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On the Evaluation of Stochastic Controllers for Systems with Variance Constraints*

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Abstract: The performance requirements of many engineering control problems are naturally described in terms of variance values of system inputs, outputs and states. The primary purpose of this paper is to deal with the problem of evaluating the stochastic controllers for systems with variance constraints. The algorithm to identify the variances of system inputs and outputs is presented. This algorithm may be used to evaluate that if the stochastic controllers can make the feedback system meet the prespecified variance constraints, so that the controllers can be appropriately adjusted prior to its implementation on practical systems and the design goal can be achieved.

Key words: stochastic control; discrete systems; variance constraints

1 Introduction

Performance requirements which are directly expressed as upper bounds on the variances of the system outputs and inputs are common in various engineering systems. For example, in large space structures[1,4,5], the vibration level at multiple points on the structure must be kept within specified bound. This performance requirement can be expressed as the variance constraints on the system inputs and outputs. In the problem of fire control[2], the main performance indices (for example, stability, cutting frequency and impulse response characteristics, etc.) of the laser communication system situated on the motional carrier can all be expressed as the upper bounds on the variances of system outputs. The energy constraint requirement upon the actuarors can be transformed to the variance contraints of system inputs. Similar examples are quite common in many stochastis control problems. Hence, a multiobjective design task may be formulated as follows. Given the variance constraints of system inputs and outputs, it is essential to design the controllers such that the inputs and outputs of the feedback system satisfy the prespecified index requirement. The prespecified variance constraint, of course, should not be less than the minimum variance value which is obtained by using the traditional LQG theory.

A design method called constrained linear quadratic gaussian (CLQG) control, which attempts to solve the above problem by using a penalty function method, has been presented in [3]. In this approach the controller is designed to minimize a weighted sum

$$v = \sum_{i=1}^{n_y} q_i E(y_i^2) + \sum_{i=1}^{n_u} r_i E(u_i^2) \tag{1.1}$$

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of the state and input variances. However, minimizing a scalar sum does not ensure that the multiple variance requirements will be satisfied. It is important but difficult to choose the acceptable weights in the CLQG methodology[7], because the choice of these weights may directly influence the synthesis of controllers and the variance values of inputs and outputs of the feedback system. In this case, it is very important to evaluate the controller prior to its implementation. Before using the controller to practical systems, we may calculate the variance values of inputs and outputs of the feedback system and evaluate this controller if it can make the feedback system satisfy the prespecified porformance requirement. If not, we can adjust the weights again. Here, the importance of performance evaluation of the stochastic controllers is clear.

To solve the above stochastic control problem with variance constraints, a more straightforward methodology called covariance control (CC) theory has been provided in[4,5]. The main idea of this theory is to assign the steady-state covariance matrix of system states to the desired value. The freedom contained in the design may be exploited to achieve the variance constraints of system inputs and outputs. The solution of the covariance assignment problem which subjects to state estimation feedback has been presented in[4,5]. It should be pointed out that, in order to ensure the existence of the controller stabilizing the feedback system, a fictitious "noise" $\varepsilon DSD^{\mathrm{T}}$ is added to the Lyapunov equation in which the steady-state covariance is satisfied. However, the synthesis of controllers depends directly on the choice of this fictitious"noise" and the solution provided by[4,5] is approximate sequentially. Thus, the controllers must not make the feedback system strictly meet the desired variance contraints. Owing to the above reasons, the performance evaluation of controllers is also important in the covariance control theory.

2 Problem Statement

Consider the stationary vector process x generated by

$$x(k+1) = Ax(k) + Bu(k) + Dw(k), \tag{2.1}$$

$$y(k) = Mx(k) + \nu(k). \tag{2.2}$$

Here y is the measure output vector; $x \in \mathrm{R}^{n_x}$, $y \in \mathrm{R}^{n_y}$, $u \in \mathrm{R}^{n_u}$, $w \in \mathrm{R}^{n_w}$, $\nu \in \mathrm{R}^{n_y}$, w and ν are the zero mean white noise processes with covariances $W > 0$ and $V > 0$, respectively. $w(k)$ and $x(0)$, $\nu(k)$ and $x(0)$ are supposed to be uncorrelated, respectively. The pairs (A,B), (A,D), (A,M) are assumed to be stabilizable, controllable, and detectable. No loss of generality, we suppose that w and ν is uncorrelated.

The input u is some state estimate feedback law

$$u(k) = G\hat{x}(k) \tag{2.3}$$

and the state estimate vector $\hat{x}$ is generated by

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K[y(k) - M\hat{x}(k)] \tag{2.4}$$

where the steady-state Kalman filter gain K is the solution of

$$K = APM^{\mathrm{T}}[V + MPM^{\mathrm{T}}]^{-1}, \tag{2.5}$$

$$P = (A - KM)PA^{\mathrm{T}} + DWD^{\mathrm{T}}, \tag{2.6}$$

and P is the covariance of the estimation error, i. e. ,

$$P = E[(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^T].$$

To this end, the problem of evaluating controllers with variance constraints may be formulated as follows. Given the controllers obtained via CLQG or CC theory, prior to their implementation on practical systems, identify the feedback outputs $\{y(k)\}$ and the feedback inputs $\{u(k)\}$, and give the algorithm to obtain the variance values of the feedback outputs and inputs. After all, we can evaluate if the controllers satisfy the prespecified variance constraints and adjust the controllers appropriately.

3 Main Results and Derivations

3.1 Identification of Feedback Outputs

Let $a(k) = x(k) - \hat{x}(k)$, then $P = E(a(k)a(k)^T)$.

Using (2.2), we have

$$\begin{aligned} y(k) &= Mx(k) + v(k) \\ &= M(a(k) + \hat{x}(k)) + v(k) \\ &= M\hat{x}(k) + b(k). \end{aligned}$$

Here

$$b(k) = Ma(k) + v(k).$$

From (2.4), it is obtained that

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + K[y(k) - M\hat{x}(k)] \\ &= (A + BG)\hat{x}(k) + Kb(k). \end{aligned} \tag{3.1}$$

(3.1) may be expressed as

$$[I - (A + BG)z^{-1}]\hat{x}(k+1) = Kb(k) \tag{3.2}$$

or

$$\hat{x}(k+1) = [I - (A + BG)z^{-1}]^{-1}Kb(k) \tag{3.2'}$$

where z is the z-transform operator and in this context z^{-1} serves as the one step backward shift operator.

Therefore the feedback output variable is given by

$$y(k) = M[I - (A + BG)z^{-1}]^{-1}Kb(k-1) + b(k). \tag{3.3}$$

It is noted that the inverse matrix appears in (3.3) and brings great difficulty in the computation. Using the approach presented in [6], we may convert this inverse matrix to the other form which is easy to compute.

Defining the following matrix polynomial

$$H(z) = H_0 + H_1z^{-1} + H_2z^{-2} + \cdots + H_nz^{-n}, \quad H_0 = I$$

where the coefficients H_i are given by

$$\begin{bmatrix} H_1^T \\ H_2^T \\ \vdots \\ H_n^T \end{bmatrix} = -\left[\begin{bmatrix} M(A+BG)^{n-1} \\ M(A+BG)^{n-2} \\ \vdots \\ M \end{bmatrix}\begin{bmatrix} M(A+BG)^{n-1} \\ M(A+BG)^{n-2} \\ \vdots \\ M \end{bmatrix}^T\right]^{-1}$$

$$\begin{bmatrix} M(A+BG)^{n-1} \\ M(A+BG)^{n-2} \\ \vdots \\ M \end{bmatrix} [M(A+BG)^n]^{\mathrm{T}} \tag{3.4a}$$

and the parameter n is the smallest integer to satisfy

$$\operatorname{rank} \begin{bmatrix} M(A+BG)^{n-1} \\ M(A+BG)^{n-2} \\ \vdots \\ M \end{bmatrix} = \operatorname{rank} \begin{bmatrix} M(A+BG)^{n} \\ M(A+BG)^{n-1} \\ \vdots \\ M \end{bmatrix} \tag{3.4b}$$

Premultiplying $H(z)$ to both sides of (3.3), we have

$$\sum_{i=0}^{n} H_i z^{-i} y(k) = \sum_{i=0}^{n} H_i z^{-i} b(k) + \sum_{i=0}^{n-1} \sum_{j=0}^{i} H_j M(A+BG)^{i-j} z^{-i} K b(k-1)$$

or

$$\sum_{i=0}^{n} H_i z^{-i} y(k) = [\sum_{i=0}^{n} H_i z^{-i} + \sum_{i=0}^{n-1} \sum_{j=0}^{i} H_j M(A+BG)^{i-j} K z^{-i-1}] b(k). \tag{3.5}$$

Here, the equation (3.5) can be directly gotten according to the approach provided by reference [6].

Remark In the equation (3.5), the matrices H_i, A, B, G, M are known. Using the assumptions, it is not difficult to obtain the conclusion that the variance of $b(k)$ is $MPM^{\mathrm{T}} + V$. Thus, we can easily obtain the variance of the feedback output $y(k)$.

3.2 Identification of Feedback Inputs

Form (3.2), we have

$$\begin{aligned} u(k) &= G\hat{x}(k) \\ &= G[I - (A+BG)z^{-1}]^{-1} K b(k-1). \end{aligned} \tag{3.6}$$

By a similar approach to the section (3.1), we can remove the inverse matrix in (3.6) and obtain

$$\sum_{i=0}^{n} H_i z^{-i} u(k) = \sum_{i=0}^{n-1} \sum_{j=0}^{i} H_j G(A+BG)^{i-j} K z^{-i-1} b(k). \tag{3.7}$$

The coefficients H_i are given as below:

$$\begin{bmatrix} H_1^{\mathrm{T}} \\ H_2^{\mathrm{T}} \\ \vdots \\ H_n^{\mathrm{T}} \end{bmatrix} = - \left[\begin{bmatrix} G(A+BG)^{n-1} \\ G(A+BG)^{n-2} \\ \vdots \\ G \end{bmatrix} \begin{bmatrix} G(A+BG)^{n-1} \\ G(A+BG)^{n-2} \\ \vdots \\ G \end{bmatrix}^{\mathrm{T}} \right]^{-1}$$

$$\begin{bmatrix} G(A+BG)^{n-1} \\ G(A+BG)^{n-2} \\ \vdots \\ G \end{bmatrix} [G(A+BG)^{n}]^{\mathrm{T}} \tag{3.8}$$

and n chosen such that n is the smallest integer to satisfy

$$\operatorname{rank}\begin{bmatrix} G(A+BG)^{n-1} \\ G(A+BG)^{n-2} \\ \vdots \\ G \end{bmatrix} = \operatorname{rank}\begin{bmatrix} G(A+BG)^{n} \\ G(A+BG)^{n-1} \\ \vdots \\ G \end{bmatrix}. \tag{3.9}$$

From the equation (3.7), the variance of feedback inputs can be gotten directly.

4 The Algorithm

In this section the algorithm to evaluate the stochastic controllers with the variance constraints is provided. It is assumed that the variance constraints of the outputs and inputs are given and the feedback controllers are obtained via CLQG theory or CC theory. We want to evaluate these controllers prior to its implementation.

Step 1 Solve the Kalman filter equation (2.5), (2.6). Obtain the steady-state Kalman filter gain K, the covariance P of the estimation error and the variance $MPM^{\mathrm{T}} + V$ of $b(k)$.

Step 2 When seeking the variance of the feedback outputs, determine the smallest integer n to satisfy (3.4) and write the matrix polynomial $H(z)$. The case of seeking the variance of the feedback inputs is similar.

Step 3 From equations (3.5) and (3.7), obtain the variances of feedback system outputs and inputs.

Step 4 Vertify that if the variances of feedback-system outputs and inputs satisfy the prespecified performance requirements. If not, adjust the weights in the case of using CLQG methods and adjust the small parameter ε in the case of using CC methods.

Step 5 Obtain the feedback controllers again and repeat the above steps until the variances of feedback outputs and inputs meet prespecified variance contraints strictly.

5 An Illustrative Example

In this section a numerical example is provided to demonstrate the directness and effectiveness of the present approach.

Consider the state-space model form

$$x(k+1) = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} u(k) + w(k), \tag{5.1a}$$

$$y(k) = [1 \quad 0] x(k) + v(k) \tag{5.2b}$$

with

$$W = I_2, \quad V = 1.$$

It is assumed that, from the CLQG theory or CC theory, the feedback controllers are obtained as follows:

$$G_1 = [-2.5 \quad -1.25], \quad G_2 = [-2.5 \quad 1].$$

Now our object is to verity the above controllers if they can make the feedback system satisfy the following steady-state output variance constraint:

$$E(y(k))^2 < 4.168$$

and the case of computing the feedback input variance is similar. By solving the Kalman filter equation (2.5), (2.6), we can easily get that

$$K = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}, \quad E(b(k))^2 = MPM^T + V = 4.$$

For the controller G_1, it is found that

$$A + BG_1 = \begin{bmatrix} 0 & 0 \\ -0.5 & 0.75 \end{bmatrix}, \quad n = 1, \quad H_1 = 0$$

where the parameter n is the smallest integer to satisfy (3.4b), and the coefficients H_i are given by (3.4a).

Therefore, the feedback output is given as below

$$y(k) = b(k), \quad E(y(k))^2 = E(b(k))^2 = 4.$$

Similarly, for the controller G_2, it is easy to obtain that

$$A + BG_2 = \begin{bmatrix} 0 & 0.9 \\ 0.5 & 1.2 \end{bmatrix}, \quad n = 2, \tag{5.2a}$$

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = -\begin{bmatrix} 0 & 0.9 \\ 1 & 0 \end{bmatrix}^{-2} \begin{bmatrix} 0 & 0.9 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -0.45 \\ 1.08 \end{bmatrix} = \begin{bmatrix} -1.2 \\ 0.45 \end{bmatrix}. \tag{5.2b}$$

Substituting (5.2a), (5.2b) into (3.5) yields

$$(1 - 1.2z^{-1} + 0.45z^{-2})y(k) = (1 - 0.2z^{-1} - 0.3z^{-2})b(k)$$

and

$$E(y(k))^2 = 16.$$

It is clear that, from the above results, the controller G_1 meets the output variance constraint while the controller G_2 does not.

6 Conclusions

This paper focuses on the problem of evaluating a class of stochastic controllers with variance constraints. The identification algorithm of feedback system outputs and inputs is given. This algorithm can be directly used to evaluate that if the controllers obtained by different methods are appropriate. Hence, we can adjust the controllers prior to their implementation. The results of this paper can be applied in the design of practical control system.

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关于约束方差控制器的评估问题的研究

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摘要：本文研究当系统的性能指标表现为系统输入及输出的方差约束时一类随机控制器的评价问题,给出了系统反馈输入及输出的辨识算法.该算法可用来评价反馈控制器能否使反馈系统满足既定的方差约束,以便在对实际系统实施控制之前及时调整控制器,从而达到设计目的.

关键词：随机控制；离散系统；方差约束

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