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Necessary and Sufficient Conditions under which a Discrete Time H_2-Optimal Control Problem Has an Unique Solution

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Abstract: A set of necessary and sufficient conditions under which a general discrete time H_2-optimal control problem has a unique solution is derived. It is shown that the solution for a discrete time H_2-optimal control problem, if it exists, is unique if and only if i) the transfer function from the control input to the controlled output is left invertible, and ii) the transfer function from the disturbance to the measurement output is right invertible.

Key words: H_2 optimal control; Q-parameterization; discrete-time disturbance decoupling; discrete-time systems

1 Introduction

Optimization theory is one of the corner stones of modern control theory. In a typical control design, the given specifications are at first transformed into a performance index, and then control laws are sought which would minimize some norm, say the H_2 or H_∞ norm, of the performance index. This paper considers discrete-time systems, and focuses on H_2 optimal control theory or otherwise known as linear quadratic gaussian (LQG) control theory. For discrete-time systems, optimal control theory based on the H_2 norm was heavily studied in the 70's and early 80's (see. e g., [1], [2], [3], [4] and [5] and references therein). This development of H_2 optimal control theory can be found in most graduate text books on control (see e. g., [6] and [7]). Although a lot of research effort has been spent in 70's and 80's, the conditions for the existence of optimal solutions for a general discrete-time H_2 optimal control problem, and a way of determining an optimal solution if it exists (again for a general problem), were not known until the very recent work of [8]. Trentelman and Stoorvogel in [8], not only obtain a set of necessary and sufficient conditions for the existence of optimal solutions to a general discrete-time H_2 optimal control problem, but also construct one such solution. This paper deals with the issue of the uniqueness of the solution to the discrete time H_2 optimal control problem. We develop a set of necessary

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and sufficient conditions for the uniqueness of the solution to the above mentioned problem. The results obtained here are analogous to those of [9] for the continuous time H_2 optimal control problem.

The paper is organized as follows. In Section 2, we introduce the problem formulation of the discrete time H_2-optimal control problem. while in Section 3, we briefly review the conditions of the existence of discrete time H_2-optimal controllers. The main results of this paper are given in Section 4. Finally, in Section 5 we draw the conclusions.

Throughout this paper, A' denotes the transpose of A and I denotes an identity matrix with appropriate dimension. $C^{\bigcirc}$ and $C^{\odot}$ respectively denote the unit circle and the open unit disc of the complex plane. Ker $[V]$ and Im $[V]$ denote, respectively, the kernel and the image of V. Given a strictly proper and stable discrete time transfer function $G(z)$, as usual, its H_2-norm is defined by $\| G \|_2$. Also, RH^s denotes the set of real-rational transfer functions which are stable and strictly proper. RH_∞ dentoes the set of real-rational transfer functions which are stable and proper.

2 Problem Statement

Consider the following standard discrete linear time invariant system,

$$\Sigma: \begin{cases} x(k+1) = Ax(k) + Bu(k) + Ew(k), \\ y(k) = C_1x(k) + D_1w(k), \\ z(k) = C_2x(k) + D_2u(k) \end{cases} \tag{2.1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $w \in R^l$ is the unknown disturbance, $y \in R^p$ is the measured output and $z \in R^q$ is the controlled output. Without loss of generality, we assume that the matrices $[C_2, D_2]$, $[C_1, D_1]$, $[B', D'_2]$ and $[E', D'_1]$ are of maximal rank. Also, consider an arbitrary proper controller Σ_F given by,

$$\Sigma_F: \begin{cases} \xi(k+1) = J\xi(k) + Ly(k), \\ u(k) = M\xi(k) + Ny(k). \end{cases} \tag{2.2}$$

The controller Σ_F is said to be admissible if it provides internal stability for the closed loop system comprising Σ and Σ_F. Let $T_{zw}(\Sigma \times \Sigma_F)$ denote the closed-loop transfer function from w to z after applying a dynamic controller Σ_F to the system Σ. The H_2-optimization problem for the discrete time system Σ is to find an admissible control law which minimizes $\| T_{zw}(\Sigma \times \Sigma_F) \|_2$. The following definitions will be convenient in the sequel.

Definition 2.1 (The regular discrete time H_2-optimization problem) A regular discrete time H_2- optimization problem refers to a problem for which the given plant Σ satisfies:

1) (A, B, C_2, D_2) is left invertible and has no invariant zeros on C°;

2) (A, E, C_1, D_1) is right invertible and has no invariant zeros on C°.

Definition 2.2 (The singular discrete time H_2-optimization problem) A singular discrete time H_2- optimization problem refers to a problem for which the given plant Σ does not satisfy either one or both of the conditions 1 and 2 in Definition 2.1.

We note that the regular vs singular characterizations for the discrete time H_2-optimization problem precisely correspond to those for the continuous time H_2-optimization problem under a bilinear mapping.

Definition 2.3 (The infimum of H_2-optimization) For a given plant Σ, the infimum of the H_2-norm of the closed-loop transfer function $T_{zw}(\Sigma \times \Sigma_F)$ over all the stabilizing proper controllers Σ_F is denoted by γ^*, namely $\gamma^* := \inf\{ \| T_{zw}(\Sigma \times \Sigma_F) \|_2 | \Sigma_F$ internally stabilizes $\Sigma\}$.

Definition 2.4 (The H_2-optimal controller) A stabilizing proper controller Σ_F is said to be an H_2-optimal controller for Σ if $\| T_{zw}(\Sigma \times \Sigma_F) \|_2 = \gamma^*$.

Definition 2.5 (Geometric subspaces) Given a system Σ_* characterized by a matrix quadruple (A,B,C,D), we define the detectable strongly controllable subspace $S_g(\Sigma_*)$ or $S_g(A,B,C,D)$ as the smallest subspace S of R^n for which there exists a linear mapping K such that the following subspace inclusions are satisfied:

$$(A + KC)S \subseteq S, \quad \mathrm{Im}(B + KD) \subseteq S \tag{2.3}$$

and such that $(A + KC) | R^n/S$ is asymptotically stable. We also define the stabilizable weakly unobservable subspace $\mathscr{V}_g(\Sigma_*)$ or $\mathscr{V}_g(A,B,C,D)$ as the largest subspace $\mathscr{V}$ for which there exists a mapping F such that the following subspace inclusions are satisfied:

$$(A + BF)\mathscr{V} \subseteq \mathscr{V}, \quad (C + DF)V = \{0\} \tag{2.4}$$

and such that $(A + BF) | \mathscr{V}$ is asymptotically stable.

The goal of this paper is to derive a set of necessary and sufficient conditions under which Σ has a unique H_2-optimal controller.

3 Existence of Optimal Controllers

Our intention in this section is to recall from Trentelman and Stoorvogel[8] the necessary and sufficient conditions under which an H_2-optimization problem has a solution. We first define the matrices C_P, D_P, E_Q and D_Q that satisfy the following conditions: i) $[C_P, D_P]$ and $[E_Q', D_Q']$ are of maximal rank, and ii)

$$F(P) = \begin{bmatrix} C_P' \\ D_P' \end{bmatrix} [C_P \quad D_P] \quad \text{and} \quad G(Q) = \begin{bmatrix} E_Q \\ D_Q \end{bmatrix} [E_Q' \quad D_Q'], \tag{3.1}$$

where

$$F(P) := \begin{bmatrix} A'PA - P + C_2'C_2 & C_2'D_2 + A'PB \\ D_2'C_2 + B'PA & D_2'D_2 + B'PB \end{bmatrix}$$

and

$$G(Q) := \begin{bmatrix} AQA' - Q + EE' & ED_1' + AQC_1' \\ D_1E' + C_1QA' & D_1D_1' + C_1QC_1' \end{bmatrix}. \tag{3.3}$$

Furthermore, here P and Q are the largest solutions of the respective matrix inequalities $F(P) \geq 0$ and $G(Q) \geq 0$. Also, let

$$R^* := (D_P')^+ (D_P'C_PQC_1' + B'PED_1')(D_Q')^+, \tag{3.4}$$

where $(\cdot)^+$ denotes the generalized inverse of $(\cdot)$.

The following theorem, which is slightly simplified from the one in Trentelman and Stoorvegel[8], gives the necessary and sufficient conditions under which the infimum, γ^*, can be attained.

Theorem 3.1 Consider the given system Σ as in (2.1). Then the infimum, γ^*, can be attained by a proper controller of the form (2.2) if and only if

1) (A,B) is stabilizable,

2) (A,C_1) is detectable,

3) $\mathrm{Im}(E_Q - BD_P^+R^*) \subseteq \mathscr{V}_g(\Sigma_P)$,

4) $\mathrm{Ker}(C_P - R^* D_Q^+ C_1) \supseteq S_g(\Sigma_Q)$,

5) $S_g(\Sigma_Q) \subseteq \mathscr{V}_g(\Sigma_P)$,

6) $(A - BD_P^+ R^* D_Q^+ C_1) S_g(\Sigma_Q) \subseteq \mathscr{V}_g(\Sigma_P)$,

where Σ_P and Σ_Q are respectively characterized by (A, B, C_P, D_P) and (A, E_Q, C_1, D_Q).

Proof It follows from Trentelman and Stoorvogel[8] that the infimum, γ^*, can be attained by a proper controller of the form (2.2) if and only if

1) (A, B) is stabilizable,

2) (A, C_1) is detectable,

3) $\mathrm{Im}(E_Q - BD_P^+ R^*) \subseteq \mathscr{V}_g(A + BNC_1, B, C_P + D_P NC_1, D_P)$,

4) $\mathrm{Ker}(C_P - R^* D_Q^+ C_1) \supseteq S_g(A + BNC_1, E_Q + BND_Q, C_1, D_Q)$,

5) $S_g(A + BNC_1, E_Q + BND_Q, C_1, D_Q) \subseteq \mathscr{V}_g(A + BNC_1, B, C_P + D_P NC_1, D_P)$,

6) $(A - BD_P^+ R^* D_Q^+ C_1) S_g(A + BNC_1, E_Q + BND_Q, C_1, D_Q) \subseteq V_g(A + BNC_1, B, C_P + D_P NC_1, D_P)$.

On the other hand, it is straightforward to see in view of their definitions that

$$\mathscr{V}_g(A + BNC_1, B, C_P + D_P NC_1, D_P) = \mathscr{V}_g(A, B, C_P, D_P) = V_g(\Sigma_P)$$

and

$$S_g(A + BNC_1, E_Q + BND_Q, C_1, D_Q) = S_g(A, E_Q, C_1, D_Q) = S_g(\Sigma_Q)$$

since $\mathscr{V}_g$ is invariant under a state feedback and S_g is invariant under an output injection, Hence the result of Theorem 3.1 follows. Q. E. D.

4 Main Results

We state in the following theorem the set of necessary and sufficient conditions under which a given plant Σ has a unique H_2-optimal controller.

Theorem 4.1 Consider a plant Σ given by (2.1). Then H_2-optimal controller for Σ is unique if and only if the following conditions hold:

1) (A, B) is stabilizable,

2) (A, C_1) is detectable,

3) $\mathrm{Im}(E_Q - BD_P^+ R^*) \subseteq \mathscr{V}_g(\Sigma_P)$,

4) $\mathrm{Ker}(C_P - R^* D_Q^+ C_1) \supseteq S_g(\Sigma_Q)$,

5) $S_g(\Sigma_Q) \subseteq \mathscr{V}_g(\Sigma_P)$,

6) $(A - BD_P^+ R^* D_Q^+ C_1) S_g(\Sigma_Q) \subseteq \mathscr{V}_g(\Sigma_P)$,

7) (A, B, C_2, D_2) is left invertible,

8) (A, E, C_1, D_1) is right invertible,

where Σ_P and Σ_Q, as before, are respectively characterized by the quadruples (A, B, C_P, D_P) and (A, E_Q, C_1, D_Q). Moreover, the unique optimal controller is given by

$$\begin{cases} \xi(k+1) = (A + BF + KC_1 - BNC_1)\xi(k) + (BN - K)y(k), \\ u(k) = (F - NC_1)\xi(k) + Ny(k), \end{cases} \tag{4.1}$$

where F and K are any constant matrices that satisfy the conditions

$$\lambda(A + BF) \subseteq \mathbb{C}^{\odot}, \quad \mathrm{Ker}[(C_P + D_P F)(zI - A - BF)^{-1}] = \mathscr{V}_g(\Sigma_P) \tag{4.2}$$

and

$$\lambda(A + KC_1) \subseteq \mathbb{C}^{\odot}, \quad \mathrm{Im}[(zI - A - KC_1)^{-1}(E_Q + KD_Q)] = S_g(\Sigma_Q), \tag{4.3}$$

respectively, and N is given by

$$N = -D_P^{-1} R^* D_Q^{-1}. \tag{4.4}$$

Also, note that there always exist F and K such that (4. 2) and (4.3) hold provided that (A,B) is stabilizable and (A,C_1) is detectable (see the construction algorithm in Chen et al [10].

Proof Our proof involves two stages. In the first stage we obtain a special parameterization of all H_2-optimal controllers (whenever at least one of them exists) for the given plant Σ. The second stage involves the examination of the set of all optimal solutions, which are identified and parameterized in the first stage, to derive the necessary and sufficient conditions for the uniqueness of the solution of the H_2-optimal control problem. Our development utilizes an interesting reformulation of the H_2-optimal control problem which was proposed by Trentelman and Stoorvogel[8]. Let us first define an auxiliary system Σ_{PQ} characterized by

$$\Sigma_{PQ}:\begin{cases} x_{PQ}(k+1) = Ax_{PQ}(k) + Bu_{PQ}(k) + E_Q w_{PQ}(k), \\ y_{PQ}(k) = C_1 x_{PQ}(k) + D_Q w_{PQ}(k), \\ z_{PQ}(k) = C_P x_{PQ}(k) + D_P u_{PQ}(k), \end{cases} \tag{4. 5}$$

where C_P, D_P, C_Q and D_Q are as defined in (3. 1). In Trentelman and Stoorvogel[8], it was shown that the controller Σ_F of (2. 2) is an optimal controller for the given plant Σ if and only if Σ_F when applied to the new system Σ_{PQ} defined by (4. 5) is internally stabilizing and the resulting closed-loop transfer function from w_{PQ} to z_{PQ} is $-R^*$, a constant matrix. The following lemma states precisely such a reformulation of the H_2-optimal control problem.

Lemma 4. 1 The following two statements are equivalent:

1) The controller Σ_F as in (2. 2) when applied to the system Σ defined by (2. 1) is internally stabilizing and the resulting closed-loop transfer function from w to z is strictly proper and has the H_2-norm γ^*. Moreover, matrix N in (2. 2) must satisfy $D_P N D_Q = -R^*$.

2) The controller Σ_F as in (2. 2) when applied to the new system Σ_{PQ} defined by (4. 5) is internally stabilizing and the resulting closed-loop transfer function from w_{PQ} to z_{PQ} is equal to $-R^*$.

Proof See Trentelman and Stoorvogel[8].

The above lemma shows that obtaining all the H_2-optimal controllers for Σ is equivalent to obtaining all the controllers that achieves a constant closed-loop transfer matrix $-R^*$. It turned out that the characterization of the controllers that achieve $-R^*$ for Σ_{PQ} is easier than that of the H_2-optimal controllers for Σ. It is well-known (see for example Maciejowski[11], that the general class of stabilizing proper controllers for Σ_{PQ} can be parameterized as,

$$\begin{cases} \zeta(k+1) = (A+BF+KC_1)\zeta(k) + By_1(k) - Ky(k), \\ u(k) = F\zeta(k) + y_1(k) \end{cases} \tag{4. 6}$$

and

$$y_1(k) = Q(z)[y(k) - C_1\zeta(k)], \tag{4. 7}$$

where F and K are any fixed gain matrices that satisfy

$$\lambda(A+BF) \subset \mathbb{C}^{\odot} \quad \text{and} \quad \lambda(A+KC_1) \subset \mathbb{C}^{\odot}, \tag{4. 8}$$

respectively, and $Q(z) \in RH_\infty$ with appropriate dimension is a free parameter. In order that the controller (4. 6) and (4. 7) achieves a constant closed-loop transfer matrix $-R^*$ for Σ_{PQ}, the free parameter $Q(z)$ must satisfy some additional conditions.

It turned out that with the choice of F and K that satisfy (4. 2) and (4. 3), respectively, the controller (4. 6) and (4. 7) achieves constant closed-loop transfer matrix for Σ_{PQ} if and only if $Q(z) \in \boldsymbol{Q}$, where

$$\boldsymbol{Q}: = \{Q(z) = Q_s(z) + N \mid Q_s(z) \in \boldsymbol{Q}_s \text{ and } N \in \boldsymbol{N}\} \tag{4.9}$$

and where

$$\begin{aligned}\boldsymbol{Q}_s: = \{Q(z) \in \mathrm{RH}^s \mid [(C_P + D_P F)(zI - A - BF)^{-1}B + D_P] \\ \cdot Q(z)[C_1(zI - A - KC_1)^{-1}(E_Q + KD_Q) + D_Q] = 0\}\end{aligned} \tag{4.10}$$

and

$$\boldsymbol{N}: = \{N \in \mathrm{R}^{m \times p} \mid D_P N D_Q = -R^*\}. \tag{4.11}$$

This claim is proved in the following lemma.

Lemma 4. 2 Consider the auxiliary system Σ_{PQ} given by (4. 5). Assume that the conditions in Theorem 3. 1 are satisifed . Then, any controller Σ_F that achieves a constant closed-loop tarnsfer matrix $-R^*$ for Σ_{PQ} if and only if it can be written in the form of (4. 6) and (4. 7) with F and K satisfying (4. 2) and (4. 3), respectively, and some $Q(z) \in \boldsymbol{Q}$.

Proof Let (A_q, B_q, C_q, N) be a state space realization of $Q(z)$. It can be shown by some simple algebraic manipulations that the controller (4. 6) and (4. 7) when applied to Σ_{PQ} yields the closed-loop transfer function from w_{PQ} to z_{PQ} as,

$$T_{z_{PQ}w_{PQ}}(\Sigma_{PQ} \times \Sigma_F) = C_e(zI - A_e)^{-1}B_e + D_e, \tag{4.12}$$

where

$$A_e = \begin{bmatrix} A + BF & BC_q & BNC_1 - BF \\ 0 & A_q & B_q C_1 \\ 0 & 0 & A + KC_1 \end{bmatrix}, \quad B_e = \begin{bmatrix} E_Q + BND_Q \\ B_q D_Q \\ E_Q + KD_Q \end{bmatrix}, \tag{4.13}$$

and

$$C_e = [C_P + D_P F \quad DC_q \quad D_P NC_1 - D_P F], \quad D_e = D_P N D_Q. \tag{4.14}$$

Thus, it is trivial to see that the closed-loop system is internally stable if and only if (4. 8) holds and $Q(z) \in \mathrm{RH}_\infty$. It is also simple to verify that

$$T_{z_{PQ}w_{PQ}}(\Sigma_{PQ} \times \Sigma_F) = T_0 - T_q + D_P N D_Q$$

where

$$\begin{aligned}T_0 = & (C_P + D_P F)(zI - A - BF)^{-1}(E_Q + BND_Q) \\ & + (C_P + D_P NC_1)(zI - A - KC_1)^{-1}(E_Q + KD_Q) \\ & - (C_P + D_P F)(zI - A - BF)^{-1}(zI - A - BNC_1)(zI - A - KC_1)^{-1}(E_Q + KD_Q)\end{aligned}$$

and

$$= [(C_P + D_P F)(zI - A - BF)^{-1}B + D_P]Q_s(z)[C_1(zI - A - KC_1)^{-1}(E_Q + KD_Q) + D_Q].$$

It follows from Lemma 4. 1 that whenever the controller achieves a constant closed-loop transfer matrix $-R^*$ for Σ_{PQ}, N must belong to the set $\boldsymbol{N}$. Also, it was shown in Trentelman and Stoorvogel[8] that the conditions 3~6 in Theorem 3. 1 are equivalent to the following:

1) $\mathrm{Im}\,(E_Q + BND_Q) \subseteq \mathscr{V}_g(\Sigma_P)$,

2) $\mathrm{Ker}\,(C_P + D_P NC_1) \supseteq S_g(\Sigma_Q)$,

3) $S_g(\Sigma_Q) \subseteq \mathscr{V}_g(\Sigma_P)$,

4) $(A + BNC_1)S_g(\Sigma_Q) \subseteq \mathscr{V}_g(\Sigma_P)$.

Thus, following the procedures of Stoorvogel and van der Woude [12], it follows that $T_0 \equiv 0$ provided the conditions in Theorem 3.1 are satisfied and F and K are such that (4.2) and (4.3) hold. Hence, $T_{z_{PQ}w_{PQ}}(\Sigma_{PQ} \times \Sigma_F) = D_P N D_Q = -R^*$ is equivalent to that $T_q = 0$ or $Q_s(z) \in Q_s$ Then the result follows.

Lemma 4.3 If equation $D_P N D_Q = -R^*$ has at least one solution, then it is unique if and only if the subsystems characterized by the matrix quadruples (A,B,C_2,D_2) and (A,E,C_1,D_1) are respectively left and right invertible. Moreover, in this case, the unique solution N is given by (4.4).

Proof It is simple to verify that $D_P N D_Q = -R^*$ has a unique solution, whenever it exists, if and only if both D_P and D_Q are respectively of maximal column and row rank. Following the results of Chen et al. [10], it is simple to show that the systems Σ_P and Σ_Q are respectively right and left invertible with no infinite zeros. These imply that D_P and D_Q are respectively of maximal row and column rank. Hence, D_P and D_Q are both invertible. Following the results of Chen et al [1] it is straingtforward to show that (A,B,C_2,D_2) and (A,E,C_2,D_1) are respectively left and right invertible.

The final step of the proof of Theorem 4.1 proceeds as follows:

($\Rightarrow$): If the H_2-optimal controller for Σ is unique, i.e., there exists a unique controller that achieves a constant closed-loop transfer matrix Σ_{PQ}, then by Theorem 3.1 conditions 1 ~6 hold. It also implies that the set N is a singleton. By Lemma 4.3, conditions 7 and 8 hold.

($\Leftarrow$): Conversely, if conditions 1)~6) hold, then Theorem 3.1 implies that there exists at least one H_2-optimal controller for Σ, which is equivalent to the existence of controllers that achieve a constant closed-loop transfer matrix $-R^*$ for Σ_{PQ}. Also, following the result of Chen et al [10], it can be shown that the conditions 7) and 8) imply that both D_P and D_P are invertible. Hence, it follows from (4.10) that the set $Q_s = \{0\}$ and from Lenma 4.3 that the set N is a singleton and is given by (4.4). Then, by Lemmas 4.1 and 4.2, the H_2-optimal controller for Σ is unique.

Finally, it is now trivial to verify from the above proof that the unique H_2-optimal controller for Σ is given by (4.1). This concludes the proof of Theorem 4.1. Q.E.D.

The following are some interesting corollaries.

Corollary 4.1 (The regular case) Consider the given system (2.1). If the following conditions are satisfied:

1) (A,B) is stabilizable,

2) (A,C_1) is detectable,

3) (A,B,C_2,D_2) is left invertible with no invariant zeros on C^o,

4) (A,E,C_1,D_1) is right invertible with no invariant zeros on C^o,

then the optimal controller exists. Moreover, it is uniquely given by

$$\begin{cases} \xi(k+1) = (A + BF + KC_1 - BNC_1)\xi(k) + (BN - K)y(k), \\ u(k) = (F - NC_1)\xi(k) + Ny(k), \end{cases}$$

where

$$F = -(B'PB + D_2'D_2)^{-1}(B'PA + D_2'C_2), \tag{4.15}$$

$$K = -(ED_1' + AQC_1')(D_1D_1' + C_1QC_1')^{-1} \tag{4.16}$$

and

$$N = -(B'PB + D_2'D_2)^{-1}[(B'PA + D_2'D_2)QC_1 + B'PED_1'](D_1D_1' + C_1QC_1')^{-1}. \tag{4.17}$$

Here P and Q are respectively the unique positive semi-definite solutions of the Riccati equations,

$$P = A'PA + C_2'C_2 - (C_2'D_2 + A'PB)(D_2'D_2 + B'PB)^{-1}(D_2'C_2 + B'PA) \tag{4.18}$$

and

$$Q = AQA' + EE' - (ED_1' + AQC_1')(D_1D_1' + C_1QC_1')^{-1}(D_1E' + C_1QA'). \tag{4.19}$$

We note that the solutions to the above Riccati equations can be obtained using the non-recursive algorithm of Chen et al[13].

Proof For the system satisfying the above conditions, it is straighforward to show that all the conditions in Theorem 4.1 are automatically satisfied. The results follow then from some simple manipulations. Q.E.D.

Corollary 4.2 (The state feedback case) Consider the given system (2.1) with $C_1 = I$ and $D_1 = 0$, i.e., the state feedback case. there exists a unique H_2-optimal controller for Σ if and only if the following conditions hold:

1) (A, B) is stabilizable,

2) (A, B, C_2, D_2) is left invertible and has no invariant zeros on C°,

3) $\mathrm{Im}(E) = R^n$.

Moreover, in this case, the unique H_2-optimal controller for Σ is given by

$$u(k) = -D_P^{-1}C_Px(k) = -(B'PB + D_2'D_2)^{-1}(B'PA + D_2'C_2)x(k), \tag{4.20}$$

where P is the unique and positive semi-definite solution of (4.18). This result coincides with the one obtained by Chen et al[10].

Proof It follows from Theorem 4.1 and the result of Chen et al[10] .

5 Conclusions

In this paper we have derived a set of necessary and sufficient conditions for the uniqueness of the solution to a general discrete time H_2-optimization problem. We have shown that the solution for a discrete time H_2-optimal control problem, if it exists is unique, if and only if the systems characterized respectively by quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1), are respectively left and right invertible. Moreover, such a unique H_2-optimal control law has been obtained.

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离散型 H_2 优化控制问题有唯一解之充分和必要条件

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摘要：本文导出一般性离散型 H_2 优化控制问题存在唯一解之充分和必要条件. 我们的结果显示，如果一离散型 H_2 优化控制问题有解，那么其解是唯一的充分和必要条件为：①从控制输入到被控制输出之传递函数是左可逆，及②从干扰输入到测量输出之传递函数是右可逆.

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