第 13 卷第 4 期 控制理论与应用 Vol. 13, No. 4
1996 年 8 月 CONTROL THEORY AND APPLICATIONS Aug., 1996

A Theorem on Dynamic Feedback Linearization in R^4*

SUN Zhendong and XIA Xiaohua
(7th Research Division, Beijing University of Aeronautics and Astronautics. Beijing, 100083, PRC)

Abstract: In this paper, we prove that invertible dynamic feedback linearizability is equivalent to linearizability by adding integrators for up to 4 dimensional affine nonlinear systems.

Key words: affine nonlinear systems; invertible feedback linearization; linearization by adding integrators

1 Preliminaries

Consider an affine nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u, \quad x \in R^n, \quad u \in R^m \tag{1}$$

with $f(0)=0$ and rank $G(0)=m$.

The class of dynamic state feedback transformations are of the form

$$\begin{cases} \dot{w} = a(x,w) + B(x,w)v, & w \in R^q, \quad a(0,0) = 0, \\ u = \alpha(x,w) + \beta(x,w)v, & v \in R^m, \quad \alpha(0,0) = 0 \end{cases} \tag{2}$$

where q is the order of the compensator. The extended system of (1) controlled by a dynamic compensator (2) can be written as (with $\bar{x}=(x,w)^T$ being the extended state)

$$\dot{\bar{x}} = \begin{pmatrix} f(x) + G(x)\alpha(x,w) \\ a(x,w) \end{pmatrix} + \begin{pmatrix} G(x)\beta(x,w) \\ B(x,w) \end{pmatrix} v = \bar{f}(\bar{x}) + \bar{G}(\bar{x})v. \tag{3}$$

If $u=\alpha(\bar{x})+\beta(\bar{x})v$ are viewed as outputs for system (3), m characteristic indices $\gamma_1, \cdots, \gamma_m$ can be defined in the usual way[1]:

$$\gamma_j = \begin{cases} 0, & \exists\ i, 1 \leqslant i \leqslant m, \text{ s.t. } \beta_{j,i}(\bar{x}) \neq 0, \\ \min\{r: \exists\ i, \text{ s.t. } L_{\bar{g}_i} L_{\bar{f}}^{r-1} \alpha_j(\bar{x}) \neq 0\}, & \beta_{j,i}(\bar{x}) = 0, \ \forall\ i. \end{cases}$$

Here we set $\min \varnothing = +\infty$. Let

$$\delta_{j,i}(\bar{x}) = \begin{cases} \beta_{j,i}(\bar{x}), & \gamma_j = 0, \\ L_{\bar{g}_i} L_{\bar{f}}^{\gamma_j - 1} \alpha_j(\bar{x}), & \gamma_j > 0. \end{cases}$$

When all γ_j are finite the $m \times m$ matrix $D(\bar{x}) = (\delta_{j,i}(\bar{x}))$ is called the decoupling matrix of the compensator (2) for system (1). We call dynamic compensator (2) invertible for system (1) if rank $D(0)=m$.

The system (1) is said to be (locally) invertible dynamic feedback linearizable, if it can be transformed into a linear controllable system

* This work was supported by NSF.
Manuscript received Apr. 4, 1995, revised Oct. 16, 1995.

$$\dot{z} = Az + Bv, \quad z \in R^{n+q}, \quad v \in R^m \tag{4}$$

via invertible dynamic compensation (2) and extended state space diffeomorphism

$$z = \varphi(x,w), \quad \varphi(0,0) = 0. \tag{5}$$

A special class of dynamic compensators are of the following form[2]:

$$u = \bar{\alpha}(x) + \bar{\beta}(x)\bar{u}, \quad \det\bar{\beta} \neq 0,$$

$$\begin{bmatrix} \bar{u}_1^{(\mu_1)} \\ \vdots \\ \bar{u}_m^{(\mu_m)} \end{bmatrix} = \alpha(x,w) + \beta(x,w)\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}, \tag{6}$$

here $\mu_i \geqslant 0, 1 \leqslant i \leqslant m, \alpha(0,0)=0, \beta(x,w)$ is of rank m around the origin, and

$$\bar{u}^\mu \overset{\text{def}}{=} \frac{d^\mu \bar{u}}{dt^\mu}, \quad w = (\bar{u}_1, \cdots, \bar{u}_1^{(\mu_1)}, \cdots, \bar{u}_m, \cdots, \bar{u}_m^{(\mu_m)}).$$

We call system (1) is linearizable by adding integrators, if it can be changed into (4) via (6) and (5). It was studied in [2,3,4]: some sufficient conditions were given. A problem naturally arise: whether linearizability by adding integrators is implied by invertible dynamic state feedback linearizability or not? In this work, we show that the answer is positive for up to 4 dimensional systems.

2 Main result

Consider system (1) with $n=4$ and $m=2$.

$$\dot{x} = f(x) + g_1(x)u_1(t) + g_2(x)u_2(t). \tag{7}$$

We assume that all functions under consideration are defined and analytic in an open neighborhood of the origin.

Proposition 1 If the nested distributions

$$\Delta_0 = \text{span}\{g_1, g_2\}, \quad \Delta_i = \Delta_{i-1} + ad_f^i\Delta_0, \quad i = 1,2,\cdots \tag{8}$$

satisfy following conditions around the origin:

1) Δ_i is involutive for $0 \leqslant i \leqslant 3$;

2) $\text{rank}(\Delta_3) = 4$.

Then the system (7) is linearizable by adding integrators.

Proof With [1, Theorem 5.2.4] in mind, we need only to consider the following case:

$$\text{rank}(\Delta_0) = 2, \quad \Delta_1 \text{ is not of constant rank}, \ \text{rank}(\Delta_2) = 4. \tag{9}$$

Hence without loss of generality, we may assume that rank$\{g_1, g_2, ad_f g_1\}$ has constant rank 3 at the origin.

Since Δ_0 is involutive and of constant rank, we can assume, up to some regular static state feedback and state change of coordinates, system (7) takes the form[5]:

$$\dot{x} = (u_1(t), u_2(t), a(x), b(x))^T.$$

Denote $\det\begin{bmatrix} \frac{\partial a}{\partial x_1} & \frac{\partial a}{\partial x_2} \\ \frac{\partial b}{\partial x_1} & \frac{\partial b}{\partial x_2} \end{bmatrix}$ by e. By (9) one has $e \neq 0, e(0)=0$, and $(\frac{\partial a}{\partial x_1}, \frac{\partial a}{\partial x_2})(0) \neq 0$. Without loss of generality, we assume that $\frac{\partial a}{\partial x_1}\neq 0$. Set

$$g_2' = g_2 - \frac{\frac{\partial a}{\partial x_2}}{\frac{\partial a}{\partial x_1}} g_1,$$

one may check that $ad_f g_2' = (\chi, 0, 0, e)^{\mathrm{T}}$, here χ is some real-valued function.

Since span$\{g_1, g_2', ad_f g_1, ad_f g_2'\} = \Delta_1$ is involutive, together with the facts that the function e is analytic and $e(0)=0$, one can deduce that $\frac{\partial e}{\partial x_1} = \frac{\partial e}{\partial x_2} = 0$, and there exist some real-value functions $p_i, q_i, i=1,2$, such that

$$\begin{pmatrix} \frac{\partial^2 a}{\partial x_1 \partial x_i} \\ \frac{\partial^2 b}{\partial x_1 \partial x_i} \end{pmatrix} = p_i \begin{pmatrix} \frac{\partial a}{\partial x_1} \\ \frac{\partial b}{\partial x_1} \end{pmatrix} + q_i \begin{pmatrix} 0 \\ e \end{pmatrix}.$$

This implies that

$$\frac{\partial p_1}{\partial x_2} = \frac{\partial p_2}{\partial x_1}, \tag{10}$$

and

$$\frac{\partial q_1}{\partial x_2} + p_1 q_2 = \frac{\partial q_2}{\partial x_1} + p_2 q_1. \tag{11}$$

Consider the following partial differential equation:

$$\begin{cases} \frac{\partial r}{\partial x_1}(x) = q_1(x) + p_1(x) r(x), \\ \frac{\partial r}{\partial x_2}(x) = q_2(x) + p_2(x) r(x), \\ r(0) = 0. \end{cases} \tag{12}$$

By (10) and (11), we get $\frac{\partial^2 r}{\partial x_1 \partial x_2} = \frac{\partial^2 r}{\partial x_2 \partial x_1}$. Therefore equation (12) has (at least) one solution. Assume r_0 satisfy (12). Define $g_1' = g_1 - r_0 g_2'$, one can verify that span $\{g_1', g_2', ad_f g_1'\}$ is involutive and has constant rank 3, and span$\{g_1', ad_f g_1', g_2', ad_f^2 g_1', a_f^2 g_2'\}$ is involutive and has constant rank 4. Applying [4, Theorem 4.2], we know system (7) is linearizable by adding integrator of order 1 to the input channel controlled by g_2'. Q.E.D.

Theorem 2 The following statements are equivalent:

i) System (7) is locally invertible feedback linearzable.

ii) System (7) is locally linearizable by adding integrators.

Proof ii)⇒i) is obvious.

i)⇒ii): Suppose some dynamic compensator (2) be such that the closedloop system (3) is transformable indices by γ_1, γ_2, and decoupling matrix by $D(\bar{x}) = (d_{ij})_{2\times 2}$. Let $y = \alpha(x, w) + \beta(x, w) v$. Compute[6]:

$$\left.\begin{aligned} y_j^{(i)} &= y_j^i(x,w) \quad i = 0,\cdots,\gamma_j, \\ y_j^{(\gamma_j)} &= y_j^{(\gamma_j)}(x,w,v) = \varphi_j(x,w) + d_j(x,w)v, \end{aligned}\right. \quad j = 1,2.$$

Note that $D=\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

Define invertible feedback transformation

$$v = D^{-1}(\bar{v} - \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}). \tag{13}$$

It is obvious that the resulted system of (3) controlled by (13) is invertible feedback linearizable.

Without loss of generality, we assume $\gamma_1 \geqslant \gamma_l \geqslant 1, l=1$ or 2. It can be verified that the vectors $\{\frac{\partial y_j^{(i)}}{\partial w}, j=1,l; i=1,\cdots,\gamma_j-1\}$ are linear independent[7]. Hence we can define state space transformation

$$\bar{w}^j = \begin{bmatrix} y_j^0 \\ \vdots \\ y_j^{\gamma_j-1} \end{bmatrix}, \quad j = 1,l; \quad \bar{\bar{w}} = \bar{w}(x,w) \tag{14}$$

such that $\{\bar{w}^j; j = 1,l; \bar{\bar{w}}\}$ together with x forms a diffemorphism of (x,w).

Simple computation shows that under transformation (13) and (14), system (3) will be changed into

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{w}} \\ \dot{\bar{\bar{w}}} \end{bmatrix} = 1 \begin{bmatrix} f + \sum_{i=1}^{l} g_i \bar{w}_1^i \\ A^c \bar{w} \\ \bar{a}(x,w,\bar{\bar{w}}) \end{bmatrix} + \begin{bmatrix} \sum_{j=l+1}^{2} g_i \bar{v}_i \\ B^c \bar{v} \\ \bar{b}(x,\bar{w},\bar{\bar{w}})\bar{v} \end{bmatrix} \tag{15}$$

where (A^c, B^c) is the Brunovsky pair with controllability indices (γ_1, γ_l).

Note that $\bar{\bar{w}}$ does not appear in the expression of $\dot{x}$ and $\dot{\bar{w}}$. Consider the follwing system:

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{w}} \end{bmatrix} = \begin{bmatrix} f + \sum_{i=1}^{l} g_i \bar{w}_1^i \\ A^c \bar{w} \end{bmatrix} + \begin{bmatrix} \sum_{j=l+1}^{2} g_i \bar{v}_i \\ B^c \bar{v} \end{bmatrix}. \tag{16}$$

By the fact that (15) is invertible feedback linearizable and [1, Theorem 5. 2. 4], one may verify that system (16) satisfies the assumptions of Proposition 1. Hence (16) is linearizable by adding integrators. Note that (16) is a prolongation of (7), it follows from [2] that (7) is linearizable by adding integrators. Q. E. D.

Remark Theorem 2 , together with [4, Theorem 2. 2, Corollary 4. 3], shows that locally invertible feedback linearizability is equivalent to locally linearizability by adding integrators for up to 4 dimensional systems. Note that if Proposition 1 holds for general case ($n \geqslant 4, m \geqslant 2$), then by the proof of Theorem 1 we know Theorem 1 also holds for general case. But whether this is true or not remains unknown.

References

[1] Isidori, A.. Nonlinear Control Systems. Berlin:Springer-Verlag, 1989

[2] Cheng, D. Z.. Geometric Theory of Nonlinear Systems. Beijing:Academic Press, 1988

[3] Charlet, B., Levine, J. and Marino, R.. On Dynamic Feedback Linearization. Sys. Contr. Lett. 1989,13:143—151

[4] Charlet, B., Levine, J. and Marino, R.. Sufficient Conditions for Dynamic State Feedback Linearization. SIAM J. Control Optim., 1991,29:38—57

[5] Marino, R.. On the Largest Feedback Linearizable Subsystem. Sys. Contr. Lett., 1986,6:345—351

[6] Di Benedetto, M. D., Grizzle, J. W. and Moog, C. H.. Rank Invariants of Nonlinear Systems. SIAM J. Control Optim., 1989,27:658—672

[7] Xia, X. H. and Gao, W. B.. The Minimal Order Dynamic Decoupling Problem of Nonlinear Systems. Acta Scientia Sinica, 1989,32:1107—1112

R^4上关于动态反馈线性化的一个定理

孙振华　夏小华

(北京航空航天大学第七研究室·北京,100083)

摘要:本文证明对低于4维的仿射非线性系统,可通过可逆动态状态反馈线性化的系统一定可通过加积分器线性化.

关键词:仿射非线性系统;可逆反馈线性化;加积分器反馈线性化

本文作者简介

孙振东　1968年生.1990年毕业于青岛海洋大学应用数学系,1993年于厦门大学系统科学系获硕士学位.现为北京航空航天大学博士研究生,IEE学生会员.研究方向为非线性系统的控制综合.

夏小华　1963年生.北京航空航天大学七研教授,博士生导师.研究兴趣包括非线性控制系统的反馈设计理论.