\frac{1}{2}g(x)h(x)]}V(x) \leqslant 0$$

and

$$L_gV(x) = h^T(x).$$

Set

$$S(x) = -L_{[f(x)-\frac{1}{2}g(x)h(x)]}V(x). \tag{3.6}$$

Obviously, it is nonnegative.

Take the performance index like (3.2) with $S(x)$ in (3.6)

$$J = \int_0^\infty [2S(x(t)) + u^T(t)u(x)]dt$$

and define $W(x) \triangleq 2V(x)$. Then we have

$$u^*(x) = -h(x) = -\frac{1}{2}(L_gW)^T. \tag{3.7}$$

By (3.6), we see that $W(x)$ and $u^*(x)$ satisfy

$$H(x,u^*) = L_fW + L_gWu^*(x) + 2S(x) + u^{*T}(x)u^*(x) = 0.$$

On the other hand, for any given $\delta u$ which constitutes $u(x) = u^*(x) + \delta u(x)$ to make corresponding closed-loop system asymptotically stable, we have

$$\begin{aligned} H(x,u) &= L_fW + L_gWu + 2S(x) + u^Tu \\ &= (\delta u)^T(\delta u) \geqslant 0. \end{aligned}$$

Hence

$$J[u^*] = W(x_0) = \int_0^\infty [2S(x^*(t)) + u^{*T}(x^*(t))u^*(x^*(t))]dt \leqslant J[u]$$

i.e $u^*(x)$ is a optimal control, where $x^*(t)$ denotes the optimal trajectory with initial state $x_0$ in $t=0$.

## 4 Nonlinear Systems with Measured Outputs

In practice, the states of the systems which are used for analysis or control goals can not be got directly from measurement. Therefore, we have to consider how to employ measured outputs to deal with problems of analysis and design.

Here we investigate a problem similar to the above section, which is restricted to use output static feedback control.

Consider (1.1) and define a control set:

$$\mathscr{U} \triangleq \{u = -\varphi(y) \mid \varphi(0) = 0,\ \varphi^T(y)y > 0,\ \text{if } y \neq 0\}. \tag{4.1}$$

**Theorem 4.1** If the following conditions hold:

1) (1.1) is passive with the storage function $V(x)$ which is positive-definite and proper;

2) (1.1) is zero-state detectable.

Then $u^*(y) \triangleq -ky \in \mathscr{U}(k>0)$ is an optimal control of (1.1) with some performance index in the form of

$$J[u] = \int_0^\infty [2S(x(t)) + y^T(t)y(t) + qu^T(t)u(t)]dt. \tag{4.2}$$

$g>0$ const.

**Proof** Since (1.1) is passive with storage function $V(x)$, we have

$$\tilde{S}(x) \triangleq -L_fV(x) \geqslant 0, \quad L_gV(x) = h^T(x).$$

For the $\tilde{S}$ and some constants $k>0$, $q>0$, we define $L(x)$ as follows:

$$L(x,u) \triangleq 2S(x) + u^Tu + qh^T(x)h(x), \tag{4.3a}$$

$$S(x) = k\tilde{S}(x). \tag{4.3b}$$

Consider Bellamn equation in the following form:

$$\begin{cases} \min\limits_{u\in R^m}\left\{\dfrac{\partial W(x)}{\partial x}f(x) + \dfrac{\partial W(x)}{\partial x}g(x)u + L(x,u)\right\} = 0, \\ W(0) = 0. \end{cases} \tag{4.4}$$

Denote $H(x,u) \triangleq L_fW + L_gW \cdot u + L(x,u)$ and take $W(x) \triangleq 2kV(x)$. It is easy to verify that $W(x)$ and $u^*(x)$

$$u^*(x) = -k[L_gV(x)]^T = -\frac{1}{2}[L_g(x)W(x)]^T \tag{4.5}$$

constitute a solution to (4.4). If we choose $q=\dfrac{k^2}{4}$, then $u^*(x)$ defined in (4.5) is just the optimal control of (1.1) and (4.2) with $S(x)$ defined in (4.3b) in the following sense:

$$J[u^*] = \int_0^\infty L(x^*(t),u^*(t))dt \leqslant \int_0^\infty L(x(t),u(t))dt = J[u],$$

$\forall\, u \in \mathscr{U}$ defined in (4.1), i.e.

$$J[u^*] = \int_0^\infty L(x^*(t),u^*(t))dt \leqslant \int_0^\infty L(x(t),u(t))dt = J[u],$$

$\forall\, u = u^* + \delta u \in \mathscr{U}$.

**Remark 4.2** Unlike the results in the above section, the inverse proposition of Theorem 4.1 is quite hard to be solved.

## References


[1] Kalman, R. E.. When Is a Linear Control System Optimal? ASME. Basic Engineering, 1964, 86(1):51—60

[2] Anderson B. D. O. and Vongpanitlerd, S.. Network Analysis and Synthesis—A Modern Systems Theory Approach. Prentice-Hall, INC. Englewood Cliffs, New Jersey, 1973

[3] Byrnes, C. J., Isidori, A. and Willems, J. C.. Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Nonlinear Systems. IEEE Trans. Automat. Contr., 1991, AC-36(11):1228—1240

[4] Picci, G. and Pinzoin, S.. On Feedback-Dissipative Systems. J. of Math. Sys. Esti. & Contr., 1992, 2(1):1—30

[5] Wang, E. and Wang, C.. Positivity and Optimality for Linear Multivariable Time-Invariant Systems. Control Theory and Applications (in Chinese), 1984, 1(1):60—67

[6] Sontag, E. D.. A 'Universal' Construction of Artstein's Theorem on Nonlinear Stabilization. Sys. Contr. Lett. 1989, 13(2):117—123

[7] Vidyasager, M.. $L_2$-Stability of Interconnected Systems Using a Reformulation of the Passivity Theorem. IEEE Trans. 1877, CAS-24(11):637—645

[8] Cheng, D.; Qin, H. and Li, S.. On the Topological Structure for Nonlinear Control Systems. Acta Automatica Sinica (in Chinese), 1991, 17(2)

[9] Kokotovic, P. V. and Sussman, H. J.. A Positive Real Condition for Global Stabilization on Nonlinear Systems. Sys. Contr. Lett. 1989, 13(2):125—134

[10] Andreini, A., Baccioti, A. and Stefani, G.. Global Stability of Homogeneous Vector Fields of Odd Degree. Syst. Contr. Lett. 1988, 10(4):251—256

[11] Kanellakopoulos, I., Kokotovic, P. V. and Morse, A. S.. A Toolkit for Nonlinear Feedback Design. Syst. Contr. Lett. 1992, 18:83—92

[12] Niemeyer, G. and Slotine, J. E.. Performance in Adaptive Manipulator Control. The International Journal of Robotics Research, 1991, 10(2):140—161

# 无源性、稳定性和最优性

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**摘要**：本文考察仿射非线性控制系统的无源性、稳定性和最优性，以及这三者之间的关系. 为分析稳定性，首先研究非线性控制系统和其线性近似的无源性之间的关系. 然后指出，就仿射非线性控制系统而言，无源性、稳定性和最优性在某种意义下是反馈等价的. 最后讨论了带量测输出的非线性系统的无源性、稳定性和最优性三者之间的关系.

**关键词**：无源性；稳定性；最优性；仿射非线性控制系统

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