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Robust Stability of the State-Space Models With Structured Perturbations

ZHANG Ying and SU Weizhou
(School of Electrical and Electronic Engineering Nanyang Technological University · Singapore, 639798)
FENG Chunbo
(Research Institute of Automation, Southeast University · Nanjing, 210096, PRC)

Abstract: This paper studies the robust stability of state-space models with structured perturbations. An necessary and sufficient condition for the stability of the perturbed system are obtained by exploiting the critical stability condition in the frequency-domain and the structural information of the system. It is shown by examples that an less conservative stability margin of the system can be achieved by using the new condition.

Key words: robustness; stability; state-space model; structure uncertainty

1 Introduction

This paper studies robust stability of state-space models with uncertain parameters in the state matrix. The system under consideration is described by

$$\dot{X} = (A + \Delta A)X, \tag{1}$$

where $X \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, \Delta A = \sum_{i=1}^{l} a_i A_i, A_i (i = 1, 2, \cdots, l) \in \mathbb{R}^{n\times n}$, and $a_i (i = 1, 2, \cdots, l)$ are uncertain perturbation parameters.

The nominal part of the system (1) denoted by

$$\dot{X} = AX \tag{2}$$

is assumed to be asymptotically stable, i.e., all the eigenvalues of matrix A lie in the open left half-plane.

The problem of the stability robustness analysis is to find the largest bound α such that the system (1) is stable for the perturbation parameters satisfying $\| a \| \leqslant \alpha$, where $a = (a_1, a_2, \cdots, a_l)^T$. The number α specifies the maximun stability robustness margin of the system.

Recently such problem has drawn much attention and many analysis methods have been proposed [1～7]. Since only the sufficient conditions on stability were exploited in those methods, the allowable perturbation bounds on the system parameters a_i were reached conservatively. Thus the problem of determining the stability margin of system (1) can not be said to be solved satisfactorily.

In this paper, the problem is treated by utilizing the critical stability conditions in frequency-domain and the structural information of the system. A new sufficient and necessary condition for the stability of the perturbed system is established. It is shown by examples

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that the stability robustness margin obtained by using this new condition is less conservative than those of the existing methods.

2 Main Results

Let us firstly consider the case where the system (1) is critically stable. In such a case, there are a pair of conjugate eigenvalues $\pm j\omega$ and a nonzero vector $X \in \mathbb{R}^n$ such that

$$(-j\omega I + A + \sum_{i=1}^{l} a_i A_i)X = 0. \tag{3}$$

Rearranging Eq. (3) gives

$$X = (j\omega I - A)^{-1} \sum_{i=1}^{l} a_i A_i X. \tag{4}$$

Let span $(\mathrm{Re}(A_1), \mathrm{Re}(A_2), \cdots, \mathrm{Re}(A_l))$ be the space spanned by $\mathrm{Re}(A_i)(i = 1, 2, \cdots, l)$, where $\mathrm{Re}(A)$ be the range space of matrix A. Then Eq. (4) implies that

$$X \in (j\omega I - A)^{-1}\mathrm{span}(\mathrm{Re}(A_1), \mathrm{Re}(A_2), \cdots, \mathrm{Re}(A_l)). \tag{5}$$

Denote $\mathcal{N}(\omega)$ be the space $(j\omega I - A)^{-1}\mathrm{span}(\mathrm{Re}(A_1), \mathrm{Re}(A_2), \cdots, \mathrm{Re}(A_l))$. The following theorem shows that the stability robustness of system (1) can be studied on the space $\mathcal{N}(\omega)$.

Theorem 1 Suppose that the nominal system given by (2) is asymptotically stable. Then the system (1) is stable if and only if

$$\sup_{\omega \in [0,\infty)} \max_{X \in \mathcal{N}(\omega) \, \| x \| = 1} \| (j\omega I - A)^{-1}(A_1 X, A_2 X, \cdots, A_l X) a \| \leqslant 1, \tag{6}$$

where the norm is compatible norm on the space $\mathcal{N}(\omega)$.

To prove Theorem 1 the following lemma is useful.

Lemma 1 Suppose the system (1) be stable when the perturbation parameter a is such that $\| a \| \in [0, b)$, where b is a positive number. If the system $\dot{X} = (A + \lambda \Delta A)X$ is critically stable then $|\lambda| \geqslant 1$.

proof The method of reduction to absurdity is used here to come up with the conclusion.

If $|\lambda| < 1$, then $0 \leqslant \| \lambda a \| < \| a \| < b$. Under the assumption in Lemma 1, we get that the system $\dot{X} = (A + \lambda \Delta A)X$ is stable, which is contradictory to the condition that the system is critically stable. Thus the conclusion of the lemma is valid. Q. E. D.

proof of Theorem 1 Sufficiency. To show the sufficent condition, a methodology of reduction to absurdity is adopted here. It will be shown firstly that if the system (1) is unstable then the inequality (6) will not hold.

Since the eignvaluees of system (1) depend continuously upon $\| a \|$, there must be a positive number $\lambda \in (0,1)$ such that the system $\dot{X} = (A + \lambda \Delta A)X$ is critically stable if the system (1) is assumed to be unstable. In such case, there is a nonzero vector $X \in \mathbb{R}^n$ satisfying.

$$(j\omega I - A - \lambda \Delta A)X = 0. \tag{7}$$

It is obvious that

$$X = \lambda (j\omega I - A)^{-1}(A_1 X_1, A_2 X, \cdots, A_l X)a \in \mathcal{N}(\omega). \tag{8}$$

Thus,

$$\| x \| = |\lambda| \, \| (j\omega I - A)^{-1}(A_1X, A_2X, \cdots, A_lX)a \|. \tag{9}$$

Since $0 < \lambda < 1$, it following from (9) that

$$\sup_{\omega\in[0,\infty)} \max_{X\in\mathcal{N}(\omega)\,\|x\|=1} \| (j\omega I - A)^{-1}(A_1X, A_2X, \cdots, A_lX)a \| > 1 \tag{10}$$

which results in a contradiction to the condition (6). The sufficient condition is thus proven.

Necessity. We will show that if the system (1) is stable then the inequality (6) holds.

Assume that the maximum stability margin of system is given by a positive real number b, i.e., the system (1) is stable with the perturbation parameter a satisfying $0 \leqslant \| a \| < b$. For a certain parameter a, there is a real number λ such that the system $\dot{X} = (A + \lambda\Delta A)X$ is critically stable. By Lemma 1, $\lambda \geqslant 1$.

Meanwhile, there are a frequency $\omega \in [0,\infty)$ and a nonzero vector X such that

$$(j\omega I - A - \lambda\Delta A)X = 0, \tag{11}$$

if the system $\dot{X} = (A + \lambda\Delta A)X$ is critically stable.

Rearranging (11) gives

$$X = \lambda(j\omega I - A)^{-1}(A_1X, A_2X, \cdots, A_lX)a. \tag{12}$$

It is obvious that

$$\frac{1}{|\lambda|} \| x \| = \| (j\omega I - A)^{-1}(A_1X, A_2X, \cdots, A_lX)a \|. \tag{13}$$

Therefore

$$\sup_{\omega\in[0,\infty)} \max_{X\in\mathcal{N}(\omega)\,\|x\|=1} \| (j\omega I - A)^{-1}(A_1X, A_2X, \cdots, A_lX)a \| \leqslant 1. \tag{14}$$

Since a is taken in $[0,b)$ arbitrarily, the above inequality is valid for all a such that $0 \leqslant \| a \| \leqslant b$. The necessity of the theorem is proven. Q. E. D.

Form Theorem 1 the following corollary can be derived easily by using properties of norm. It provides us sufficient condition for determining the bound on the perturbations.

Corollary 1 Suppose the nominal system (2) is asympototically stable. Then the system (1) is stable if

$$\sup_{\omega\in[0,\infty)} \| (j\omega I - A)^{-1} \| \max_{X\in\mathcal{N}(\omega)\,\|x\|=1} \| (A_1X, A_2X, \cdots, A_lX) \| \, \| a \| \leqslant 1. \tag{15}$$

In this corollary the norm of a may be anycomatible norm on $\mathcal{N}(\omega)$ such as H_1, H_2 and H_∞.

3 Application: The Euclidean norm

The results of the last section are now applied to the case where the norm of the perturbation parameter a is taken as the Euclidean norm

$$\| X \|_2 = [\sum_{i=1}^{n} |X_i|^2]^{\frac{1}{2}}, \quad X \in \mathbb{R}^n). \tag{16}$$

Suppose that an orthogonal basis of the space span $(\mathrm{Re}(A_1), \mathrm{Re}(A_2), \cdots, \mathrm{Re}(A_l))$ is given by $e_1, e_2, \cdots, e_k$, where k is the dimension of the space $\mathcal{N}(\omega)$ and $e_i \in \mathbb{R}^n, (i = 1, 2, \cdots, k)$. Then each of the perturbation matrices $A_i (i = 1, 2, \cdots, l)$ can be described by

$$A_i = (e_1, e_2, \cdots, e_k)\begin{bmatrix} l_{1i}^{\mathrm{T}} \\ l_{2i}^{\mathrm{T}} \\ \vdots \\ l_{ki}^{\mathrm{T}} \end{bmatrix}, \quad (i = 1, 2, \cdots, l), \tag{17}$$

where $l_{ki} \in \mathbb{R}^n$ and T denotes the transpose operator of a matrix or a vector.

Defining $T(X) = (A_1X, A_2X, \cdots, A_lX)^* (A_1X, A_2X, \cdots, A_lX)$, we get

$$T(X) = \mathrm{diag}(X^*, \cdots, X^*)[\sum_{i=1}^{k}(l_{i1}, l_{i2}, \cdots, l_{il})\begin{pmatrix} l_{i1}^{\mathrm{T}} \\ l_{i2}^{\mathrm{T}} \\ \vdots \\ l_{il}^{\mathrm{T}} \end{pmatrix}]\mathrm{diag}(X, \cdots, X), \tag{18}$$

where $*$ denotes the conjugate transpose openator of a matrix or a vector.

By the definition of the 2-norm, it follows that

$$\| (A_1X, A_2X, \cdots, A_lX) \|_2^2 = \lambda_{\max}[T(X)], \tag{19}$$

where $\lambda_{\max}[T(X)]$ denotes the maximum eigenvalue of the matrix $T(X)$.

Similarly, we can obtain

$$\| (j\omega I - A)^{-1} \|_2^{-2} = \min_{X \in \mathscr{N}(\omega) \| x \| = 1} X^*(-j\omega I - A)^{\mathrm{T}}(j\omega I - A)X. \tag{20}$$

Then the results presented in the above section can now be restated in terms of the Euclidean norm.

Theorm 2 Suppose that the system (2) is stable. Then the system (1) is stable if the perturbation parameters satisfy

$$\sum_{i=1}^{l} a_i^2 \leqslant \inf_{\omega \in [0, \infty)} \frac{\min\limits_{X \in \mathscr{N}(\omega) \| x \| = 1} X^*(-j\omega I - A)^{\mathrm{T}}(j\omega I - A)X}{\max\limits_{X \in \mathscr{N}(\omega) \| x \| = 1} \lambda_{\max}(T(X))}. \tag{21}$$

Proof This conclusion can follow directly from corollary 1, (19) and (20). Q. E. D.

The right side of (21) gives a bound of the stability robustness margin. The expressions of denominator and the nominator of such a bound reveal that the stability robustness margin should be determined by using both the frequency features of the nominal system and the structure of perturbation matrices.

4 Examples

Example 1 Consider the system in [1,4] and [5] with structured perturbations given by

$$\dot{X} = \left[\begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} + \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \end{bmatrix}\right]X, \tag{22}$$

where $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

It is obvious that

$$\mathrm{span}(\mathrm{Re}(A_1), \quad \mathrm{Re}(A_2)) = R^2, \tag{23}$$

and

$$\mathscr{N}(\omega) = \begin{cases} C^2, & \omega \neq 0, \\ R^2, & \omega = 0 \end{cases} \tag{24}$$

where C^2 is the complex space of two dimensions.

Taking the following orthogonal basis of the space span $(\mathrm{Re}(A_1), \mathrm{Re}(A_2))$

$$e_1^{\mathrm{T}} = (1,0) \text{ and } e_2^{\mathrm{T}} = (0,1), \tag{25}$$

we obtain

$$l_{11}^{\mathrm{T}} = (1,0) \quad l_{21}^{\mathrm{T}} = (0,0) \quad l_{12}^{\mathrm{T}} = (0,0), \quad l_{12}^{\mathrm{T}} = (0,1), \tag{26}$$

and

$$T(X) = 1. \tag{27}$$

On the other hand,

$$\min_{X \in \mathcal{N}(\omega)\, \| x \| = 1} X^*(-j\omega I - A)^{\mathrm{T}}(j\omega I - A)X = \omega^2 + 1. \tag{28}$$

From Theorem 2, we know that

$$\| a \|_2 \leqslant 1. \tag{29}$$

Therefore the stability robustness margin is given by $\alpha = 1$, which is actually the maximum stability robustness margin. Comparing with the margins 0.6575, 0.6667 and 0.9150 obtained in [1,4]and [5], we can see that the presented result is better than the existing ones.

Example 2 Consider the system

$$\dot{X} = \begin{bmatrix} -2+k_1 & 0 & -1+k_1 \\ 0 & -3+k_2 & 0 \\ -1+k_1 & -1+k_2 & -4+k_1 \end{bmatrix} X, \tag{30}$$

where $A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

This example was used in [1,2],[4] and [5]. The stability margin given by these methods are 0.8151, 0.5207, 1.5533, and 1.75 respectively. It will be shown that the largest bound on the perturbation parameters can be calculated by using (21).

Taking the following orthogonal basis of span $(\mathrm{Re}(A_1), \mathrm{Re}(A_2))$

$$e_1^{\mathrm{T}} = (1,0,1) \text{ and } e_2^{\mathrm{T}} = (0,1,1), \tag{31}$$

we get that

$$\begin{aligned} & l_{11}^{\mathrm{T}} = (1,0,1), \quad l_{21}^{\mathrm{T}} = (0,0,0), \quad l_{12}^{\mathrm{T}} = (0,0,0), \quad l_{22}^{\mathrm{T}} = (0,1,0). \\ & \lambda_{\max} T(X) = 1, \end{aligned} \tag{32}$$

and

$$\inf_{\omega \in [0,\infty)} \max_{X \in \mathcal{N}(\omega)\, \| x \| = 1} X^*(-j\omega I - A)^{\mathrm{T}}(j\omega I - A)X = 1.75. \tag{33}$$

Therefore, the bound is given by $\alpha = 1.75$, which is just the maximum stability robustness margin of the system.

Both the examples demonstrate that an improveed stability robustness margin can be obtained by using the method presented in this paper. Applying our method to other examples also yields better bounds than those of other methods.

5 Conclusion

The stability robustness of state-space models with structured perturbations is studied in this paper. The results show that the stability margin of the system is related not only to the perturbation structure but also to the frequency features of the nominal system. An improved or the exact bound on the margin can be obtained by using the structural information of the system and the perturbation matrices.

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含结构扰动的状态空间模型的鲁棒性分析

张 颖 苏为州

(南洋理工大学,电工电子工程学院·新加坡,639798)

冯纯伯

(东南大学自动化研究所·南京,210096)

摘要:本文研究了具有结构扰动的系统的鲁棒性.根据系统频域的临界条件以及扰动系统的结构信息,给出了一个判定系统稳定的充分必要条件,并利用此结果提出了一种确定系统鲁棒稳定上界的方法.与现有的方法相比,本文方法可得保守性小的鲁棒稳定上界.

关键词:鲁棒稳定性;参数扰动;鲁棒稳定上界

本文作者简介

张 颖 分别于1989年,1992年和1995年于东南大学获学士、硕士和博士学位.现在新加坡 Nanyang Technological University 从事博士后研究工作.

苏为州 分别于1983年1986年在东南大学获学士和硕士学位.目前在澳大利亚 Newcastle 大学从事科学研究工作.

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