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Practical Stability of Uncertain Linear Control Systems*

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Abstract: This paper is devoted to the investigation of practical stability of uncertain linear control systems by Lyapunov functions and Riccati matrix equations. In the paper, a comaprison principle is first established, then a criterion for practical stability of uncertain linear control systems is obtained. A wider type of admissible sets is proposed in the paper. An example is given to illustrate the application of the main result of the paper.

Key words: uncertain systems; control; practical stability; Riccati matrix equations

1 Introduction

The stabilization of various uncertain systems has been widely investigated and a lot of results have been obtained, while it should be pointed out that, in the previous literature, the problem was investigated in the sense of Lyapunov stability, few result in the sense of practical stability has been obtained. But, as we know, practical stability is more practical for more engineering problems[1~2], thus it is necessary to investigate the stabilization of uncertain systems in the sense of parctical stability.

In the present paper, a comparison principle (Theorem 1) is first estabilished such that the main result can be carried out. The conception of admissible sets for the controls is generalized so that a wider range of problems, including the stabilization of control systems with uncertain disturbances in the controls, can be investigated and the results carried out are applicable in engineering practice.

2 Definition and Problem

Consider the control system

$$x = f(t,x,u), \quad x(t_0) = x_0, \quad u \in \Omega, \tag{1}$$

where $f \in C\{\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n\}$, $u \in \mathbb{R}^m$ is the control function, Ω is called the admissible set, it can be determined in accordance with the need of the practical problems, a typical form of admissible set is given in Section 3 of the paper, which is determined with the bounedeness of a type of functional of the controls.

Definition i) The control system (1) is said to be practically stable with respect to (λ, A, Ω) with $0<\lambda<A$, if for any $u \in \Omega$, $|x_0|<\lambda$ implies $|x(t)|<A (t \geqslant t_0)$ for some $t_0 \in \mathbb{R}^+$.

ii) the control system (1) is said to be uniformly practically stable with respect to (λ, A, Ω), if for any $u \in \Omega$, $|x_0| < \lambda$ implies $|x(t)| < A (t \geqslant t_0)$ for any $t_0 \in \mathbb{R}^+$.

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Remark 1 The asyptotic practical stability of control systems can also be defined in a manner similar to that above.

For the practical stability of the control ststem (1), an important problem is to check if, given boundedness index A for the state of the system, an admissible set Ω exists and how small λ should be? This paper will centralizes on this problem and some concrete criterion is obtained for the uncertain time-invariant linear control systems.

3 Comparison Principle

Let $U(t,u(\cdot))$ be a functional of $u(t)$, $\lambda_0(t) \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a given function. Consider the admissible set

$$\Omega_1 = \{u \in \mathbb{R}^m : U(t,u(\cdot)) \leqslant \lambda_0(t), t \geqslant t_0\}. \tag{2}$$

Such an admissible set gives a contraint to the bound of a related factor of the control of the system (1).

Theorem 1 (Comparison priciple I) Assume that

i) $0 < \lambda < A$ are given;

ii) $V \in C\{\mathbb{R}^+ \times S(A), \mathbb{R}^+\}$, $V(t,x)$ is locally Lipschitzian in x and

$$b(|x|) \leqslant V(t,x) \leqslant a(|x|), \quad (t,x) \in \mathbb{R}^+ \times S(A), \tag{3}$$

where $a(\cdot), b(\cdot) \in K$, $S(A) = \{x : |x| < A\}$;

iii) for $(t,x) \in \mathbb{R}^+ \times S(A)$ and $u \in \Omega_1$,

$$\begin{aligned} D^+ V(t,x) &= \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x+hf(t,x,u(t))) - V(t,x)] \\ &\leqslant g(t, V(t,x), U(t,u(\cdot))), \end{aligned} \tag{4}$$

where $g \in C\{\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}\}$ and $g(t,w,v)$ is nondecreasing in v for each (t,w);

iv) $a(\lambda) < b(A)$ holds.

Then the practical stability properties of the comparison equation

$$\dot{w} = g_0(t,w), \quad w(t_0) = w_0 \geqslant 0, \tag{5}$$

where $g_0(t,w) = g(t,w,\lambda_0(t))$ imply the corresponding practical stability properties of the control system (1).

Proof Because of the similarity, here only the part of the proof for the case of practical stability is given.

Denote $m(t) = V(t,x(t))$, where $x(t)$ is the solution of system (1) with given control function $u(t)$. By the assumption ii), for $u \in \Omega_1$,

$$D^+ m(t) \leqslant g(t, m(t), U(t,u(\cdot))) \leqslant g(t, m(t), \lambda_0(t)),$$

i.e.

$$D^+ m(t) \leqslant g_0(t, m(t)), \quad t \geqslant t_0. \tag{6}$$

By the comparison theorem for the solution of ordinary differential equations and differential inequalities[2],

$$m(t) \leqslant V(t,x(t)) \leqslant w(t), \quad t \leqslant t_0,$$

where $w(t)$ is the maximal solution of the comparison equation (5) with initial value $w(t_0) = V(t_0,x_0)$. Set $|x_0| < \lambda$. By the assumptions of the theorem, $w(t_0) = V(t_0,x_0) \leqslant a(|x_0|) \leqslant a(\lambda)$, then $w(t) < b(A)$, thus, by

$$b(|x(t)|) \leqslant V(t,x(t)) \leqslant w(t) < b(A),$$

we have $b(|x(t)| < b(A), |x(t)| < A$ for $t \leqslant t_0$. The proof is complete.

By way, a similiar result as follows is obtained.

Theoerem 2 Comparison principle II) If the assumptions i)~iv) of Theorem 1 hold for

$$u \in \Omega_2 = \{u \in \mathbb{R}^m : U(t,u(\cdot)) \leqslant r(t), t \geqslant t_0\}, \tag{7}$$

where $r(t)$ is the maximal solution of the comparison equation

$$\dot{w} = g(t,w,w), \quad w(t_0) = w_0, \tag{8}$$

with initial value $w_0 = V(t_0,x_0)$, then the conclusion of Theorem 1 remains to be valid with the comparison equation (8).

Proof The proof is similar to the proof of Theorem 1, one needs only notice the following fact:

$$D^+ m(t) \leqslant g(t,m(t),r(t)), \quad t \geqslant t_0. \tag{9}$$

By the comparison theorem for the solution of ordinary differential inequalities and differential equations, $m(t) = V(t,x(t)) \leqslant w(t)$, where $w(t)$ is the solution of the comparison equation

$$\dot{w} = g(t,w,r(t)), \quad w(t_0) = V(t_0,x_0). \tag{10}$$

This $w(t)$ is just the function $r(t)$, because $r(t)$ satisfies

$$\dot{r} = g(t,r,r), \quad r(t_0) = V(t_0,x_0).$$

The proof is complete.

Remark 2 Theorem 1 and Theorem 2 are improvement on the corresponding results in [2].

4 Main Result

Consider the uncertain linear control system

$$\dot{x} = Ax + Bu, \quad u = u_c + u_\delta, \tag{11}$$

where $x \in \mathbb{R}^n$, $u_c, u_\delta \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$; A and B are constant matrices, (A,B) is a controllable pair.

In system (11), u_δ is the disturbance to the control, u_c will be chosen as $u_c = -R^{-1}B^{\mathrm{T}}Px$, the admissible set will be the form

$$\Omega_3 = \{u = u_c + u_\delta : u_c = -R^{-1}B^{\mathrm{T}}Px, u_\delta \in \mathbb{R}^m, U(u(\cdot)) \leqslant \gamma = \text{const}\}, \tag{12}$$

where $U(u(\cdot)) = (Bu_\delta)^{\mathrm{T}}P(Bu_\delta)$, P is a positive definite marix, $\gamma > 0$.

Theorem 3 Assume that (A,B) is controllable, $R \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$ are given positive definite weighting matrices, P is the positive definite solution matrix of the Riccati matrix equation

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q = 0, \tag{13}$$

u_c is chosen as $u_c = -R^{-1}B^{\mathrm{T}}Px$.

If λ, A are such that

$$0 < \lambda < \sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}}A, \tag{14}$$

where $\lambda_m(P) = \lambda_{\min}(P)$, $\lambda_M(P) = \lambda_{\max}(P)$, then there exists a constant $\gamma > 0$ such that the control system (11) is uniformly practically stable with respect to (λ, A, Ω_3).

Proof Define a matrix function

$$\xi(\alpha) = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + \alpha P. \tag{15}$$

Since $\lim\limits_{\alpha\to 0^+} \xi(\alpha) = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}R = -Q < 0$, there exists a constant $\alpha > 0$ such that

$$PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + \alpha P = -\tilde{Q} \leqslant 0. \tag{16}$$

Define another function

$$\varphi(\gamma) = \sqrt{\frac{\lambda_m(P)A^2 - \dfrac{r^2\gamma}{r\alpha - 1}}{\lambda_M(P)}}, \quad \left(r \geqslant \frac{1}{\alpha}\right). \tag{17}$$

Since $\lim\limits_{\gamma\to 0^+} = \sqrt{\dfrac{\lambda_m(P)}{\lambda_M(P)}} > \lambda$, there exists a constant $\gamma > 0$ such that $\varphi(\gamma) > \lambda$, i.e. we have

$$\lambda_M(P)\lambda^2 < \lambda_m(P)A^2 - \frac{r^2\gamma}{r\alpha - 1}. \tag{18}$$

Define Lyapunov function $V(t,x) = x^{\mathrm{T}}Px$, then $a(\lambda) = \lambda_M(P)\lambda^2$, $b(A) = \lambda_m(P)A^2$. With the inequality $2X^{\mathrm{T}}PY \leqslant rX^{\mathrm{T}}PX + \frac{1}{r}Y^{\mathrm{T}}PY$[3], along the solution of the closed-loop system of (11) with $u \in \Omega_3$ we have

$$\begin{aligned} D^+ V &= x^{\mathrm{T}}[(A - BR^{-1}B^{\mathrm{T}}P)^{\mathrm{T}}P + P(A - BR^{-1}B^{\mathrm{T}}P)]x + 2(Bu_\delta)^{\mathrm{T}}Px \\ &= x^{\mathrm{T}}[PA + A^{\mathrm{T}}P - 2PBR^{-1}B^{\mathrm{T}}P]x + 2(Bu_\delta)^{\mathrm{T}}Px \\ &= x^{\mathrm{T}}[-\alpha P - PBR^{-1}B^{\mathrm{T}}P - \tilde{Q}]x + 2(Bu_\delta)^{\mathrm{T}}Px \\ &\leqslant -\alpha x^{\mathrm{T}}Px + r(Bu_\delta)^{\mathrm{T}}P(Bu_\delta) + \frac{1}{r}x^{\mathrm{T}}Px \end{aligned}$$

$$D^+ V \leqslant g(V, U(u(\cdot))), \tag{19}$$

where

$$g(V,U) = -\left(\alpha - \frac{1}{r}\right)V + rU, U(u(\cdot)) = (Bu_\delta)^{\mathrm{T}}P(Bu_\delta). \tag{20}$$

One can easily verify that the conditions i)~iv) of Theorem 1 are satisfied. Thus, by Theorem 1, the uniform practical stability of the comparison equation

$$\dot{w} = g(w, w), \quad w(t_0) = V(t_0, x_0), \tag{21}$$

with respect to $(a(\lambda), b(A))$ implies the uniform practical stability of the closed-loop system of control system (11) with respect to (λ, A). To complete the proof the theorem, one meeds only show the uniform practical stability of (20) with respect to $(a(\lambda), b(A))$.

In fact, the soluition of (20) is given by

$$\begin{aligned} w(t) &= w_0 e^{-(\alpha - \frac{1}{r})(t - t_0)} + \frac{r^2\gamma}{\alpha r - 1} e^{-(\alpha - \frac{1}{r})(t - t_0)} \\ &\leqslant w_0 + \frac{r^2\gamma}{\alpha r - 1}, \quad t \geqslant t_0. \end{aligned}$$

One can easily verify that (21) is uniformly practically stable with respect to $(a(\lambda), b(A))$ under the condition (18). The proof is complete.

Remark 3 If r is chosen as $r = \dfrac{2}{\alpha}$, then (18) becomes

$$0 < \gamma < \frac{\alpha^2}{4}[\lambda_m(P)A^2 - \lambda_M(P)\lambda^2]. \tag{22}$$

5 Optimizaton Problem for γ

An larger constant γ to ensure the practical stability of the colsed-loop system of (11) admits a larger admissible set for the disturbance u_δ. Thus, an important problem is to find an upper bound as large as poossible for γ.

In view of (22), one needs only find a constant α, with (16) being satisfied, as large as possible.

In fact, (16) is just $-Q + \alpha P \leqslant 0$, i.e. $\alpha P \leqslant Q$. This inequality is equivalent to $\alpha I \leqslant P^{-\frac{1}{2}}QP^{-\frac{1}{2}}$, where $P^{-\frac{1}{2}}$ is the positive definite square root of the matrix P. Thus, α should be chosen as

$$\alpha = \lambda_m(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}), \tag{23}$$

or

$$\alpha = \lambda_m(Q)\lambda_M^{-1}(P). \tag{24}$$

The estimations for γ corresponding to (23) and (24) are

$$0 < \gamma < \frac{1}{4}\lambda_m^2(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})(\lambda_m(P)A^2 - \lambda_M(P)\lambda^2), \tag{25}$$

$$0 < \gamma < \frac{1}{4}\lambda_m^2(Q)\lambda_M^{-2}(P)(\lambda_m(P)A^2 - \lambda_M(P)\lambda^2), \tag{26}$$

respectively.

6 Example

Consider control system (11) with $n = 2, m = 1$,

$$A = \begin{bmatrix} 0.00000 & 1.00000 \\ -0.47446 & -0.45916 \end{bmatrix}, \quad B = \begin{bmatrix} 0.00000 \\ 1.00000 \end{bmatrix}.$$

Where (A,B) is controllable. Choosing $R = [0.50000]$, and

$$Q = \begin{bmatrix} 1.04868 & -0.00138 \\ -0.00138 & 1.08372 \end{bmatrix}.$$

Then the positive definite solution of (13) is

$$P = \begin{bmatrix} 1.44050 & 0.52554 \\ 0.52554 & 0.54084 \end{bmatrix}. \tag{27}$$

We have $\lambda_m(P) = 0.28641$, $\lambda_M(P) = 1.70593$, $\lambda_m(Q) = 1.04866$. In this case, the estimation (26) for γ is

$$0 < \gamma < 0.02706A^2 - 0.16116\lambda^2. \tag{28}$$

7 Conclusion

Practical stability of uncertain linear control systems is investigated with Lyapunov functions and Ricati matrix equations. A comprison principle associated with the Lyapunov function method is established, and a criterion for practical stability of uncertain linear conrtrol systems is obtained. A wider type of admissible sets for the control action is proposed in the paper. It is shown that the practical stability of the uncertain linear control systems can be ensured if the disturbances to the controls are restricted by suitable upper bounds.

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不确定线性控制系统的实用稳定性

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摘要：本文利用 Lyapunov 函数和 Riccati 矩阵方程研究不确定线性控制系统的实用稳定性，文中首先建立了一个比较原理，然后对不确定性线性控制系统的实用稳定性得到了一个判据．文中定义了一类更广泛的容许集．文中给出了一个例子以说明本文主要结果的应用方法．

关键词：不确定系统；控制；实用稳定性；Riccati 矩阵方程

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