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Exponential Stability and L_p Exponential Stability of Continuous-Time Systems with Time-Varying Parameters*

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Abstract: This paper presents conditions guaranteeing the exponential stability of continuous-time linear homogeneous systems with slowly time-varying parameters and the L_p exponential stability of continuous-time linear homogeneous systems with randomly time-varying parameters. The conditions are motivated by those used in the stability analysis of adaptive control systems with unmodeled dynamics, deterministic or stochastic disturbances as well as time-varying paramters.

Key words: continuous-time systems; time-varying parameters; exponential stability; L_p exponential stability

1 Introduction

During the last decade, stability and robustness issue of adaptive control systems has been drawing much attention from researchers (for example, see [1]). An adaptive controller often leads to a closed-loop system which has time-varying parameters. It is well known that an exponentially stable system can tolerate a certain amount of disturbances and unmodeled dynamics. Therefore, obtaining exponential stability of the corresponding homogeneous systems with time-varying parameters is very important in the analysis of the stability and robustness of adaptively controlled closed-loop systems.

Consider the time-varying system in $\mathbb{R}^n$

$$\dot{x}(t)=A(t)x(t), \quad x(0)=x_0, \tag{1.1}$$

where $A(t)$ is an $n\times n$ piecewise continuous matrix function of t. The zero solution of (1.1) is said to be exponentially stable if and only if there exist positive constants M_0 and β_0 such that

$$\|\Phi(t,u)\| \leqslant M_0 e^{-\beta_0(t-u)}, \quad \forall\, t\geqslant u, \tag{1.2}$$

where $\Phi(t,u)$ is the fundamental matrix of $A(t)$, i.e.

$$\frac{d\Phi(t,0)}{dt} = A(t)\Phi(t,0), \quad \Phi(0,0) = I, \quad \Phi(t,u) = \Phi(t,0)\Phi^{-1}(u,0). \tag{1.3}$$

Consider the time-varying system (1.1) under the assumption that $A(t)$ is an $n\times n$ random and piecewise continuous matrix function of t. The zero solution of (1.1) is said to be $L_p(p>1)$ exponentially stable if and only if there exist positive constants M_1 and β_1 such that

$$\{E\|\Phi(t,u)\|^p\}^{1/p} \leqslant M_1 e^{-\beta_1(t-u)}, \quad \forall\, t\geqslant u. \tag{1.4}$$

For the case $A(t)\equiv A$, the eigenvalues of the constant matrix A determine the stability

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behavior of the system (1. 1). When the system parameter matrix $A(t)$ varies arbitrarily, many examples have been presented to show that the stability of the frozen time system (i. e., the real parts of the eigenvalues of $A(t)$ at every time instant are over bounded by $-\beta$, $\beta>0$) does not imply the stability of the time-varying system. (for example, see[4]) For the case where $A(t)$ is differentiable with $\sup_{t\geqslant 0}\|A(t)\|$ being bounded and $\sup_{t\geqslant 0}\|\dot{A}(t)\|$ being sufficiently small, results in [2] show that the stability of the frozen time system at every time instant is sufficient for the exponential stability of the time-varying system (1. 1). For the case where $A(t)$ is continuous, but not necessarily differentiable, the exponential stability conditions for the time-varying system (1. 1) are given in [3] for the case where the condition that $\sup_{t\geqslant 0}\|\dot{A}(t)\|$ is sufficiently small is replaced by the more general condition that $\|A(t_1)-A(t_2)\|\leqslant L|t_1-t_2|^{\alpha}$ for all $t_1,t_2\geqslant 0,\alpha>0$, with $L>0$ being sufficiently small.

In many situations of interest, $A(t)$ is discontinuous (for example, see[4]). For this case, Zhang[4] shows that (1. 1) is exponentially stable provided that there is a real number sequence $\{t_k\}$ satisfying

$$t_k<t_{k+1}\to\infty,\quad \text{as}\quad k\to\infty,\quad \text{and}\quad \sup_{k\geqslant 0}(t_{k+1}-t_k)<\infty,$$

such that

$$\sup_{k\geqslant 1}\|A(t_k)\|\leqslant C<\infty,\quad \max\{\mathrm{Re}\lambda_i(A(t_k)),i=1,\cdots,n;k=1,2,\cdots\}\leqslant -\alpha$$

and
$$\lim_{l\to\infty}\sup\lim_{k\to\infty}\sup\frac{1}{l}\int_{t_k}^{t_k+l}\|A(t)-A(t_k)\|\mathrm{d}t\leqslant v,\tag{1.5}$$

where $\lambda_i(A)$ denotes the ith-eigenvalue of the matrix A, $\mathrm{Re}\lambda_i(A)$ denotes the real part of $\lambda_i(A)$ and $v>0$ is sufficiently small.

Paralled attention has also been paid to the stability analysis of the systems with randomly time-varying parameters. Most of the existing results focus on the system with time-varying parameters being stationary processes[5], or jump processes modeled as a finite Markov chain[6] or processes satisfying φ-mixing condition[7].

This paper aims at presenting some new conditions both for the exponential stability and the L_p exponential stability of the system (1. 1) with slowly time-varying parameters. The conditions are motivated by those used in the stability analysis of adaptive control systems with unmodeled dynamics, deterministic or stochastic disturbances and time-varying parameters. For the case where the parameters are deterministic, the conditions we present in section 2 for exponential stability of the system (1. 1) are more general than those used in recent papers [3] and [4].

2 Conditions for Exponential Stability

The main purpose of this section is to provide some sufficient conditions, which is more general than those used in [2~4], for the exponential stability of the system (1. 1).

Theorem 2. 1 For the linear time-varying system(1. 1), assume that there are a real matrix process $\{A_0(t)\}$ and a real number sequence $\{t_k,k=0,1,\cdots\}$ satisfying

$$t_0=0,\quad t_k<t_{k+1}\to\infty,\quad \text{as}\quad k\to\infty;\quad \sup_{k\geqslant 0}(t_{k+1}-t_k)=T<\infty,$$

$$\sup_{k\geqslant 0}\|A_0(t_k)\|\leqslant C<\infty,\tag{2.1}$$

and
$$\max\{\mathrm{Re}\lambda_i(A_0(t_k)), i=1,\cdots,n; k=0,1,\cdots\} \leqslant -\alpha \tag{2.2}$$
such that
$$\int_{t_j}^{t_k} \|A(t) - A_0(t_j)\| \, dt \leqslant vf(t_k - t_j) + K, \quad \text{for all} \quad k \geqslant j, \tag{2.3}$$
where $C>0$, $\alpha>0$ and $K \geqslant 0$ are constants, and $f(t)$ is an increasing function of $t \in [0,\infty)$.

Then there exists a real number v_0 such that the system (1.1) is exponentially stable provided that $v \in [0, v_0)$.

Proof By (2.1), (2.2) and the argument used in [2,3], we may conclude that there exist constants $M \geqslant 1$ and $\rho > 0$ such that
$$\|\exp\{A_0(t_k)(t-s)\}\| \leqslant Me^{-\rho(t-s)}, \quad \forall\, t \geqslant s \geqslant 0, \tag{2.4}$$
for any k. (for example, see [3], we may choose $\rho = \frac{\alpha}{2}$ and $M = 1 + \frac{2(2^n-1)}{\pi\rho^n} C(3C)^{n-1}$).

Define $\tilde{T} = 2\rho^{-1}(\ln M + KM)$, $T_0 = 0$ and $T_{k+1} = \max_{i \geqslant 0}\{t_i, t_i \in [T_k + \tilde{T}, T_k + \tilde{T} + T]\}$. Then it is clear that
$$\tilde{T} \leqslant T_{k+1} - T_k \leqslant \tilde{T} + T, \tag{2.5}$$
$$\|\exp\{A_0(T_k)(t-s)\}\| \leqslant Me^{-\rho(t-s)}, \quad \forall\, t \geqslant s \geqslant 0, \tag{2.6}$$
and
$$\int_{T_k}^{T_{k+1}} \|A(t) - A_0(T_k)\| \, dt \leqslant vf(T_{k+1} - T_k) + K. \tag{2.7}$$

For any fixed $t \geqslant u \geqslant 0$, it is clear that there exist k_t and k_u, $k_t \geqslant k_u$, such that $t \in [T_{k_t}, T_{k_t+1})$ and $u \in [T_{k_u}, T_{k_u+1})$.

From (1.3) we have
$$\frac{d\Phi(t,u)}{dt} = A_0(T_{k_t})\Phi(t,u) + [A(t) - A_0(T_{k_t})]\Phi(t,u), \quad \Phi(u,u) = I, \tag{2.8}$$
which implies that
$$\Phi(t,u) = e^{A_0(T_{k_t})(t-T_{k_t})}\Phi(T_{k_t},u) + \int_{T_{k_t}}^{t} e^{A_0(T_{k_t})(t-s)}[A(s) - A_0(T_{k_t})]\Phi(s,u)ds. \tag{2.9}$$
This, together with (2.6), yields
$$\|\Phi(t,u)\| \leqslant Me^{-\rho(t-T_{k_t})}\|\Phi(T_{k_t},u)\| + M\int_{T_{k_t}}^{t} e^{-\rho(t-s)}\|A(s) - A_0(T_{k_t})\| \cdot \|\Phi(s,u)\| ds. \tag{2.10}$$

Multiplying both sides of (2.10) by $e^{\rho t}$ and applying the Bellman-Gronwall lemma (see, [8]), we obtain
$$\|\Phi(t,u)\| \leqslant M\exp\left\{-\rho(t - T_{k_t}) + M\int_{T_{k_t}}^{t} \|A(s) - A_0(T_{k_t})\| ds\right\} \|\Phi(T_{k_t},u)\|. \tag{2.11}$$

Similarly, for any fixed T_k, $k_u + 2 \leqslant k \leqslant k_t$, we can derive that
$$\|\Phi(T_k,u)\| \leqslant M\exp\left\{-\rho(T_k - T_{k-1}) + M\int_{T_{k-1}}^{t} \|A(s) - A_0(T_{k-1})\| ds\right\} \|\Phi(T_{k-1},u)\|, \tag{2.12}$$
which together with (2.5), (2.7) leads to
$$\begin{aligned}\|\Phi(T_k,u)\| &\leqslant M\exp\{-\rho(T_k - T_{k-1}) + M[vf(T_k - T_{k-1}) + K]\} \|\Phi(T_{k-1},u)\| \\ &\leqslant \exp\{\ln M + KM - \rho\tilde{T} + vMf(\tilde{T} + T)\} \|\Phi(T_{k-1},u)\|\end{aligned}$$

behavior of the system (1. 1). When the system parameter matrix $A(t)$ varies arbitrarily, many examples have been presented to show that the stability of the frozen time system (i. e., the real parts of the eigenvalues of $A(t)$ at every time instant are over bounded by $-\beta$, $\beta>0$) does not imply the stability of the time-varying system. (for example, see[4]) For the case where $A(t)$ is differentiable with $\sup_{t\geqslant 0}\|A(t)\|$ being bounded and $\sup_{t\geqslant 0}\|\dot{A}(t)\|$ being sufficiently small, results in [2] show that the stability of the frozen time system at every time instant is sufficient for the exponential stability of the time-varying system (1. 1). For the case where $A(t)$ is continuous, but not necessarily differentiable, the exponential stability conditions for the time-varying system (1. 1) are given in [3] for the case where the condition that $\sup_{t\geqslant 0}\|\dot{A}(t)\|$ is sufficiently small is replaced by the more general condition that $\|A(t_1)-A(t_2)\|\leqslant L|t_1-t_2|^{\alpha}$ for all $t_1,t_2\geqslant 0,\alpha>0$, with $L>0$ being sufficiently small.

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such that

$$\sup_{k\geqslant 1}\|A(t_k)\|\leqslant C<\infty,\quad \max\{\mathrm{Re}\lambda_i(A(t_k)),i=1,\cdots,n;k=1,2,\cdots\}\leqslant-\alpha$$

and
$$\limsup_{l\to\infty}\limsup_{k\to\infty}\frac{1}{l}\int_{t_k}^{t_k+l}\|A(t)-A(t_k)\|\mathrm{d}t\leqslant v,\tag{1.5}$$

where $\lambda_i(A)$ denotes the ith-eigenvalue of the matrix A, $\mathrm{Re}\lambda_i(A)$ denotes the real part of $\lambda_i(A)$ and $v>0$ is sufficiently small.

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2 Conditions for Exponential Stability

The main purpose of this section is to provide some sufficient conditions, which is more general than those used in [2~4], for the exponential stability of the system (1. 1).

Theorem 2. 1 For the linear time-varying system(1. 1), assume that there are a real matrix process $\{A_0(t)\}$ and a real number sequence $\{t_k,k=0,1,\cdots\}$ satisfying

$$t_0=0,\quad t_k<t_{k+1}\to\infty,\quad \text{as}\quad k\to\infty;\quad \sup_{k\geqslant 0}(t_{k+1}-t_k)=T<\infty,$$

$$\sup_{k\geqslant 0}\|A_0(t_k)\|\leqslant C<\infty,\tag{2.1}$$

$$\max\{\mathrm{Re}\lambda_i(A_0(t_k)), i=1,\cdots,n; k=0,1,\cdots\} \leqslant -\alpha \tag{2.2}$$

and such that

$$\int_{t_j}^{t_k} \|A(t)-A_0(t_j)\| \,\mathrm{d}t \leqslant vf(t_k-t_j)+K, \quad \text{for all} \quad k \geqslant j, \tag{2.3}$$

where $C>0$, $\alpha>0$ and $K\geqslant 0$ are constants, and $f(t)$ is an increasing function of $t\in[0,\infty)$.

Then there exists a real number v_0 such that the system (1.1) is exponentially stable provided that $v\in[0,v_0)$.

Proof By (2.1), (2.2) and the argument used in [2,3], we may conclude that there exist constants $M\geqslant 1$ and $\rho>0$ such that

$$\|\exp\{A_0(t_k)(t-s)\}\| \leqslant M\mathrm{e}^{-\rho(t-s)}, \quad \forall\, t\geqslant s\geqslant 0, \tag{2.4}$$

for any k. (for example, see [3], we may choose $\rho=\dfrac{\alpha}{2}$ and $M=1+\dfrac{2(2^n-1)}{\pi\rho^n}C(3C)^{n-1}$).

Define $\tilde{T}=2\rho^{-1}(\ln M+KM)$, $T_0=0$ and $T_{k+1}=\max\limits_{i\geqslant 0}\{t_i, t_i\in[T_k+\tilde{T}, T_k+\tilde{T}+T]\}$. Then it is clear that

$$\tilde{T}\leqslant T_{k+1}-T_k\leqslant \tilde{T}+T, \tag{2.5}$$

$$\|\exp\{A_0(T_k)(t-s)\}\| \leqslant M\mathrm{e}^{-\rho(t-s)}, \quad \forall\, t\geqslant s\geqslant 0, \tag{2.6}$$

and

$$\int_{T_k}^{T_{k+1}} \|A(t)-A_0(T_k)\| \,\mathrm{d}t \leqslant vf(T_{k+1}-T_k)+K. \tag{2.7}$$

For any fixed $t\geqslant u\geqslant 0$, it is clear that there exist k_t and k_u, $k_t\geqslant k_u$, such that $t\in[T_{k_t}, T_{k_t+1})$ and $u\in[T_{k_u}, T_{k_u+1})$.

From (1.3) we have

$$\frac{\mathrm{d}\Phi(t,u)}{\mathrm{d}t} = A_0(T_{k_t})\Phi(t,u)+[A(t)-A_0(T_{k_t})]\Phi(t,u), \quad \Phi(u,u)=I, \tag{2.8}$$

which implies that

$$\Phi(t,u) = \mathrm{e}^{A_0(T_{k_t})(t-T_{k_t})}\Phi(T_{k_t},u)+\int_{T_{k_t}}^{t}\mathrm{e}^{A_0(T_{k_t})(t-s)}[A(s)-A_0(T_{k_t})]\Phi(s,u)\mathrm{d}s. \tag{2.9}$$

This, together with (2.6), yields

$$\|\Phi(t,u)\| \leqslant M\mathrm{e}^{-\rho(t-T_{k_t})}\|\Phi(T_{k_t},u)\| + M\int_{T_{k_t}}^{t}\mathrm{e}^{-\rho(t-s)}\|A(s)-A_0(T_{k_t})\|\cdot\|\Phi(s,u)\|\mathrm{d}s. \tag{2.10}$$

Multiplying both sides of (2.10) by $\mathrm{e}^{\rho t}$ and applying the Bellman-Gronwall lemma (see, [8]), we obtain

$$\|\Phi(t,u)\| \leqslant M\exp\left\{-\rho(t-T_{k_t})+M\int_{T_{k_t}}^{t}\|A(s)-A_0(T_{k_t})\|\mathrm{d}s\right\}\|\Phi(T_{k_t},u)\|. \tag{2.11}$$

Similarly, for any fixed T_k, $k_u+2\leqslant k\leqslant k_t$, we can derive that

$$\|\Phi(T_k,u)\| \leqslant M\exp\left\{-\rho(T_k-T_{k-1})+M\int_{T_{k-1}}^{t}\|A(s)-A_0(T_{k-1})\|\mathrm{d}s\right\}\|\Phi(T_{k-1},u)\|, \tag{2.12}$$

which together with (2.5), (2.7) leads to

$$\begin{aligned}\|\Phi(T_k,u)\| &\leqslant M\exp\{-\rho(T_k-T_{k-1})+M[vf(T_k-T_{k-1})+K]\}\|\Phi(T_{k-1},u)\| \\ &\leqslant \exp\{\ln M+KM-\rho\tilde{T}+vMf(\tilde{T}+T)\}\|\Phi(T_{k-1},u)\|\end{aligned}$$

$$=\exp\{vMf(\tilde{T}+T)-\frac{\rho}{2}\tilde{T}\}\,\|\Phi(T_{k-1},u)\|. \tag{2.13}$$

Therefore, if we take $v_0=\dfrac{\rho\tilde{T}}{2Mf(\tilde{T}+T)}$, then for any fixed $v\in[0,v_0)$, we have

$$\|\Phi(T_k,u)\|\leqslant e^{-\delta}\|\Phi(T_{k-1},u)\|,\quad \delta=-vMf(\tilde{T}+T)+\frac{\rho}{2}\tilde{T}>0. \tag{2.14}$$

From (1. 3), it easy to see that

$$\|\Phi(\tau_1,\tau_2)\|\leqslant\exp\left\{\int_{\tau_2}^{\tau_1}\|A(s)\|\,ds\right\},\quad\forall\,\tau_1\geqslant\tau_2\geqslant 0. \tag{2.15}$$

This, combined with (2. 1) and (2. 7) yields

$$\begin{aligned}\|\Phi(T_{k_u+1},u)\|&\leqslant\exp\left\{\int_{T_{k_u}}^{T_{k_u+1}}\|A(s)\|\,ds\right\}\\ &\leqslant\exp\left\{\int_{T_{k_u}}^{T_{k_u+1}}\|A_0(T_{k_u})\|\,ds+\int_{T_{k_u}}^{T_{k_u+1}}\|A(s)-A_0(T_{k_u})\|\,ds\right\}\\ &\leqslant\exp\{vf(\tilde{T}+T)+K+C(\tilde{T}+T)\}.\end{aligned} \tag{2.16}$$

Combining this with (2. 11) and (2. 14) leads to

$$\begin{aligned}&\|\Phi(t,u)\|\\ &\leqslant M\exp\left\{M\int_{T_{k_t}}^{t}\|A(s)-A_0(T_{k_t})\|\,ds-\delta(k_t-k_u-1)+vf(\tilde{T}+T)+K+C(\tilde{T}+T)\right\}\\ &\leqslant M\exp\{(M+1)[vf(\tilde{T}+T)+K]+C(\tilde{T}+T)-\delta(k_t-k_u-1)\}.\end{aligned} \tag{2.17}$$

From (2. 5) and the definitions of k_t and k_u, we see that

$$t-u\leqslant T_{k_t+1}-T_{k_u}\leqslant(k_t-k_u+1)(\tilde{T}+T),$$

which means that $\qquad k_t-k_u\geqslant(\tilde{T}+T)^{-1}(t-u)-1.$

Substituting this into (2. 17), we finally obtain

$$\begin{aligned}\|\Phi(t,u)\|&\leqslant M\exp\{(M+1)[vf(\tilde{T}+T)+K]+C(\tilde{T}+T)\\ &\qquad-\delta[(\tilde{T}+T)^{-1}(t-u)-2]\}\\ &\leqslant M\exp\{(M+1)[vf(\tilde{T}+T)+K]+C(\tilde{T}+T)+2\delta\}\exp\left\{-\frac{\delta}{\tilde{T}+T}(t-u)\right\}.\end{aligned}$$

Then we see that (1. 2) holds for the system (1. 1) with

$$M_0=M\exp\{(M+1)[vf(\tilde{T}+T)+K]+C(\tilde{T}+T)+2\delta\},\quad \beta_0=\frac{\delta}{\tilde{T}+T}>0.$$

Remark 2. 1 It is easy to varify that the conditions given in Theorem 2. 1 for the exponential stability of the system (1. 1) is more general than those used in [2~4].

Remark 2. 2 From the proof of Theorem 2. 1, we see that the result in Theorem 2. 1 holds if we choose $v_0=\dfrac{\alpha\tilde{T}}{4Mf(\tilde{T}+T)}$ with $M=1+\dfrac{2^{n+1}(2^n-1)}{\pi\alpha^n}C\,(3C)^{n-1}$ and $\tilde{T}=4\alpha^{-1}$ $(\ln M+KM)$.

Example 1 Consider the system (1. 1) with

$$A(t)=\begin{bmatrix}-1 & b(t)\\ a(t) & -1\end{bmatrix}, \tag{2.18}$$

where
$$a(t)=\begin{cases}1, & \text{if } t\in[k,k+10^{-k}),\quad k=1,2,\cdots,\\ 0, & \text{otherwise},\end{cases}$$
and
$$b(t)=\begin{cases}10^{-12}(t-2j\cdot 10^{13}), & \text{if } t\in[2j\cdot 10^{13},(2j+1)10^{13}), j=0,1,\cdots,\\ 10-10^{-12}[t-(2j+1)10^{13}], & \text{if } t\in[(2j+1)\cdot 10^{13},2(j+1)10^{13}), j=0,1,\cdots.\end{cases}$$

It is easy to see that $A(t)$, given in (2.18), is piecewise continuous and does not satisfy (1.5) with $v=1$ for any real number sequence $\{t_k\}$, so the results of [2~4] can not be applied to judge whether the system is exponentially stable or not. If we choose
$$A_0(t)=\begin{bmatrix}-1 & b(t)\\ 0 & -1\end{bmatrix},$$
and $t_k=k, k=0,1,2,\cdots$, we derive that the conditions in Theorem 2.1 hold with $C=102, \alpha=1, K=\frac{1}{9}, v=10^{-12}$ and $f(t)=\frac{t^2}{2}$. It is easy to verify that $v=10^{-12}<v_0=\frac{\tilde{T}}{2M(\tilde{T}+1)^2}$, where $M=1+\frac{2^{n+1}(2^n-1)}{\pi\alpha^n}C(3C)^{n-1}$ and $\tilde{T}=4\alpha^{-1}(\ln M+KM)$. Therefore, from Theorem 2.1 and Remark 2.2, we see that the system in this example is exponentially stable.

3 Conditions for L_p exponential Stability

In this section, we will present some new sufficient conditions for the L_p exponential stability of the system (1.1) without assuming that the time-varying parameters are stationary processes, or jump processes modeled as a finite Markov chain, or processes satisfying φ-mixing condition.

Theorem 3.1 Consider the linear time-varying system (1.1) under the assumption that $A(t)$ is an $n\times n$ random real matrix process adapted to a nondecreasing family of σ-algebra $\mathscr{F}_t$. Suppose that there are a random real matrix process $\{A_0(t)\}$ adapted to $\mathscr{F}_t$ and a deterministic real number sequence $\{t_k, k=0,1,\cdots\}$ satisfying
$$t_0=0,\quad t_k<t_{k+1}\to\infty,\quad \text{as}\quad k\to\infty;\quad \sup_{k\geqslant 0}(t_{k+1}-t_k)=T<\infty,$$
$$\sup_{k\geqslant 0}\|A_0(t_k)\|\leqslant C<\infty,\quad \text{a.s.},\tag{3.1}$$
and
$$\max\{\mathrm{Re}\lambda_i(A_0(t_k)), i=1,\cdots,n; k=0,1,\cdots\}\leqslant -\alpha,\quad \text{a.s.},\tag{3.2}$$
such that
$$\left\{E\left(\exp\{N\int_{t_j}^{t_k}\|A(t)-A_0(t_j)\|\,dt\}\,|\mathscr{F}_{t_j}\right)\right\}^{1/N}\leqslant \exp\{vf(t_k-t_j)+K\},\quad \text{a.s.},\tag{3.3}$$
for all $t\geqslant j$, where $f(t)$ is an increasing function of $t\in[0,\infty), C>0, \alpha>0, K\geqslant 0$ and $N>0$ are constants, satisfying
$$N>M=1+\frac{2^{n+1}(2^n-1)}{\pi\alpha^n}C(3C)^{n-1}.\tag{3.4}$$

Then there exists a real number v_0 such that for any $v\in[0,v_0), p\in(1,N/M)$, the system (1.1) is L_p exponentially stable. Furthermore, if the condition (3.3) is replaced by
$$\left\{E\left(\exp\{N\int_u^t\|A(s)-A_0(t_k)\|\,ds\}\,|\mathscr{F}_u\right)\right\}^{1/N}\leqslant \exp\{vf(t-u)+K\},\ \text{a.s.},\ k=0,1,\cdots,\tag{3.5}$$
for all $t\geqslant t_k\geqslant u$, then there exist positive constants M_1 and β_1 such that

$$\{E(\|\Phi(t,u)\|^p|\mathscr{F}_u\}^{1/p}\leqslant M_1\mathrm{e}^{-\beta'(t-u)},\quad \text{a.s.},\quad \forall t\geqslant u. \tag{3.6}$$

Proof We first show that (1.4) holds for some positive constants M_1 and β_1 under the assumption (3.1)~(3.4).

Let $\tilde{T},\{T_k,k=0,1,2,\cdots\}$ be the same as those in the proof of Theorem 2.1. For any fixed T_k, by the argument used in the proof of Theorem 2.1, we obtain

$$\|\Phi(t,T_k)\|\leqslant M\exp\left\{-\rho(t-T_k)+M\int_{T_k}^{t}\|A(s)-A_0(T_k)\|\,\mathrm{d}s\right\},\quad \text{a.s.},\quad \forall t\geqslant T_k, \tag{3.7}$$

where M is defined in (3.4).

For $p\in(1,N/M)$, by (3.7), (3.3) and the Hölder inequality, we have

$$\begin{aligned}E(\|\Phi(T_{k+1},T_k\|^p|\mathscr{F}_{T_k})&\leqslant M^p\mathrm{e}^{-p\rho(T_{k+1}-T_k)}\cdot E\left(\exp\{pM\int_{T_k}^{T_{k+1}}\|A(s)-A_0(T_k)\|\,\mathrm{d}s\}|\mathscr{F}_{T_k}\right)\\&\leqslant M^p\mathrm{e}^{-p\rho(T_{k+1}-T_k)}\cdot\left\{E\left(\exp\{N\int_{T_k}^{T_{k+1}}\|A(s)-A_0(T_k)\|\,\mathrm{d}s\}|\mathscr{F}_{T_k}\right)\right\}^{pM/N}\\&\leqslant M^p\mathrm{e}^{-p\rho(T_{k+1}-T_k)+pM[\nu f(T_{k+1}-T_k)+K]}\\&\leqslant\mathrm{e}^{p[\ln M+KM-\rho\tilde{T}+\nu Mf(\tilde{T}+T)]}=\mathrm{e}^{p(\nu Mf(\tilde{T}+T)-\frac{\rho}{2}\tilde{T})},\quad \text{a.s.}.\end{aligned} \tag{3.8}$$

Therefore, if we take $v_0=\dfrac{\rho\tilde{T}}{2Mf(\tilde{T}+T)}$, then for any fixed $v\in[0,v_0)$, we have

$$E(\|\Phi(T_{k+1},T_k\|^p|\mathscr{F}_{T_k})\leqslant\mathrm{e}^{-p\delta},\quad \text{a.s.}, \tag{3.9}$$

where $\delta>0$ is defined as in the proof of Theorem 2.1.

For any fixed $t\geqslant u\geqslant 0$, we see that there exist k_t and k_u, $k_t\geqslant k_u$, such that $t\in[T_{k_t},T_{k_t+1}]$ and $u\in[T_{k_u},T_{k_u+1})$.

If $k_t=k_u$, then it is easy to see from (2.5) that $0\leqslant t-u\leqslant\tilde{T}+T$. By (2.15), (3.3) and the Hölder inequality, we derive that

$$\begin{aligned}\{E\|\Phi(t,u)\|^p\}^{1/p}&\leqslant E\exp\{p\int_{T_{k_t}}^{T_{k_t+1}}\|A(s)\|\,\mathrm{d}s\}\}^{1/p}\\&\leqslant\left\{E\exp\{p\left[\int_{T_{k_t}}^{T_{k_t+1}}\|A_0(T_{k_t})\|\,\mathrm{d}s+\int_{T_{k_t}}^{T_{k_t+1}}\|A(s)-A_0(T_{k_t})\|\,\mathrm{d}s\right]\}\right\}^{1/p}\\&\leqslant\mathrm{e}^{C(\tilde{T}+T)}\left\{E\exp\{N\int_{T_{k_t}}^{T_{k_t+1}}\|A(s)-A_0(T_{k_t})\|\,\mathrm{d}s\}\right\}^{1/N}\\&\leqslant\mathrm{e}^{C(\tilde{T}+T)+\nu f(\tilde{T}+T)+K}\\&\leqslant\mathrm{e}^{C(\tilde{T}+T)+\nu f(\tilde{T}+T)+K+\delta}\cdot\exp\left\{-\frac{\delta}{\tilde{T}+T}(t-u)\right\}.\end{aligned} \tag{3.10}$$

For the case where $k_t\geqslant k_u+1$, from the definition of $\Phi(t,u)$, we see that

$$\Phi(t,u)=\Phi(t,T_{k_t})\Phi(T_{k_t},T_{k_u+1})\Phi(T_{k_u+1},u). \tag{3.11}$$

Since $\{A(t),\mathscr{F}_t\}$ is an adapted matrix process, it is easy to see that so is $\{\Phi(t,t_0),\mathscr{F}_t\}$ for any $0\leqslant t_0\leqslant t$. Then, by (3.9) and the Hölder's inequality, for $p\in(1,N/M)$, $k_t>k_u+1$, we have

$$E(\|\Phi(T_{k_t},T_{k_u+1}\|^p|\mathscr{F}_{T_{k_u+1}}\}\leqslant E(E(\|\Phi(T_{k_t},T_{k_t-1}\|^p|\mathscr{F}_{T_{k_t-1}})\|\Phi(T_{k_t-1},T_{k_u+1}\|^p|\mathscr{F}_{T_{k_u+1}})$$

$$\leqslant e^{-p\delta} E(\|\Phi(T_{k_t-1}, T_{k_u+1})\|^p | \mathscr{F}_{T_{k_u+1}}), \quad \text{a.s.}.$$

Repeating this argument $k_t-(k_u+1)$ times and using the fact that $\|\Phi(T_{k_u+1}, T_{k_u+1})\|=1$, a. s. , we obtain that

$$E(\|\Phi(T_{k_t}, T_{k_u+1})\|^p | \mathscr{F}_{T_{k_u+1}}) \leqslant e^{-p\delta(k_t-k_u-1)}, \quad \text{a.s.}, \quad \text{for} \quad k_t \geqslant k_u+1. \tag{3.12}$$

By the argument similar to that used in (3. 10), we derive that

$$\begin{aligned} &E(\|\Phi(t, T_{k_t})\|^p | \mathscr{F}_{T_{k_t}}) \leqslant e^{p[C(\tilde{T}+T)+vf(\tilde{T}+T)+K]}, \\ &E\|\Phi(T_{k_u+1}, u)\|^p \leqslant e^{p[C(\tilde{T}+T)+vf(\tilde{T}+T)+K]}. \end{aligned} \tag{3.13}$$

Then, by (3. 12), (3. 13), the Hölder inequality and the fact that $k_t-k_u \geqslant (\tilde{T}+T)^{-1}(t-u)-1$ for $p\in(1,N/M)$, we obtain that

$$\begin{aligned} \{E\|\Phi(t,u)\|^p\}^{1/p} &\leqslant \{E(\|\Phi(t,T_{k_t})\|^p \|\Phi(T_{k_t},T_{k_u+1})\|^p \|\Phi(T_{k_u+1},u)\|^p)\}^{1/p} \\ &\leqslant \{E(E(\|\Phi(t,T_{k_t})\|^p | \mathscr{F}_{T_{k_t}} \|\Phi(T_{k_t},T_{k_u+1})\|^p \|\Phi(T_{k_u+1},u)\|^p)\}^{1/p} \\ &\leqslant e^{C(\tilde{T}+T)+vf(\tilde{T}+T)+K} \{E(E(\|\Phi(T_{k_t},T_{k_u+1})\|^p | \mathscr{F}_{T_{k_u+1}}) \|\Phi(T_{k_u+1},u)\|^p)\}^{1/p} \\ &\leqslant e^{C(\tilde{T}+T)+vf(\tilde{T}+T)+K-\delta[k_t-k_u-1]} \{E\|\Phi(T_{k_u+1},u)\|^p\}^{1/p} \\ &\leqslant e^{2C(\tilde{T}+T)+2vf(\tilde{T}+T)+2K-\delta[k_t-k_u-1]} \\ &\leqslant e^{2C(\tilde{T}+T)+2vf(\tilde{T}+T)+2K+2\delta} \cdot \exp\left\{-\frac{\delta}{\tilde{T}+T}(t-u)\right\}. \end{aligned} \tag{3.14}$$

From (3. 10) and (3. 14), we see that (1. 4) holds for the system (1. 1) with

$$M_1 = e^{2C(\tilde{T}+T)+2vf(\tilde{T}+T)+2K+2\delta}, \quad \beta_1 = \frac{\delta}{\tilde{T}+T} > 0,$$

i. e. , (1. 1) is L_p exponentially stable.

Similarly, if the condition (3. 3) is replaced by (3. 5), we can obtain (3. 6). Q. E. D.

Remark 3. 1 From the proof of Theorem 3. 1, we see that the result in Theorem 3. 1 holds if we choose $v_0=\dfrac{\alpha\tilde{T}}{4Mf(\tilde{T}+T)}$ with $M=1+\dfrac{2^{n+1}(2^n-1)}{\pi\alpha^n}C(3C)^{n-1}$ and $\tilde{T}=4\alpha^{-1}(\ln M+KM)$.

Example 2 Consider the system (1. 1) with

$$A(t) = \begin{bmatrix} -1 & b(t) \\ a(t) & -1 \end{bmatrix}, \tag{3.15}$$

where $a(t)=10^{-14}w_k^2$, for $t\in[k,k+1), k=0,1,2,\cdots$, with w_k being a Gaussian white noise sequence with variance 1, and

$$b(t) = \begin{cases} 10^{-8}(t-2j\cdot 10^9), & \text{if } t\in[2j\cdot 10^9, (2j+1)10^9), j=0,1,2,\cdots, \\ 10-10^{-8}[t-(2j+1)10^9], & \text{if } t\in[(2j+1)\cdot 10^9, 2(j+1)10^9), j=0,1,2,\cdots. \end{cases}$$

If we choose $$A_0(t) = \begin{bmatrix} -1 & b(t) \\ 0 & -1 \end{bmatrix},$$

and $t_k=0,1,2,\cdots$, and use the fact that $E\exp\{\varepsilon w_k^2\} \leqslant e^{M_w}$ for any $k\geqslant 0, 0\leqslant\varepsilon<\frac{1}{2}, M_w \geqslant \frac{\varepsilon}{1-2\varepsilon}$ (see [9]), then we derive that conditions in Theorem 3. 1 hold for the system with $C=102$,

$\alpha=1, K=0, v=10^{-8}, N=10^6, M=24\cdot 102^2$ and $f(t)=\frac{t^2}{2}+\frac{10^{-6}}{1-2\cdot 10^{-8}}t$. It is easy to verify that $v=10^{-8}<v_0=\frac{\tilde{T}}{4M(\tilde{T}+1)^2}$ with $\tilde{T}=4\ln M$. Therefore, applying the result in Theorem 3.1 and Remark 3.1, we see that the system in this example is L_p exponentially stable for $p\in(1, N/M)$.

4 Conclusion

In this paper, we have presented general conditions for the exponential stability of continuous-time linear homogeneous systems with slowly time-varying parameter and the L_p exponential stability of continuous-time linear homogeneous systems with randomly time-varying parameter. The results can be applied to investigate robust stability of adaptive control systems.

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连续时间时变参数系统的指数稳定性和 L_p 指数稳定性

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摘要: 本文给出具有慢时变参数的连续时间线性齐次系统指数稳定的一个一般性条件,并给出具有随机时变参数的连续时间线性齐次系统 L_p 指数稳定的一个充分条件.这些结果可用来分析带有未建模动态,确定性噪声或随机噪声及慢时变参数的自适应控制系统的稳定性问题.

关键词: 连续时间系统; 时变参数; 指数稳定性; L_p 指数稳定性

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