

Identification of Non-Linear Distributed Parameter Systems via Block-Pulse Functions and the Optimal Selection of the Truncated Terms*

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Abstract: A method for identifying a class of non-linear distributed systems is presented by using two-dimension block-pulse functions. An error analysis for the approximation is emphasized and made. The optimal selection of the numbers of the truncated terms is discussed by using non-linear integer programming. Appropriate examples are included to illustrate the ideas.

Key words: non-linear distributed parameter systems; error analysis; identification; block-pulse functions

1 Introduction

The identification of non-linear distributed parameter systems (DPS) is more difficult than that of linear DPS. But at present, the identification of non-linear DPS has become an important problem in modern control engineering and many other areas. A few researchers have studied the problem by using Walsh functions^[1], block-pulse functions^[2], and Laguerre polynomials^[3]. A critical review for all these publications was given in [4].

In this paper, the problem of non-linear DPS identification via two-dimension block-pulse functions is considered as usual. An error analysis for the approximation is emphasized first, it seems few papers have noticed the problem. Based on the analysis, a non-linear integer programming model is established to select the numbers of the truncated terms so that the approximation error may be less than a given error level and the balance between the number of measurement points and sample period may be attained. Two examples are given to illustrate our work.

2 Preliminaries

A set of two-dimension block-pulse functions is defined as^[5]

$$H_{ij}(x, t) = H_i(x)H_j(t) = \begin{cases} 1, & (i-1)L/M < x \leq iL/M, (j-1)T/N < t \leq jT/N; \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

A function $u(x, t)$ which is absolutely integrable in the region $E_2 \triangleq \{(x, t); 0 \leq x \leq L, 0 \leq t \leq T\}$

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$\leq T\}$ may be approximated as

$$u(x, t) \simeq \sum_{i=1}^M \sum_{j=1}^N u_{ij} H_{ij}(x, t) = \sum_{i=1}^M \sum_{j=1}^N u_{ij} H_i(x) H_j(t) = \mathbf{H}_M^T(x) \mathbf{U} \mathbf{H}_N(t) \quad (2.2)$$

where τ means transpose, and $\mathbf{U} = (u_{ij})_{M \times N}$ is the two-dimension block-pulse function coefficient matrix of the function $u(x, t)$, $\mathbf{H}_M(x) = (H_1(x), \dots, H_M(x))^T$, $\mathbf{H}_N(t) = (H_1(t), \dots, H_N(t))^T$.

The coefficients u_{ij} minimize

$$\epsilon = \left\| u(x, t) - \sum_{i=1}^M \sum_{j=1}^N H_i(x) u_{ij} H_j(t) \right\|_{L^2(E_2)}^2 \quad (2.3)$$

which gives

$$u_{ij} = \frac{MN}{LT} \iint_{\Delta_{ij}} u(x, t) dx dt, \quad 1 \leq i \leq M, \quad 1 \leq j \leq N, \quad (2.4)$$

where $\Delta_{ij} \triangleq \{(x, t) : (i-1)L/M < x \leq iL/M, (j-1)T/N < t \leq jT/N\}$, Hence, u_{ij} is the integral mean value of $u(x, t)$ over the subregion Δ_{ij} .

$\mathbf{H}_M(x)$ and $\mathbf{H}_N(t)$ have the following properties^{[5]7}

$$\int_0^x \mathbf{H}_M(x) dx = \mathbf{P}_M \mathbf{H}_M(x); \quad (2.5)$$

$$\int_0^t \mathbf{H}_N(t) dt = \mathbf{P}_N \mathbf{H}_N(t) \quad (2.6)$$

where \mathbf{P}_M and \mathbf{P}_N are both known matrices.

Using properties (2.2), (2.5) and (2.6), we get^[2]

$$\underbrace{\int_0^x \dots \int_0^x}_{\alpha \text{ times}} \underbrace{\int_0^t \dots \int_0^t}_{\beta \text{ times}} u(x, t) \underbrace{dx \dots dx}_{\alpha \text{ times}} \underbrace{dt \dots dt}_{\beta \text{ times}} \simeq \mathbf{H}_M^T(x) (\mathbf{P}_M^T)^\alpha \mathbf{U} \mathbf{P}_N^\beta \mathbf{H}_N(t) \quad (2.7)$$

Using properties (2.1) and (2.2), we get

$$u^\rho(x, t) \simeq \mathbf{H}_M^T(x) \mathbf{L} \mathbf{H}_N(t), \quad \text{with } \mathbf{L} = [l_{ij}]_{M \times N}, \quad l_{ij} = u_{ij}^\rho. \quad (2.8)$$

3 Error Analysis

Although $\{u_{ij}\}$ given by (2.4) minimize the integral square error ϵ in (2.3), they are determined just in the condition of given M and N . Suppose one require ϵ be less than a certain index (say ϵ_0). In general, the inequality

$$\epsilon \leq \epsilon_0 \quad (3.1)$$

is hardly held for given M and N if ϵ_0 was smaller. Therefore, how to select M and N such that (3.1) holds remains one problem. On the other hand, $u(x, t)$ in (2.4) is unknown, but distributed measurements may give the information of it. As all known, it is impossible to make distributed measurements physically. Point-wise measurements are often used. So, using point-wise record in the subregion Δ_{ij} to determine u_{ij} is necessary. This will introduce errors to u_{ij} , hence affect the accuracy of approximate expression (2.2). However, the effect of these errors may be alleviated by proper selection of M and N . In the section, the problem is discussed, and a formula estimating error ϵ is deduced based on point-wise measurements.

Theorem 1 Assume $u(x, t) \in L^2(E_2)$, with $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}$ continuous on $L^2(E_2)$. Then

$$\epsilon \simeq \frac{1}{24} \frac{TL}{MN} \sum_{i=1}^M \sum_{j=1}^N (\tilde{u}_{ij}^2 + \tilde{u}_{i-1,j}^2 + \tilde{u}_{i,j-1}^2 + \tilde{u}_{i-1,j-1}^2 - 2\tilde{u}_{ij}\tilde{u}_{i-1,j-1} - 2\tilde{u}_{i-1,j}\tilde{u}_{i,j-1}) \quad (3.2)$$

where $\tilde{u}_{ij} = u(x_i, t_j)$ are records of $u(x, t)$ in x_i and at t_j .

Proof Let ϵ_{ij} such that

$$\epsilon = \sum_{i=1}^M \sum_{j=1}^N \epsilon_{ij} \quad (3.3)$$

where

$$\epsilon_{ij} = \| u(x, t) - \mathbf{H}_M^T(x) \mathbf{U} \mathbf{H}_N(t) \|_{L^2(\Delta_{ij})}^2.$$

It is easy to show^[5]

$$\epsilon_{ij} = \iint_{\Delta_{ij}} u^2(x, t) dx dt - \frac{NM}{TL} \left[\iint_{\Delta_{ij}} u(x, t) dx dt \right]^2. \quad (3.4)$$

From Taylor formula, we have

$$u(x, t) = u(x_{i-1}, t_{j-1}) + \frac{\partial}{\partial x} u(\xi_i, \eta_j)(x - x_{i-1}) + \frac{\partial}{\partial t} u(\xi_i, \eta_j)(t - t_{j-1}) \quad (3.5)$$

where

$$x_i = \frac{iL}{M}, \quad t_j = \frac{jT}{N}, \quad \xi_i = x_{i-1} + \theta_1(x - x_{i-1}), \quad \eta_j = t_{j-1} + \theta_2(t - t_{j-1}),$$

$$0 < \theta_1, \quad \theta_2 < 1.$$

Substituting (3.5) into (3.4), and using the continuity of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}$, we get

$$\epsilon_{ij} = \frac{1}{12} \left[\frac{\partial}{\partial x} u(x_{i-1}, t_{j-1}) \right]^2 \frac{L^3 T}{M^3 N} + \frac{1}{12} \left[\frac{\partial}{\partial t} u(x_{i-1}, t_{j-1}) \right]^2 \frac{LT^3}{MN^3}, \quad (M \rightarrow \infty, N \rightarrow \infty). \quad (3.6)$$

Using second order difference scheme to approximate $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ gives

$$\frac{\partial}{\partial x} u(x_{i-1}, t_{j-1}) \simeq \frac{M}{2L} [\tilde{u}_{ij} - \tilde{u}_{i-1,j} + \tilde{u}_{i,j-1} - \tilde{u}_{i-1,j-1}],$$

$$\frac{\partial}{\partial t} u(x_{i-1}, t_{j-1}) \simeq \frac{N}{2T} [\tilde{u}_{ij} - \tilde{u}_{i,j-1} + \tilde{u}_{i-1,j} - \tilde{u}_{i-1,j-1}].$$

Then (3.6) becomes

$$\epsilon_{ij} = \frac{1}{24} \frac{LT}{MN} [\tilde{u}_{ij}^2 + \tilde{u}_{i-1,j}^2 + \tilde{u}_{i,j-1}^2 + \tilde{u}_{i-1,j-1}^2 - 2\tilde{u}_{ij}\tilde{u}_{i-1,j-1} - 2\tilde{u}_{i-1,j}\tilde{u}_{i,j-1}]. \quad (3.7)$$

Substituting (3.7) into (3.3) gives the result (3.2).

From (3.2), one can easily determine M and N such that (3.1) holds.

Remark Theorem 1 is a revision of Theorem 3.6 given by [5], but the former is more direct and useful in estimating error.

Example 1 Consider the following non-linear partial differential equation

$$a_1 \frac{\partial u}{\partial t} + a_2 \frac{\partial u}{\partial x} + a_3 u^2(x, t) = r(x, t),$$

$$u(0, t) = 0, \quad 0 \leq t \leq 1, \quad u(x, 0) = 0, \quad 0 \leq x \leq 1$$

where $a_1 = 2, a_2 = 1, a_3 = 4$, and $r(x, t) = 2x + 4x^2 t^2 + t$. It can be seen that $u(x, t) = xt$.

Now, given records $\{\tilde{u}_{ij}\}$ of $u(x, t)$ in x_i and at t_j and let u_{ij} in (2.4) equal to it, i.e.

$$u_{ij} = i/M \times j/N, \quad i = 1, 2, \dots, M, \quad j = 1, 2, \dots, N.$$

Using (3.2), one can select M and N for a given ϵ_0 . The results are listed in the Table 1.

4 Identification Process

Consider a non-linear time-invariant distributed parameter system described by the following second-order partial differential equation

$$\begin{aligned} a_1 \frac{\partial^2 u^{p_1}(x,t)}{\partial x^2} + a_2 \frac{\partial^2 u^{p_2}(x,t)}{\partial x^2} \\ + a_3 \frac{\partial u^{p_3}(x,t)}{\partial x \partial t} + a_4 \frac{\partial u^{p_4}(x,t)}{\partial t} \\ + a_5 \frac{\partial u^{p_5}(x,t)}{\partial x} + a_6 u^{p_6}(x,t) \\ = r^{p_7}(x,t) \end{aligned} \quad (4.1)$$

where $p_i (i = 1, 2, \dots, 7)$ are integers and $a_i (i = 1, 2, \dots, 6)$ are unknown parameters.

Integrating (4.1) twice with respect to t and twice with respect to x , one obtains

$$\begin{aligned} a_1 \int_0^x \int_0^x u^{p_1}(x,t) dx dx + a_2 \int_0^t \int_0^t u^{p_2}(x,t) dt dt + a_3 \int_0^x \int_0^t u^{p_3}(x,t) dt dx \\ + a_4 \int_0^x \int_0^x \int_0^t u^{p_4}(x,t) dt dx dx + a_5 \int_0^t \int_0^x \int_0^x u^{p_5}(x,t) dx dx dt \\ + a_6 \int_0^t \int_0^x \int_0^x \int_0^t u^{p_6}(x,t) dx dx dx dt - a_1 \int_0^x \int_0^x f(x) dx dx - a_2 \int_0^t \int_0^t g(t) dt dt \\ - a_3 u^{p_3}(0,0) \int_0^t \int_0^x dx dt - \int_0^t \int_0^x \int_0^x h(x) dx dx dt - \int_0^t \int_0^x \int_0^x s(t) dx dx dt \\ = \int_0^t \int_0^x \int_0^x r^{p_7}(x,t) dx dx dt \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} f(x) &= u^{p_1}(x,0), \quad g(t) = u^{p_2}(0,t), \\ h(x) &= a_1 \frac{\partial u^{p_1}(x,t)}{\partial t} \Big|_{t=0} + a_3 \frac{\partial u^{p_3}(x,0)}{\partial x} + a_4 u^{p_4}(x,0), \\ s(t) &= a_2 \frac{\partial u^{p_2}(x,t)}{\partial x} \Big|_{x=0} + a_3 \frac{\partial u^{p_3}(0,t)}{\partial t} + a_5 u^{p_5}(0,t). \end{aligned}$$

We now approximate $u^{p_i}(x,t), i = 1, 2, \dots, 6, u^{p_1}(x,0), u^{p_2}(0,t), c = a_3 u^{p_3}(0,0), h(x), s(t)$ and $r^{p_7}(x,t)$ in terms of block-pulse functions, substitute the approximations into (4.2) and make use of properties (2.5)~(2.8) to yield

$$\begin{aligned} a_1 (P_M^T)^2 L_1 + a_2 L_2 P_N^2 + a_3 (P_M^T) L_3 P_N + a_4 (P_M^T)^2 L_4 P_N + a_5 (P_M^T) L_5 P_N^2 + a_6 (P_M^T)^2 L_6 P_N^2 \\ - (P_M^T)^2 \sum_{i=1}^M f_i E_i - \sum_{j=1}^N \hat{g}_j E_j P_N^2 - \hat{c} D - (P_M^T)^2 \sum_{i=1}^M h_i E_i P_N - P_M^T \sum_{j=1}^N s_j E_j P_N^2 \\ = (P_M^T)^2 L_7 P_N^2 \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} L_k &= [l_{kij}]_{M \times N}, \quad l_{kij} = u^{p_k}_{ij}, \quad k = 1, 2, \dots, 6, \quad L_7 = [l_{7ij}]_{M \times N}, \quad l_{7ij} = r^{p_7}_{ij}, \\ D &= [d_{ij}]_{M \times N}, \quad d_{ij} = \frac{LT}{MN} \frac{(2i-1)(2j-1)}{4}, \end{aligned}$$

$$f_i = a_1 f_i, \quad f_i \text{ such that } u^{p_1}(x, 0) = \sum_{i=1}^M f_i H_i(x),$$

$$\hat{g}_j = a_2 g_j, \quad g_j \text{ such that } u^{p_2}(0, t) = \sum_{j=1}^N g_j H_j(t)$$

and E_{ij} is an $M \times N$ matrix having the (ij) th element unity and the remaining elements zero.

Now (4.3) may be rewritten in the form

$$A\theta = V \quad (4.4)$$

and θ can be obtained by using the least-square technique

$$\theta = (A^*A)^{-1}A^*V \quad (4.5)$$

where $A = \{a_{ij}\} \in \mathbb{R}^{MN \times K}$, $\theta \in \mathbb{R}^{K \times 1}$, $V \in \mathbb{R}^{MN \times 1}$.

The value of K depends on specific problems:

Case I If the initial and the boundary conditions in (4.1) are both known, then $K = 6$.

Case II If the initial condition in (4.1) is known, then $K = 7 + 2N$.

Case III If the boundary condition in (4.1) is known, then $K = 7 + 2M$.

Case IV If the initial and the boundary conditions are both unknown, then $K = 7 + 2M + 2N$.

5 Selection of M, N

In theory, the greater M, N are, the less the error caused by approximant (2.2) is. This means that more measurement points and smaller time interval Δt are needed. In practice, this requires more sensors (i.e. M should be large), and better high speed sampling performance of measurement system. This would be expensive. Therefore, how to attain the balance among the accuracy of approximant (2.2), the solvability of equation (4.4) and the arrangement of measurement points is a practical problem we should study.

Let the cost function, which is related to M and N , be $C(M, N)$. Note that relation (4.4) is solvable if and only if the rank of matrix A is equal to K (the number of parameters to be identified). That is, the following condition should be held

$$MN \geq K. \quad (5.1)$$

Suppose the accuracy index be ϵ_0 . Thus, we can establish a non-linear integer programming model

$$\left. \begin{array}{l} \min C(M, N), \\ \text{s. t. } MN \geq K, \epsilon \leq \epsilon_0, M > 0, N > 0, \text{ both are integers.} \end{array} \right\} \quad (5.2)$$

Using the techniques of operational research, one can find out M^* and N^* — the optimal selection of M and N — which minimize the cost function $C(M, N)$.

Example 2^[2] Consider the non-linear distributed system described by

$$a_1 \frac{\partial u(x, t)}{\partial x} + a_2 \frac{\partial^2 u(x, t)}{\partial t^2} + a_3 u(x, t) = r(x, t) \quad (0 \leq x \leq 1, 0 \leq t \leq 1),$$

$$u(x, 0) = 0, \quad u(0, t) = 0$$

with $a_1 = 2$, $a_2 = 2$, and $a_3 = 1$, $r(x, t) = 4x^2t + 2t + xt$. For given records of $u(x, t)$ and $r(x, t)$, the problem is to estimate the $u(x, 0)$ and the parameters a_1, a_2, a_3 . Suppose $C(M, N) = 2M + 5N + 10MN$ and $\epsilon_0 = 0.01$.

From Section 3 and 4, we may select $M \geq 2, N \geq 2$, and $K = 4 + M$, thus, have the model

$$\left. \begin{aligned} \min C(M, N) &= 2M + 5N + 10MN, \\ \text{s.t. } MN &\geq 4 + M, M \geq 2, N \geq 2, \text{ both are integers.} \end{aligned} \right\} \quad (5.3)$$

The solution of model (5.3) is $M^* = 4, N^* = 2$, i.e. the measurement points are located in 0.25, 0.5, 0.75 and 1.00; the sample period $\Delta t = 0.5$. Using the algorithm mentioned in Section 4, we get estimation values of parameters and initial condition as list in Table 2.

Table 2 Estimated results of parameters and initial condition

| Parameter | a_1 | a_2 | a_3 | b_1 | b_2 | b_3 | b_4^{**} |
|--------------------|--------|--------|--------|--------|--------|--------|------------|
| Estimated values | 2.0817 | 1.9998 | 1.0003 | 0.0007 | 0.0051 | 0.0087 | 0.0103 |
| Estimated values * | 1.9999 | 2.0625 | 1.0000 | 0.0005 | 0.0044 | 0.0122 | 0.0239 |
| True values | 2.0000 | 2.0000 | 1.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

* The results obtained by Hsu and Cheng (1982).

** $b_i, i=1, 2, 3, 4$ is the block-pulse function coefficient of $u(x, 0) = \sum_{i=1}^4 b_i H_i(x)$.

Comparing the results with that given by Hsu and Cheng, one may find that the same results are obtained, but here used fewer M and N .

6 Conclusion

A method to identify non-linear distributed systems is presented by using two dimension block-pulse functions. A modified algorithm is proposed. The error analysis for the approximation, which seems few researchers have noticed it, is emphasized first. A formula estimating error ϵ is deduced based on point-wise measurements. Specially, a non-linear integer programming model is established to help select the values of M and N . Examples show that good estimation results may also be gained by using fewer M and N .

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非线性分布参数系统的块脉冲函数辨识法 及截断项数的最佳选取

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摘要: 本文介绍了利用二维块脉冲函数辨识一类非线性分布参数系统的方法, 强调并进行了这种近似处理下的误差分析, 通过构建一非线性整数规划模型讨论了截断项数的最佳选取问题, 给出了两个实例以分别说明有关的概念和方法.

关键词: 非线性分布参数系统; 误差分析; 辨识; 块脉冲函数

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