The Convergence of the Forgetting Factor Algorithm for Identifying Time-Varying Systems

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Abstract: It is well-known that the recursive forgetting factor algorithm (RFFA) may be used to identify the parameters of time-varying systems and it has good tracking performance. In this paper the convergence and stability of RFFA are analyzed by means of stochastic process theory, the upper and lower bounds of the parameter tracking error are given.

Key words: time-varying systems; parameter estimation; forgetting factor method

1 Introduction

Consider the following linear time-varying system[1,2]

$$\begin{aligned} y(t) &= \varphi^{\mathrm{T}}(t)\theta(t) + v(t), \\ \varphi^{\mathrm{T}}(t) &= [-y(t-1), \cdots, -y(t-n_a), u(t-1), \cdots, u(t-n_b)], \\ \theta^{\mathrm{T}}(t) &= [a_1(t), \cdots, a_{n_a}(t), b_1(t), \cdots, b_{n_b}(t)] \end{aligned} \tag{1}$$

where $\{u(t)\}$, $\{y(t)\}$ and $\{v(t)\}$ are the input, output and stochastic noise sequences of the system, respectively, superscript T denotes matrix transpose.

In developing the estimation algorithms of $\theta(t)$, many authors assumed that the parameter vector $\theta(t)$ satisfies certain special relation[1] or conditions[2]. But, just as Wittenmark[1] pointed out, it is even more difficult to know whether the parameters vector $\theta(t)$ satisfies these special relation or conditions than to estimate the parameters themselve. In this paper no restrictions on the parameter vector $\theta(t)$ are imposed, and the only assumption is that the system (1) is a stable time-varying process. The objective of this paper is, by means of RFFA, to obtain the real-time estimation of the time-varying parameter vector $\theta(t)$ by utilizing the observations up to and including time t. Lozano[3] has analyzed the convergence of RFFA. Unfortunately, in his analysis, as the forgetting factor becomes unity, the covariance matrix and parameter estimation error (PEE) grow without limits even for time invariant systems. Thus this result lacks consistency with previously known results. Canetti & Espana[4] have proved the convergence of RFFA of identifying time-varying systems, but the upper and lower bounds of the PEE can not be given. The chief results of this paper is that it is possible to find out the upper and lower bounds of the PEE, which appear to result in quite good tracking performance in the simulation examples.

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2 The Convergence of RFFA

RFFA of estimating the parameter vector of the model (1) can be described as

$$\hat{\theta}(t+1) = \hat{\theta}(t) + P(t+1)\varphi(t)[y(t) - \varphi^{\mathrm{T}}(t)\hat{\theta}(t)], \tag{2a}$$

$$P^{-1}(t+1) = \lambda P^{-1}(t) + \varphi(t)\varphi^{\mathrm{T}}(t), \tag{2b}$$

$$P(1) = aI \ (a>0), \quad \hat{\theta}(1) = \text{a very small real vector } (10^{-4})$$

where $\hat{\theta}(t)$ denotes the estimates of $\theta(t)$, λ is the forgetting factor $(0<\lambda<1)$.

Theorem 1 Assume that i) $\{v(t)\}$ is an independent random variable sequence with zero mean and mean square bounded, i. e.

A1) $$E[v(t)v(s)] = \delta_{ts}\sigma^2 < \infty, \quad \delta_{ij} = \begin{cases} 1, & i=j, \\ 0, & i \neq j. \end{cases}$$

ii) The following persistent excitation condition holds:

A2) $$\lim_{t\to\infty} \frac{1}{t}\sum_{s=1}^{t} \varphi(s)\varphi^{\mathrm{T}}(s) = E[\varphi(s)\varphi^{\mathrm{T}}(s)] = R > 0,$$

$$0 < m \leqslant \|\varphi(t)\|^2 = \mathrm{tr}[\varphi(t)\varphi^{\mathrm{T}}(t)] \leqslant M < \infty, \quad \text{a. s.}$$

where tr(*) denotes the trace of (*).

and iii) the parameter change rate $\alpha(t) = \theta(t) - \theta(t-1)$ is bounded, and $\alpha(t)$ and $v(t)$ are independent, ie.

A3) $$E\|\alpha(t)\|^2 \leqslant M_1 < \infty, \quad E[\alpha(t)v(s)] = 0,$$

then as $t\to\infty$, $\hat{\theta}(t)$ given by RFFA (2) satisfies

$$A_2 \leqslant \lim_{t\to\infty} E\|\hat{\theta}(t) - \theta(t)\|^2 \leqslant B_2$$

where

$$A_2 = \frac{\mathrm{tr}(R)(1-\lambda)\sigma^2}{M^2(1+\lambda)}, \quad B_2 = \frac{\mathrm{tr}(R)(1-\lambda)\sigma^2}{m^2(1+\lambda)} + \frac{M^2M_1}{m^2(1-\lambda)^2}, \quad 0<\lambda<1.$$

Theorem 2 For the time invariant deterministic system $y(t) = \varphi^{\mathrm{T}}(t)\theta[\theta(t) \equiv \theta, v(t) \equiv 0]$, the assumption A2) holds, then for $\{\hat{\theta}(t)\}$ given by RFFA (2), as $t\to\infty$, we have

i) $E\|\hat{\theta}(t) - \theta\|^2 \to 0$ exponentially fast, and

ii) $\hat{\theta}(t) \to \theta$, a. s. exponentially fast.

where $\hat{\theta}(1)$ is a random variable with $E\|\hat{\theta}(1)\|^2 \leqslant M_0 < \infty$.

Theorem 3 For the time invariant stochastic system $y(t) = \varphi^{\mathrm{T}}(t)\theta + v(t)[\theta(t) \equiv \theta]$, the assumption A1)~A3) hold, then $\{\hat{\theta}(t)\}$ given by RFFA (2), as $t\to\infty$, satisfies

$$A_1 \leqslant \lim_{t\to\infty} E\|\hat{\theta}(t) - \theta\|^2 \leqslant B_1$$

where $$A_1 = \frac{\mathrm{tr}(R)(1-\lambda)\sigma^2}{M^2(1+\lambda)}, \quad B_1 = \frac{\mathrm{tr}(R)(1-\lambda)\sigma^2}{m^2(1+\lambda)}, \quad 0<\lambda<1.$$

Theorem 3 shows that RFFA can not give uniform estimates of the parameters of time invariant stochastic systems, while the ordinary least squares algorithm (LS algorithm for short) may give uniform estimates of its parameters[5].

Theorem 2 and Theorem 3 may be used to estimate the error of the estimates $(\hat{\theta}(t))$ given by RFFA.

The proofs of these three theorems are given in what follows.

Define the parameter tracking error as $\tilde{\theta}(t)=\hat{\theta}(t)-\theta(t)$ and assume that $\tilde{\theta}(1)$ and $v(t)$ are independent, and using the relations (1) and (2), we get

$$\begin{aligned}\hat{\theta}(t+1) &=\hat{\theta}(t+1)-\theta(t+1)=\hat{\theta}(t+1)-(\theta(t)+\alpha(t+1)) \\ &=[I-P(t+1)\varphi(t)\varphi^{\mathrm{T}}(t)]\tilde{\theta}(t)+P(t+1)\varphi(t)v(t)-\alpha(t+1) \\ &=\lambda P(t+1)P^{-1}(t)\tilde{\theta}(t)+P(t+1)\varphi(t)v(t)-\alpha(t+1) \\ &=\gamma_1+\gamma_2+\gamma_3 \end{aligned} \tag{3}$$

where
$$\gamma_1=\lambda^t P(t+1)P^{-1}(1)\tilde{\theta}(1),$$
$$\gamma_2=P(t+1)\sum_{s=1}^{t}\lambda^{t-s}\varphi(s)v(s),$$
$$\gamma_3=-P(t+1)\sum_{s=2}^{t+1}\lambda^{t+1-s}P^{-1}(s)\alpha(s).$$

From (2b) and A3), we have

$$\frac{m(1-\lambda^t)}{1-\lambda}I+\lambda^t P^{-1}(1)\leqslant P^{-1}(t+1)\leqslant\frac{M(1-\lambda^t)}{1-\lambda}I+\lambda^t P^{-1}(1) \tag{4}$$

as $t\to\infty$, it gives

$$\frac{m(1-\lambda^t)}{1-\lambda}I\leqslant P^{-1}(t+1)\leqslant\frac{M(1-\lambda^t)}{1-\lambda}I\leqslant\frac{M}{1-\lambda}I \tag{5}$$

or $t\to\infty$
$$\frac{1-\lambda}{M(1-\lambda^t)}I\leqslant P(t+1)\leqslant\frac{1-\lambda}{m(1-\lambda^t)}I. \tag{6}$$

The proof of theorem 1 Taking $\|*\|^2$ of (3) and using (5) and (6), we have

$$\begin{aligned}&\lim_{t\to\infty}E\|\hat{\theta}(t+1)-\theta(t+1)\|^2=\lim_{t\to\infty}E\|\gamma_1+\gamma_2+\gamma_3\|^2 \\ &=\lim_{t\to\infty}[E\|\gamma_1\|^2+2E(\gamma_1^{\mathrm{T}}\gamma_2)+E\|\gamma_2\|^2+2E(\gamma_1^{\mathrm{T}}\gamma_3)+2E(\gamma_2^{\mathrm{T}}\gamma_3)+E\|\gamma_3\|^2],\end{aligned} \tag{7}$$

$$E\|\gamma_1\|^2=E[a^2\lambda^{2t}\tilde{\theta}^{\mathrm{T}}(1)P^2(t+1)\tilde{\theta}(1)]\leqslant\frac{a^2M_0}{m^2}\lambda^{2t}\left(\frac{1-\lambda}{1-\lambda^t}\right)^2 \tag{8}$$

so, as $t\to\infty$, it gives

$$\lim_{t\to\infty}E\|\gamma_1\|^2=\lim_{t\to\infty}O(\lambda^{2t})\to 0,\quad 0<\lambda<1, \tag{9}$$

$$\begin{aligned}\lim_{t\to\infty}E\|\gamma_2\|^2&=\lim_{t\to\infty}E[\sum_{s,i}^{t}\lambda^{2t-s-i}\varphi^{\mathrm{T}}(s)P^2(t+1)\varphi(i)v(s)v(i)] \\ &=\lim_{t\to\infty}\sum_{s=1}^{t}\lambda^{2(t-s)}E[\varphi^{\mathrm{T}}(s)P^2(t+1)\varphi(s)v^2(s)] \\ &=\lim_{t\to\infty}\frac{1-\lambda^{2t}}{1-\lambda^2}E[\varphi^{\mathrm{T}}(s)P^2(t+1)\varphi(s)v^2(s)].\end{aligned} \tag{10}$$

Substituting (6) into (10), it is not difficult to get

$$\lim_{t\to\infty}\frac{\mathrm{tr}(R)\sigma^2(1+\lambda^t)(1-\lambda)}{M^2(1+\lambda)(1-\lambda^t)}\leqslant\lim_{t\to\infty}E\|\gamma_2\|^2\leqslant\lim_{t\to\infty}\frac{\mathrm{tr}(R)\sigma^2(1+\lambda^t)(1-\lambda)}{m^2(1+\lambda)(1-\lambda^t)} \tag{11}$$

or
$$A_1\leqslant\lim_{t\to\infty}E\|\gamma_2\|^2\leqslant B_1,\quad 0<\lambda<1, \tag{12}$$

$$\lim_{t\to\infty} E\|\gamma_3\|^2 = \lim_{t\to\infty} E\Big[\sum_{s,i=2}^{t+1} \lambda^{2(t+1)-s-i}\alpha^{\mathrm{T}}(s)P^{-1}(s)P^2(t+1)P^{-1}(i)\alpha(i)\Big]$$

$$\leqslant \lim_{t\to\infty}\sum_{s,i=1}^{t}\lambda^{2t-s-i}\frac{1}{m^2}\Big(\frac{1-\lambda}{1-\lambda^t}\Big)^2\Big(\frac{M}{1-\lambda}\Big)^2 M_1 = \frac{M^2M_1}{m^2(1-\lambda)^2},\quad 0<\lambda<1, \tag{13}$$

$$\lim_{t\to\infty} E[\gamma_2^{\mathrm{T}}\gamma_3]^2 = -\lim_{t\to\infty} E\Big[\sum_{s=1}^{t}\sum_{i=2}^{t+1}\lambda^{2t+1-s-i}\varphi^{\mathrm{T}}(s)P^2(t+1)P^{-1}(i)\alpha(i)v(s)\Big] = 0. \tag{14}$$

Using inequality $E\|AB\| \leqslant \sqrt{E\|A\|^2E\|B\|^2}$, from (9), (12)~(14) and (7), we may obtain

$$A_2 \leqslant \lim_{t\to\infty} E\|\hat{\theta}(t)-\theta(t)\|^2 \leqslant B_2,\ 0<\lambda<1. \tag{15}$$

This proves the assertion of theorem 1.

The proof of theorem 2 Since $v(t)\equiv 0$, $\alpha(t)\equiv 0$, from (7) and (9) it is easy to obtain

$$\lim_{t\to\infty} E\|\hat{\theta}(t+1)-\theta\|^2 = \lim_{t\to\infty} E\|\gamma_1\|^2 = \lim_{t\to\infty} O(\lambda^{2t}) \to 0,\ 0<\lambda<1. \tag{16}$$

or

$$E\|\hat{\theta}(t)-\theta\|^2 = O(\lambda^{2t}) \to 0,\ 0<\lambda<1, \tag{17}$$

$$\hat{\theta}(t)-\theta \to 0,\ 0<\lambda<1.$$

This proves theorem 2.

If in (8), $\lambda \to 1$ (ie. LS algorithm), we get, $t\to\infty$

$$E\|\hat{\theta}(t)-\theta\|^2 = O\Big(\frac{1}{t^2}\Big) \to 0,\ \lambda = 1. \tag{18}$$

From (11) and (18) we may reach the conclusions for time-invariant deterministic systems: i) the PEE given by LS algorithm converges to zero at the rate of $\left(\frac{1}{t}\right)$, and ii) the PEE given by RFFA converges to zero at an exponential rate. It is clear that RFFA has faster convergence rate than LS algorithm.

The proof of theorem 3 Since $\alpha(t)\equiv 0$, from (7) we have

$$\lim_{t\to\infty} E\|\hat{\theta}(t+1)-\theta\|^2 = \lim_{t\to\infty} E\|\gamma_1+\gamma_2\|^2. \tag{19}$$

From (9), (12) and (19) we may obtain

$$A_1 \leqslant \lim_{t\to\infty} E\|\hat{\theta}(t)-\theta\|^2 \leqslant B_1,\ 0<\lambda<1. \tag{20}$$

This completes the proof of theorem 3.

If in (11), $\lambda\to 1$ (LS algorithm), then from (9), (11) and (19), we have

$$\lim_{t\to\infty}\frac{\mathrm{tr}(R)\sigma^2}{M^2t} \leqslant \lim_{t\to\infty} E\|\hat{\theta}(t)-\theta\|^2 \leqslant \lim_{t\to\infty}\frac{\mathrm{tr}(R)\sigma^2}{m^2t},\quad \lambda = 1. \tag{21}$$

The expressions (20) and (21) show that for the time-invariant stochastic systems: i) PEE given by LS algorithm converges to zero under the mean square sense, and its convergence rate is of $\left(\frac{1}{\sqrt{t}}\right)$, ii) RFFA can not give uniform and unbiased estimates, but it gives a bounded mean square PEE.

For the time-varying stochastic systems, the mean square PEE given by RFFA is bounded. If in (15), $\lambda\to 1$ (LS algoirthm), then the right-hand side of (15) goes to infinity, that is to say, the mean square PEE given by LS algorithm is unbounded, so LS algorithm is unable to track the time-varying parameters.

For the variable forgetting factor λ_t, as $\lambda_{\min} \leqslant \lambda_t \leqslant \lambda_{\max}$, Theorems 1 to 3 hold.

3 Simulation Studies

Example 1 $y(t) - 1.35y(t-1) + 0.85y(t-2) = 0.68u(t-1) + 0.32u(t-2) + v(t)$ where $\{u(t)\}$ is taken as a zero mean and unit variance random variable sequence, and $\{v(t)\}$ is a white noise sequence with zero mean and variance $\sigma^2 = 0.1^2$. The data length $L = 1000$, simulation results are as follows:

$$\lambda = 0.95, \quad A_1 = 2.284 \times 10^{-6}, \quad B_1 = 6.344 \times 10^{-4},$$
$$E\|\hat{\theta}(L) - \theta\|^2 \approx \|\hat{\theta}(L) - \theta\|^2 = 4.439 \times 10^{-4}.$$

Example 2 $y(t) + a(t)y(t-1) = b(t)u(t-1) + v(t)$,

$$a(t) = 0.15 + 0.21\sin(0.1t), \quad b(t) = 0.20 + 0.03\sqrt{t}.$$

Simulation conditions are the same as those of the example 1, and

$$\lambda = 0.8, \quad A_2 = 1.192 \times 10^{-4}, \quad B_2 = 0.5663, \quad \|\hat{\theta}(L) - \theta(L)\|^2 = 7.743 \times 10^{-3}$$

Example 3 $y(t) + a(t)y(t-1) = b(t)u(t-1) + v(t)$,

$$a(t) = \begin{cases} 0.3, & t \leqslant 80, \\ 0.5, & t > 80, \end{cases} \qquad b(t) = 0.45.$$

Simulation conditions are the same as those of the example 1, and

$$\lambda = 0.8, \quad A_2 = 2.426 \times 10^{-4}, \quad B_2 = 5.955, \quad \|\hat{\theta}(L) - \theta(L)\|^2 = 4.451 \times 10^{-3}.$$

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辨识时变系统遗忘因子算法的收敛性分析

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摘要: 著名的递推遗忘因子算法(RFFA)可以用来辨识时变系统参数,具有良好的跟踪性能. 本文借助于随机过程理论分析了 RFFA 的收敛性和稳定性,给出了参数跟踪误差的上下界.

关键词: 时变系统; 参数估计; 遗忘因子方法

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更　正

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"$> L(\gamma^1)[m_{ij}(k_2) - m_{ij}(1)] + (k_1 - 1)w(\gamma^1) + (k_1 - 1)[m_{ij}(k$

$$\frac{-\ m_{ij}(1)}{L(\gamma^1) + k_2 - 1}, \qquad (3.6)"$$

应改为

"$> L(\gamma^1)[m_{ij}(k_2) - m_{ij}(1)] + (k_1 - 1)w(\gamma^1) + (k_1 - 1)[m_{ij}(k_2) - m_{ij}(1)]$,

上式与下面的不等式等价

$$\frac{w(\gamma^1) + m_{ij}(k_1) - m_{ij}(1)}{L(\gamma^1) + k_1 - 1} > \frac{w(\gamma^1) + m_{ij}(k_2) - m_{ij}(1)}{L(\gamma^1) + k_2 - 1}, \qquad (3.6)"$$

特此更正, 谨此向作者深表歉意.

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A New Method of Discrete Variable Structure Control and Application to Hydraulic Servo System

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Abstract: This paper deals with the thorough investigation for the discrete variable structure control. Aiming at the special requirement of servos, a new kind of sliding control——IVSC is put forward, it comprises an integral controller followed by a variable structure controller, so the system can acquire the accurate servotracking property and fair robustness to parameter variations and external disturbances. The digital simulation result of pump-controlled-motor velocity servo system using the control law presented shows that the control effect is quite satisfactory.

Key words: variable structure control; servo systems; discrete system; robustness; hydraulic servo system

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