2 Basic Lagrangian Relaxation Approach to Job Shop Scheduling

2.1 Problem Formulation

The job-shop scheduling problem considered in this paper is to schedule n jobs on m machines such that a weighted quadratic tardiness cost function of the jobs is minimized, where each job requires a sequence of operations for completion, and each operation may be performed on one of a set of machine types. The jobs have different due dates and may have different weights. The following variables will be used for the formulation of the scheduling problem, where operation j of job i is referred to as operation (i,j).

s_{ij}: Starting time of operation (i,j); c_{ij}: Completion time of operation (i,j); C_i: Completion time of job i; D_i: Due date of job i; w_i: Weight (value of importance) of job i; n: Number of jobs; n_i: Number of operations for job i; m: Number of machines; M: Number of machine types; m_i: Number of machines of type i; M_{ij}: Set of machine types capable of performing operation (i,j); m_{ij}: Machine type selected to process operation (i,j); $m_{ij} \in M_{ij}$; I_{ij}: Set of operations of job i immediately following operation (i,j); t_{ijr}: Processing time of operation (i,j) on machine type $r \in M_{ij}$; H: Time horizon under consideration; T_i: Tardiness of job i, defined as $\max\{0, C_i - D_i\}$; J: Objective function to be optimized.

It is assumed that the precedence constraints of a job form a directed acyclic graph and, without loss of generality, that each job ends with a single operation so that $C_i = c_{in_i}$. All operations are assumed to be nonpreemptive and the time horizon H is long enough to complete all the jobs. All jobs are assumed available for processing at time 0 and, for simplicity, all machines are assumed available throughout the whole time horizon from time 0 to time H. Note that all results of this paper can be applied to the situation where some machines may be not available at some time slots within the time horizon.

The decison variables of the scheduling problem are the starting times of all operations $\{s_{ij}\}$ and the machine types $\{m_{ij}\}$. The cost function to be minimized is a weighted quadratic tardiness function of the jobs.

$$J = \sum_{i=1}^{n} w_i T_i^2. \tag{1}$$

Let

$$\varphi(\tau) = \begin{cases} 1, & \text{if } \tau \geqslant 0, \\ 0, & \text{if } \tau < 0, \end{cases}$$

$$O_k = \{(i,j) \mid m_{ij} = k\}.$$

The deterministic scheduling problem can now be formulated as follows

$$P: \quad \min_{\{m_{ij}\},\{s_{ij}\}} J, \tag{2}$$

subject to precedence constraints:

$$s_{ij} + t_{ijm_{ij}} \leqslant s_{ir}, \quad (i = 1,2,\cdots,n; \quad j = 1,2,\cdots,n_{i-1}; \quad r \in I_{ij}) \tag{3}$$

capacity constraints:

$$\sum_{(i,j)\in O_k} (\varphi(\tau - s_{ij}) - \varphi(\tau - s_{ij} - t_{ijk})) \leqslant m_k, \quad (k = 1,2,\cdots,M; \tau = 0,1,\cdots,H). \tag{4}$$

2.2 Solution Methodology

In this subsection, we briefly introduce the Lagrangian relaxation framework to job shop scheduling problems proposed by Peter B. Luh and his olleagues.

The capacity constraints (4) and (3) can be relaxed by using nonnegative Lagrange multipliers $\pi_{k\tau}(k = 1,2,\cdots,M;\tau = 0,1,\cdots,H)$ and $\lambda_{ijr}(i = 1,2,\cdots,n; j = 1,2,\cdots,n_i - 1; r \in M_{ij})$. This leads to the following relaxed problem:

$$\text{RP:}\quad \min_{\{m_{ij}\},\{s_{ij}\}}\{\sum_i\{w_iT_i^2 + \sum_{k,\tau}[\pi_{k\tau}(\sum_{(i,j)\in O_k}(\varphi(\tau - s_{ij}) - \varphi(\tau - s_{ij} - t_{ijk})) - m_k)] + \sum_{j,r\in I_{ij}}[\lambda_{ijr}(s_{ij} + t_{ijm_{ij}} - s_{ir})]\}\}. \tag{5}$$

The Lagrangian dual to problem P is

$$\text{DP:}\quad \max_{\pi,\lambda\geqslant 0}\{-\sum_{k,\tau}\pi_{k\tau}m_k + \sum_i \min_{\{m_{ij}\},\{s_{ij}\}}\{w_iT_i^2 + \sum_j[\sum_{\tau=s_{ij}}^{s_{ij}+t_{ijm_{ij}}-1}\pi_{m_{ij}\tau} + \sum_{r\in I_{ij}}\lambda_{ijr}(s_{ij} + t_{ijm_{ij}} - s_{ir})]\}\}. \tag{6}$$

RP can be decomposed into the following minimization subproblems:

$$\text{RP}_i:\quad \min_{\{m_{ij},s_{ij}\}}\{w_iT_i^2 + \sum_j[\sum^{s_{ij}} + t_{ijm_{ij}-1_{\tau=s_{ij}}}\pi_{m_{ij}\tau} + \sum_{r\in I_{ij}}\lambda_{ijr}(s_{ij} + t_{ijm_{ij}}) - \sum_{r:j\in I_{ir}}\lambda_{irj}s_{ij})\},\quad i = 1,2,\cdots,n. \tag{7}$$

For a particular operation (i,j) and a particular machine type $r \in I_{ij}$, (7) can be further decomposed to the following operation level subproblems:

$$\text{RP}_{ij}:\quad \min_{\{m_{ij},s_{ij}\}}\{w_iT_i^2\Delta_{ij} + \sum_{\tau=s_{ij}}^{s_{ij}+t_{ijm_{ij}}-1}\pi_{m_{ij}\tau} + (\sum_{r\in I_{ij}}\lambda_{ijr} - \sum_{r:j\in I_{ir}}\lambda_{irj})s_{ij}\},\quad j = 1,2,\cdots,n_i \tag{8}$$

where $\Delta_{ij} = 1$ if j is the last operation of the job i and 0 otherwise.

The basic Lagrangian relaxation approach (LR) to solve the job shop scheduling problem now can be described as follows:

1) Solve the dual problem (i. e., maximize the dual objective function) by using the subgradient method;

2) Construct a feasible schedule from the solution to the dual problem by using a list-scheduling method;

3) Evaluate the obtained feasible schedule by using the approximate relative duality gap.

At each iteration of the subgradient algorithm to maximize the dual objective function, an operation level subprolem for each operation is solved for given multipliers $\{\pi_{k\tau}\}$ and $\{\lambda_{ijr}\}$.

The operation level subproblem is enumeratively solved for each candidate machine type $r \in I_{ij}$. The starting time and machine type associated with the smallest is selected and used to update the multipliers. However, the selection of s_{ij} strongly depends on the sign of $\sum_{r\in I_{ij}}\lambda_{ijr} - \sum_{r:j\in I_{ir}}\lambda_{irj}$, which is a constant for the operation level subproblem. As a consequence, s_{ij} is very large when this term is negative and very small when it is positive. This results in solution oscillation when we solve the dual problem by using the subgradient algorithm.

3 Sequential Lagrangian Relaxation Approach

In this section, we propose a sequential Lagrangian relaxation approach (SLR) to the job shop scheduling problem, which can not only avoid the solution oscillation, but also generate high-quality near-optimal schedules.

3.1 Structure of SLR Algorithm

Let S be the vector whose components are $s_{ij}, i = 1,2,\cdots,n, j = 1,2,\cdots,n_i$. We introduce a set of constrainted minimization problems as follows:

$$P^k: \quad \min_S J^k(S,S^{k-1}) = J + \rho_k \| S - S^{k-1} \|^2, \tag{9}$$

$$S \text{ is subject to } (3),(4).\ k = 1,2,\cdots,$$

where $S^k = \arg\min_S \{J^k(S,S^{k-1}) \mid, S \text{ is subject to (3) and (4)}\}$, S^0 is taken to be a feasible solution of the constraints (3) and (4), $\rho_k > 0, k = 0,1,\cdots$ are positive real numbers.

The SLR algorithm can be described as follows:

SLR Algorithm

Step 0 Take S^0 to be a feasible solution of the constraints (3) and (4), and set $k = 1$;

Step 1 Solve P^k using LR approach to obtain a near-optimal solution of P^k and denote the solution by S^k;

Stpe 2 Calculate $e = \| S^k - S^{k-1} \|$. If $e < \varepsilon$(ε is a given admissible error), then S^k is taken as the solution of the scheduling problem and stop. Otherwise, $k = k + 1$, goto step 1.

Note that the second term of J^k, $\rho_k \| S - S^{k-1} \|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n_i}(s_{ij} - s_{ij}^{k-1})$, is decomposable with respect to $s_{ij}, i = 1,2,\cdots,n, j = 1,2,\cdots,n_i$, so the relaxed problem of P^i can be decomposed into a number of operation level subproblems, so LR approach introduced in the previous section is applicable to P^i. More importantly, this term (a quadratic term) prevents LR approach from solution oscillation.

3.2 Properties of SLR Algorithm

Let $E = \{S \mid S \text{ is subject to (3) and (4)}\}$. Without loss of generality, we assume that $w_i, d_i, t_{ijr}, i = 1,2,\cdots,n, j = 1,2,\cdots,n_i, r \in M_{ij}$ are all integers. Suppose that S^k is solution of P^k obtained by using LR approach at step 1, that $J^{k^*}(S^{k-1})$ is the optimal objective value of P^k and that g_k is the gap (difference) between $J^k(S^k,S^{k-1})$ and $J^{k^*}(S^{k-1})$.

Firstly, if $\rho_k > g_k$ for $k \geqslant K$ (K is a sufficiently large positive integer), then we can prove that

$$\lim_{k\to\infty} S^k = S^*, \quad S^* \in E.$$

From (9), we have:

$$
\begin{aligned}
& J(S^k) + \rho_k \| S^k - S^{k-1} \|^2 \\
& = J^{k^*}(S^{k-1}) + g_k \\
& \leqslant J(S^{k-1}) + \rho_k \| S^{k-1} - S^{k-1} \|^2 + g_k = J(S^{k-1}) + g_k \\
& \Rightarrow \rho_k \| S^k - S^{k-1} \|^2 \leqslant J(S^{k-1}) - J(S^k) + g_k \\
& \quad \leqslant J(S^{k-1}) - J(S^k) + g_k \| S^k - S^{k-1} \|^2, \quad \text{for } k \text{ with } S^k \neq S^{k-1} \\
& \Rightarrow (\rho_k - g_k) \| S^k - S^{k-1} \|^2 \leqslant J(S^{k-1}) - J(S^k), \quad \text{for } k \text{ with } S^k \neq S^{k-1} \\
& \Rightarrow \sum_{k=K}^{\infty} (\rho_k - g_k) \| S^k - S^{k-1} \|^2 \leqslant J(S^{K-1}) - J(S^{+\infty}) < +\infty \\
& \Rightarrow \lim_{k \to \infty} \| S^k - S^{k-1} \| = 0.
\end{aligned}
$$

Since $S^k \in E, k = 0,1,2,\cdots$, and E is a finite compact set, then from $\lim\limits_{k\to\infty} \| S^k - S^{k-1} \| = 0$, it follows that there is $S^* \in E$ such that $\lim\limits_{k\to\infty} S^k = S^*$.

Note that if $\lim\limits_{k\to\infty} S^k = S^*$, then from the fact that $S^k, k = 0,1,\cdots, S^*$ are all taken from the finite discrete set E, it follows that there is a positive integer K_1 such that for any $k \geqslant K_1, S^k = S^*$.

Secondly, if $\lim\limits_{k\to\infty} S^k = S^*$ and $\rho_k < (g_k + 1)/h$ for $k \geqslant K_2$(K_2 is a sufficiently large positive integer), where $h = \left(\sum\limits_{i=1}^{n} n_i\right) H^2$, then we can prove that $J(S^*) \leqslant J(S) + \min\limits_{k \geqslant K^*} g_k$ for any $S \in E$, where $K^* = \max(K_1 + 1, K_2)$.

If the assertion is not true, then there is $k \geqslant K^*$ and $S' \in E, S' \neq S^*$ such that $J(S') < J(S^*) - g_k$. Since w_i, d_i, t_{ijr} are all integers, so are $J(S')$ and g_k. It follows that $J(S') \leqslant J(S^*) - g_k - 1$.

For any $k \geqslant K^*$, note that $S^k = S^{k-1} = S^*$, we have:

$$
\begin{aligned}
J(S^k) + \rho_k \| S^k - S^{k-1} \|^2 &= J(S^*) \leqslant J(S') + \rho_k \| S' - S^* \|^2 + g_k \\
&\leqslant J(S^*) - g_k - 1 + \rho_k \| S' - S^* \|^2 < J(S^*).
\end{aligned}
$$

This is a contradiction. Thus, $J(S^*) \leqslant J(S) + \min\limits_{k \geqslant K} g_k$ for any $S \in E$.

The above analysis suggests that the choice of ρ_k must compromise between the convergence of SLR algorithm and the performance of its obtained solution. In practice, we take $\rho_k, k = 1,2\cdots$ to be an increasing sequence of positive real numbers, for example $\rho_k = \rho_0\theta^k, \theta > 1$. At beginning several iterations, ρ_k is taken small to ensure that the solution of P_k approaches an optimal solution of P, and at final several iterations, ρ_k is taken large to ensure the convergence of the algorithm.

As LR approach, the final solution of SLR algorithm is generally associated with an infeasible schedule. A feasible schedule can be constructed using the same method as in the previous section and the value of the dual objective function $\varphi^* = \varphi(\pi^*)$ (where π^* is the Lagrange multiplier vector obtained by SLR algorithm, but φ is the dual objective function in LR approach) is a lower bound of the optimal cost of scheduling problem P, which can be used to evaluate suboptimality of the obtained schedule.

4 Computational Experience

We have tested SLR algorithm by ten problems whose machine number and job number are distributed from 3 to 10 and 8 to 20 respectively. The computational results show that the algorithm can not only avoid the solution oscillation but also generate high-quality near-optimal schedules with relative duality gap no more that 10%.

5 Conclusion

In this paper, we propose a sequential Lagrangian relaxation approach to job shop scheduling problems. Computational experience shows that the approach can not only avoid the solution oscillation in basic Lagrangian relaxation approach but also generate high-quality near optimal schedules.

One problem we investigate now is to reduce the computation time and memory of SLR algorithm when a scheduling problem has a long time horizon.

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Job Shop 调度的序列拉格朗日松驰法

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摘要：拉格朗日松驰法为求解复杂调度问题次最优解的一种重要方法，陆宝森等人把这种方法推广到 Job Shop 调度问题，但他们的方法存在解振荡问题. 本文提出一种序列拉格朗日松驰法，它能避免解振荡.

关键词：Job Shop 调度；拉格朗日松驰；解振荡

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