Stability of Solutions of Singular Systems with Delay *

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Abstract: This paper presents new definitions of stability of solutions to singular systems with delay, and establishes theorems on stability and instability. Then, as an example, this paper discusses a class of n-th order constant coefficient linear singular systems with delay, obtains some results of stability and unstability.

Key words: singular systems; delay; consistency condition; stability

1 Introduction

Many practical problems are modeled by singular systems, such as optimal control problems and constrained control problems, electrical circuits, some population growth models and singular perturbations. Since delay often occurs in these problems, therefore the research of singular systems with delay runs important role in practice and theory. On the discussion of stability of singualr systems, compared with that of nonsingular systems, there are three main new difficulties: the first is that it isn't easy to satisfy the existence and uniqueness of solutions, since the initial conditions may not be consistent; the second is that it is difficult to calculate the derivatives of Liapunov functions; the third is that there often happen impulses and jumps in the solutions.

In the second section of this paper, we presents new definitions of stability of singular systems with delay using the thoughts of papers [1] and [2], and establish two theorems on stability and instability of solutions to the following singular systems with delay,

$$A\dot{x}(t) = f(t, x_t), \tag{1.1}$$

where A is an $n \times n$ constant singular matrix, $x_t(\theta) = x(t+\theta), \theta \in [-r, 0], r > 0, f(t, \phi) \in c([0, +\infty) \times c([-r, 0], \mathbb{R}^n), \mathbb{R}^n), f(t, 0) = 0$, for any $t \geqslant t_0 \geqslant 0$.

The initial condition of Eq. (1.1) is

$$x_{t_0} = \psi, \quad \psi \in c([-\gamma, 0], \mathbb{R}^n). \tag{1.2}$$

In the third section, we discuss stability and instability of zero solution to the following singular systems with delay,

$$\begin{cases} \dot{X}_1(t) = A_{11}X_1(t) + A_{12}X_2(t) + B_{11}X_1(t-r) + B_{12}X_2(t-r), \\ 0 = A_{21}X_1(t) + X_2(t) + B_{21}X_1(t-r) + B_{22}X_2(t-r), \end{cases} \tag{1.3}$$

where, A_{ij}, B_{ij} are $n_i \times n_j$ constant matrices, $X_i \in \mathbb{R}^{n_i}, n_i + n_j = n (i, j = 1, 2)$, r is a positive constant, $t \geqslant t_0 \geqslant 0$.

Suppose that the initial condition of Eq. (1.3) is

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$$X_{t_0} = \begin{bmatrix} X_{1t_0} \\ X_{2t_0} \end{bmatrix} = \phi, \quad \phi \in c([-r,0],\mathbb{R}^n). \tag{1.4}$$

2 Stability of Singular Systems with Delay

At first, we introduce the following notations,

$T_k = [0, +t_k]$, where $0 < t_k \leqslant +\infty$; $q(t,x) \in C^1([0,+\infty) \times \mathbb{R}^n, \mathbb{R}^m)$;

$S_k(t_0,t_k)$ is a set of all consistency initial functions, and $\forall\, \phi_1 \in S_k(t_0,t_k)$, there exists a continuous solution of Eq. (1.1) in $[t_0 - r, t_k)$ through (t_0,ϕ_1) at least;

$$B(0,\delta) = \{\phi \in C([-r,0],\mathbb{R}^n);\ \|\phi\| < \delta, \delta > 0\}.$$

Definition 2.1 If $\forall\, t_0 \in T_k, \forall\, \varepsilon > 0$, there always exists $\delta(t_0,\varepsilon) > 0$, such that $\forall\, \psi \in B(0,\delta) \cap S_k(t_0,t_k)$, the solution $x(t,t_0,\psi)$ of Eqs. (1.1) and (1.2) satisfies that $\|q(t,x(t))\| \leqslant \varepsilon, \forall\, t \in [t_0,t_k)$, then the zero solution of Eq. (1.1) is said to be stable on $\{q(t,x), T_k\}$. If δ is only related to ε and has nothing to do with t_0, then the zero solution is said to be uniformly stable on $\{q(t,x),T_k\}$.

Definition 2.2 1) If the zero solution of Eq. (1.1) is stable on $\{q(t,x),[0,+\infty)\}$, and $\forall\, t_0 \in [0,+\infty)$, there exists a $\Delta(t_0) > 0$, such that $\forall\, \psi \in B(0,\Delta(t_0)) \cap S_k(t_0,+\infty)$, $\lim\limits_{t\to+\infty} \|q(t,x(t,t_0,\psi))\| = 0$, then the zero solution of Eq. (1.1) is said to be asymptotically stable on $\{q(t,x),[0,+\infty)\}$.

2) If the zero solution of Eq. (1.1) is uniformly stable on $\{q(t,x),[0,+\infty)\}$, and there exists a $\Delta > 0, \forall\, t_0 \in [0,+\infty), \forall\, \psi \in B(0,\Delta) \cap S_k(t_0,+\infty)$, $\lim\limits_{t\to+\infty} \|q(t,x(t,t_0,\psi))\| = 0$, and which is uniform on $(t_0,\psi) \in [0,+\infty) \times (B(0,\Delta) \cap S_k(t_0,+\infty))$, then the zero solution of Eq. (1.1) is said to be uniformly asymptotically stable on $\{q(t,x),[0,+\infty)\}$.

Now we give two theorems on stability and instability.

Theorem 2.1 Suppose that for any solution $x(t)$ of Eq. (1.1), $\dot{q}(t,x(t))$ is bounded when $q(t,x(t))$ is bounded, and that there exist two wedge functions $u(\cdot)$ and $v(\cdot)$, and nonnegative and nondecreasing function $\omega(\cdot)$, continuous V functional $V(t,\varphi): [0,+\infty) \times C([-r,0],\mathbb{R}^n) \to \mathbb{R}$, which satisfies

i) $u(\|q(t,x(t))\|) \leqslant V(t,x_t) \leqslant v(\|x_t\|)$; ii) $D^+V(t,x_t) \leqslant -\omega(\|q(t,x(t))\|)$.

Then the zero solution of Eq. (1.1) is uniformly stable on $\{q(t,x),T_k\}$, where $t_k \leqslant +\infty$. If $\omega(s) > 0$ when $s > 0$, then the zero solution of Eq. (1.1) is asymptotically stable on $\{q(t,x),[0,+\infty)\}$.

Proof 1) $\forall\, t_0 \in [0,t_k), \forall\, \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ which is only related to ε, such that $\forall\, \psi \in B(0,\delta), v(\|\psi\|) < v(\delta) \leqslant u(\varepsilon)$.

By that $D^+V(t,x_t) \leqslant 0$, the solution of Eq. (1.1) through (t_0,ψ) satisfies

$$u(\|q(t,x(t,t_0,\psi))\|) \leqslant V(t,x_t(t_0,\psi)) \leqslant V(t_0,\psi) \leqslant v(\|\psi\|) \leqslant u(\varepsilon).$$

Thus, $\|q(t,x(t,t_0,\psi))\| < \varepsilon, \forall\, t \in [t_0,t_k)$. Hence the zero solution of Eq. (1.1) is uniformly stable on $\{q(t,x),[0,t_k)\}$.

2) For that the zero solution of Eq. (1.1) is uniformly stable on $\{q(t,x),[0,t_k)\}$, we set $t_k = +\infty, \varepsilon = 1$. Then there exists a $\delta_0 > 0$ such that $\forall\, t_0 \in [0,+\infty), \psi \in (0,\delta_0) \cap S_k(t_0, +\infty)$, we have

$$\| q(t,x(t,t_0,\psi)) \| < 1, \quad \forall\, t \in [t_0,t_k).$$

By conditions of the theorem, there exists an $L > 0$, such that $\| \dot{q}(t,x(t,t_0,\psi)) \| < L, \forall\, t \in [t_0,t_k)$.

If $\lim\limits_{t\to+\infty} \| q(t,x(t,t_0,\psi)) \| = 0$ is not satisfied, then there are an $\varepsilon_0 \in (0,1)$ and a sequence $\{t_m\}$, $\lim\limits_{m\to+\infty} t_m = +\infty$, such that $\forall\, m$, $\| q(t_m,x(t_m,t_0,\psi)) \| > \varepsilon_0$. We can suppose that $\forall\, m, t_m - t_{m-1} \geqslant 2$.

Take a $\delta_1 \in (0,1)$, such that $L\delta_1 < \frac{\varepsilon_0}{2}$, then $\forall\, m, \forall\, t \in [t_m - \delta_1, t_m + \delta_1]$,

$$\begin{aligned} \| q(t,x(t,t_0,\psi)) \| &\geqslant \| q(t_m,x(t_m,t_0,\psi)) \| - \| q(t_m,x(t_m,t_0,\psi)) - q(t,x(t,t_0,\psi)) \| \\ &\geqslant \varepsilon_0 - L\delta_1 \geqslant \frac{\varepsilon_0}{2}. \end{aligned}$$

Thus $D^+ V(t,x_t(t_0,\varphi)) \leqslant -\omega(\frac{\varepsilon_0}{2}), \forall\, t \in [t_m - \delta_1, t_m + \delta_1]$. Hence there exists an m_0 big enough, such that, when $t > t_{m_0}$,

$$V(t,x_t(t_0,\psi)) \leqslant V(t_0,\psi) - 2\sum_{m=1}^{m_0} \omega(\frac{\varepsilon_0}{2})\delta_1 = V(t_0,\psi) - 2m_0\omega(\frac{\varepsilon_0}{2})\delta_1 < 0$$

which is in contradiction with the definition of V. Thus $\lim\limits_{t\to+\infty} \| q(t,x(t,t_0,\psi)) \| = 0$, and the zero solution of Eq. (2.1) is asymptotically stable on $\{q(t,x), [0,+\infty)\}$.

Theorem 2.2 Consider Eq. (1.1), suppose that $V(\psi)$ is a continuous bounded functional defined in $C([-r,0],\mathbb{R}^n)$, and that there exist an $\alpha > 0$ and an open subset U in $C([-r,0],\mathbb{R}^n)$, such that

i) $\forall\, \phi \in U, V(\phi) > 0, \forall\, \phi \in \partial U, V(\phi) = 0$;

ii) $0 \in \mathrm{col}(U \cap B(0,\alpha))$;

iii) $V(\phi) \leqslant u(\| q(t,\phi(0) \|), \forall\, \phi \in U \cap B(0,\alpha)$;

iv) $\forall\, \phi \in U \cap B(0,\alpha) \cap S_k(t_0, +\infty)$,

$$\dot{V}_-(x_t(t_0,\psi)) = \lim_{h\to 0^+} \inf \frac{1}{h}[V(x_{t+h}(t_0,\psi)) - V(x_t(t_0,\psi))] \geqslant \omega(\| q(t,x(t,t_0,\psi)) \|),$$

where $u(\cdot)$ and $\omega(\cdot)$ are wedge functions. Then the zero solution of Eq. (1.1) is unstable on $\{x, [0,+\infty)\}$. Specially, $\forall\, \delta \in (0,\alpha), \forall\, \psi \in U \cap B(0,\delta) \cap S_k(t_0,+\infty)$, there always exists a $t_1 > t_0$, such that the solution $x_t(t_0,\psi)$ reaches the boundary of $B(0,\alpha)$ at the moment t_1.

Proof $\forall\, \psi \in U \cap B(0,\alpha) \cap S_k(t_0,+\infty)$, we have $V(\psi) > 0$. From condition iv), $V(x_t(t_0,\psi)) \geqslant V(\psi)$, thus $\| q(t,x(t,t_0,\psi)) \| \geqslant u^{-1}(V(x_t,t_0,\psi))) \geqslant u^{-1}(V(\psi))$.

And by condition iv),

$$\dot{V}_-(x_t(t_0,\psi)) \geqslant \omega(\| q(t,x(t,t_0,\psi)) \| \geqslant \omega(u^{-1}(V(\psi))) > 0,$$

for $x_t(t_0,\psi) \in U \cap B(0,\alpha)$.

Set $\eta = \omega(u^{-1}(V(\psi)))$, then

$$V(x_t,t_0,\psi)) \geqslant V(\psi) + \eta(t - t_0),$$

provided that $x_\theta(t_0,\psi) \in U \cap B(0,\alpha)$ for $\theta \in [t_0,t]$.

Since $V(\phi)$ is bounded in $U \cap B(0,\alpha)$, by the discussion above, it is impossible for $x_t(t_0,\psi)$ to belong to $U \cap B(0,\alpha)$ for all $t \in [t_0,+\infty)$. And for that $V(\phi)$ is 0 in ∂U, it is impossible for $x_t(t_0,\psi)$ to go through the boundary of U. Hence there exists a $t_1 \geqslant t_0$, such that $x_{t_1}(t_0,\psi) \in \partial B(0,\alpha)$. Then the zero solution of Eq. (1.1) is unstable on $\{x, [0,+\infty)\}$.

3 Analysis of Stability of Eq. (1.3)

Now we discuss stability and instability of zero solution of Eq. (1.3) using Theorems 2.1 and 2.2. From Eq. (1.3), we have:

$$X_2(t) = -[A_{21}X_1(t) + B_{21}X_1(t-r) + B_{22}X_2(t-r)]. \tag{3.1}$$

If the initial function ϕ satisfies the following consistency condition, the Eq. (1.3) has a unique continuous solution in $[t_0 - r, +\infty)$ through (t_0,ϕ) ([4]),

$$0 = A_{21}\phi_1(0) + \phi_2(0) + B_{21}\phi_1(-r) + B_{22}\phi_2(-r), \tag{3.2}$$

where $\phi(\theta) = \begin{bmatrix}\phi_1(\theta)\\ \phi_2(\theta)\end{bmatrix} \in C([-r,0],\mathbb{R}^n)$.

Thus $\forall\, t_0 \in [0,+\infty)$, $S_k(t_0,+\infty) = \{\psi \in C([-r,0],\mathbb{R}^n); \psi \text{ satisfies (3.2)}\}$.

In this section, if the matrix $F = (f_{ij})_{n\times n}$, then we set $\| F \| = \sum_{i,j=1}^{n} |f_{ij}|$.

Theorem 3.1 For Eq. (1.3), if all eigenvalues of $A_{11} - A_{12}A_{21}$ have negative real parts, $\| B_{ij} \|$, $(i,j = 1,2)$ are small enough, and $\| B_{22} \| < 1$, then the zero solution of Eq. (1.3) is uniformly stable and asymptotically stable on $\{X_1^{\mathrm T}, X_1, [0,+\infty)\}$. Furthermore, the zero solution of Eq. (1.3) is uniformly stable and asymptotically stable on $\{X, [0,+\infty)\}$.

Proof 1) From Eq. (1.3), we have

$$\dot X_1(t) = (A_{11} - A_{12}A_{21})X_1(t) + (B_{11} - A_{12}B_{21})X_1(t-r) + (B_{12} - A_{12}B_{22})X_2(t-r). \tag{3.3}$$

Since all eigenvalues of $A_{11} - A_{12}A_{21}$ have negative real parts, for any given negative definite matrix D, there exists a positive definite matrix E whose smallest eigenvalue is denoted by λ_E, such that

$$(A_{11} - A_{12}A_{21})^{\mathrm T}E + E(A_{11} - A_{12}A_{21}) = D. \tag{3.4}$$

Set $q(t,X) = X_1^{\mathrm T}X_1$, $V(t,\phi) = \phi_1^{\mathrm T}(0)E\phi_1(0) + \int_{-r}^{0} [\phi_1^{\mathrm T}(s),\phi_2^{\mathrm T}(s)]G\begin{bmatrix}\phi_1(s)\\ \phi_2(s)\end{bmatrix}\mathrm{d}s$, where G is an $n\times n$ positive definite matrix which is to be determined.

$\forall\, \psi \in S_k(t_0,+\infty)$, the solution of Eq. (1.3) through (t_0,ψ) is denoted by $X(t)$, then

$$V(t,X_t) = X_1^{\mathrm T}(t)EX_1(t) + \int_{t-r}^{t} [X_1^{\mathrm T}(s),X_2^{\mathrm T}(s)]G\begin{bmatrix}X_1(s)\\ X_2(s)\end{bmatrix}\mathrm{d}s.$$

And

$$\begin{aligned}\| E \| \, \| q(t,X(t)) \| &\leqslant V(t,x_t) \leqslant X_1^{\mathrm T}(t)EX_1(t) + \| G \| \cdot \| x_t \|^2 r\\ &\leqslant (\| E \| + \| G \| r) \| x_t \|^2.\end{aligned}$$

By Eq. (3.3),

$$\begin{aligned}\dot V(t,X_t) = {} & \dot X_1^{\mathrm T}(t)EX_1(t) + X_1^{\mathrm T}(t)E\dot X_1(t) + [X_1^{\mathrm T}(t),X_2^{\mathrm T}(t)]G\begin{bmatrix}X_1(t)\\ X_2(t)\end{bmatrix}\\ & - [X_1^{\mathrm T}(t-r),X_2^{\mathrm T}(t-r)]G\begin{bmatrix}X_1(t-r)\\ X_2(t-r)\end{bmatrix}\\ = {} & X_1^{\mathrm T}(t)[A_{11} - A_{12}A_{21}]^{\mathrm T}EX_1(t) + X_1^{\mathrm T}(t)E[A_{11} - A_{12}A_{21}]X_1(t)\end{aligned}$$

$$
\begin{aligned}
&+ X_1^{\mathrm{T}}(t-r)[B_{11} - A_{12}B_{21}]^{\mathrm{T}} E X_1(t) + X_2^{\mathrm{T}}(t-r)[B_{12} - A_{12}B_{22}]^{\mathrm{T}} E X_1(t) \\
&+ X_1^{\mathrm{T}}(t) E [B_{11} - A_{12}B_{21}] X_1(t-r) + X_1^{\mathrm{T}}(t) E [B_{12} - A_{12}B_{22}] X_2(t-r) \\
&+ [X_1^{\mathrm{T}}(t), -[A_{21}X_1(t) + B_{21}X_1(t-r) + B_{22}X_2(t-r)]^{\mathrm{T}}] G \\
&\cdot \begin{bmatrix} X_1(t) \\ -A_{21}X_1(t) - B_{21}X_1(t-r) - B_{22}X_2(t-r) \end{bmatrix} \\
&- [X_1^{\mathrm{T}}(t-r), X_2^{\mathrm{T}}(t-r)] G \begin{bmatrix} X_1(t-r) \\ X_2(t-r) \end{bmatrix} \\
&= [X_1^{\mathrm{T}}(t), X_1^{\mathrm{T}}(t-r), X_2^{\mathrm{T}}(t-r)][A_1 + B_1 + C_1] \begin{bmatrix} X_1(t) \\ X_1(t-r) \\ X_2(t-r) \end{bmatrix},
\end{aligned}
$$

where

$$
A_1 = \begin{bmatrix} D & E[B_{11} - A_{12}B_{21}] & E[B_{12} - A_{12}B_{22}] \\ [B_{11} - A_{12}B_{21}]^{\mathrm{T}} E & 0 & 0 \\ [B_{12} - A_{12}B_{22}]^{\mathrm{T}} E & 0 & 0 \end{bmatrix}. \tag{3.5}
$$

Set $G = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}$, then

$$
B_1 = \begin{bmatrix} G_{11} + A_{21}^{\mathrm{T}} G_{22} A_{21} & A_{21}^{\mathrm{T}} G_{22} B_{21} & A_{21}^{\mathrm{T}} G_{22} B_{22} \\ B_{21}^{\mathrm{T}} G_{22} A_{21} & B_{21}^{\mathrm{T}} G_{22} B_{21} & B_{21}^{\mathrm{T}} G_{22} B_{22} \\ B_{22}^{\mathrm{T}} G_{22} A_{21} & B_{22}^{\mathrm{T}} G_{22} B_{21} & B_{22}^{\mathrm{T}} G_{22} B_{22} \end{bmatrix}, \tag{3.6}
$$

$$
C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -G_{11} & 0 \\ 0 & 0 & -G_{22} \end{bmatrix}, \tag{3.7}
$$

$$
A_1 + B_1 + C_1 = \begin{bmatrix} D + G_{11} + A_{21}^{\mathrm{T}} G_{22} A_{21} & E[B_{11} - A_{12}B_{21}] + A_{21}^{\mathrm{T}} G_{22} B_{21} & E[B_{11} - A_{12}B_{22}] + A_{21}^{\mathrm{T}} G_{22} B_{22} \\ [B_{11} - A_{12}B_{21}]^{\mathrm{T}} E + B_{21}^{\mathrm{T}} G_{22} A_{21} & B_{21}^{\mathrm{T}} G_{22} B_{21} - G_{11} & B_{21}^{\mathrm{T}} G_{22} B_{22} \\ [B_{11} - A_{12}B_{22}]^{\mathrm{T}} E + B_{22}^{\mathrm{T}} G_{22} A_{21} & B_{22}^{\mathrm{T}} G_{22} B_{21} & B_{22}^{\mathrm{T}} G_{22} B_{22} - G_{22} \end{bmatrix}. \tag{3.8}
$$

If $B_{ij} = 0, i, j = 1, 2$, then

$$
A_1 + B_1 + C_1 = \begin{bmatrix} D + G_{11} + A_{21}^{\mathrm{T}} G_{22} A_{21} & 0 & 0 \\ 0 & -G_{11} & 0 \\ 0 & 0 & -G_{22} \end{bmatrix}. \tag{3.9}
$$

Since D is negative definite, we can take a positive definite matrix G with norm small enough, such that $A_1 + B_1 + C_1$ in (3.9) is negative definite, for instance take diagonal matrix G. Thus when $\| B_{ij} \|$, $(i, j = 1, 2)$ are small enough, there exists a positive definite symmetric matrix G, such that $A_1 + B_1 + C_1$ in (3.8) is negative definite. So there exists a $\lambda > 0$, such that

$$
\begin{aligned}
\dot{V}(t, X_t) &\leqslant -\lambda [X_1^{\mathrm{T}}(t) X_1(t) + X_1^{\mathrm{T}}(t-r) X_1(t-r) + X_2^{\mathrm{T}}(t-r) X_2(t-r) \\
&\leqslant -\lambda X_1^{\mathrm{T}}(t) X_1(t).
\end{aligned}
$$

Thus $\forall\, \varepsilon > 0$, there exists a $\delta(\varepsilon) \in (0, \varepsilon)$, $\forall\, t_0 \geqslant 0$, $\forall\, \psi \in S_k(t_0, +\infty) \cap B(0, \delta)$, the solution of Eq. (1.3) through (t_0, ψ) satisfies:

$$X_1^{\mathrm{T}}(t)X_1(t) \leqslant \varepsilon^2.$$

If $q(t,X(t))$ is bounded in $[t_0, +\infty)$, and the boundary is $M^2, M>0$, then $\| X_1(t) \| \leqslant M, \forall\, t \in [t_0, +\infty)$. We can suppose that the inequality is satisfied for $t \in [t_0 - r, +\infty)$, then

$$|\dot{q}(t,X(t))| = 2|X_1^{\mathrm{T}}(t)[A_{11}X_1(t) + A_{12}X_2(t) + B_{11}x_1(t-r) + B_{12}x_2(t-r)]|$$
$$\leqslant 2[\| A_{11} \| M^2 + \| A_{12} \| M|\, \| X_2(t) \| + \| B_{11} \| M^2 + \| B_{12} \| M \| x_2(t-r) \|].$$

By the second equation of Eq. (1.3), $\forall\, t \in [t_0 + (k-1)r, t_0 + kr]$,

$$\begin{aligned}\| X_2(t)) \| \leqslant & (\| A_{21} \| + \| B_{21} \|)M \\ & + \| B_{22} \| [(\| A_{21} \| + \| B_{21} \|)M \\ & + \| B_{22} \| \| X_2(t-2r) \|] \\ \leqslant & \beta + \| B_{22} \| \beta + \| B_{22} \|^2 \beta + \cdots + \| B_{22} \|^k \beta,\end{aligned}$$

where $\beta = \max\{(\| A_{21} \| + \| B_{21} \|)M, \max\{ \| \psi_2(\theta) \| , \theta \in [-r, 0]\}\}$.

Since $\| B_{22} \| < 1$, then

$$\| X_2(t) \| \leqslant \frac{\beta}{1 - \| B_{22} \|}, \forall\, t \in [t_0 - r, +\infty).$$

Thus

$$|\dot{q}(t,X(t))| \leqslant 2(\| A_{11} \| + \| B_{11} \|)M^2 + 2(\| A_{12} \| + \| B_{12} \|)M \cdot \frac{\beta}{1 - \| B_{22} \|}.$$

Hence, if $q(t,X(t))$ is bounded in $[t_0, +\infty)$, then $|\dot{q}(t,X(t))|$ is bounded. By Theorem 2.1, the zero solution of Eq. (1.3) is uniformly stable and asymptotically stable on $\{X_1^{\mathrm{T}}X_1, [0, +\infty)\}$.

2) Since the zero solution of Eq. (1.3) is uniformly stable on $\{X_1^{\mathrm{T}}X_1, [0, +\infty)\}$, therefore $\forall\, \varepsilon > 0$, we can take a $\delta \in (0, \varepsilon)$, such that $\forall\, \psi \in B(0,\delta) \cap S_k(t_0, +\infty)$,

$$\| q(t,X(t)) \| = X_1^{\mathrm{T}}(t)X_1(t) < \varepsilon^2.$$

By the same manners of proving boundedness of $|\dot{q}(t,X(t))|$ in 1), we can prove that

$$\| X_2(t) \| \leqslant \frac{\beta_1}{1 - \| B_{22} \|}, \quad \forall\, t \in [t_0, +\infty),$$

where, $\beta_1 = \max\{(\| A_{21} \| + \| B_{21} \|)\varepsilon, \varepsilon\}$.

Thus the zero solution of Eq. (1.3) is uniformly stable on $\{X, [0, +\infty)\}$.

Since $X_2(t) = -[A_{21}X_1(t) + B_{21}x_1(t-r) + B_{22}x_2(t-r)]$, if $B_{22} = 0$, then $\lim\limits_{t\to+\infty} \| X_2(t) \| = 0$ when $\lim\limits_{t\to+\infty} \| X_1(t) \| = 0$.

If $B_{22} \neq 0$, let $\delta_0 > 0$, such that $\forall\, \psi \in B(0,\delta_0) \cap S_k(t_0, +\infty)$, $\| X(t) \| \leqslant 1, \forall\, t \in [t_0, +\infty)$.

If $\lim\limits_{t\to+\infty} X_2(t) = 0$ is not true, since $\| X_2(t) \| \leqslant 1, \forall\, t \in [t_0, +\infty)$, thus there exists a sequence $\{t_n\}, t_n \to +\infty$, such that $X_2(t_n) \to \alpha_0$, as $n \to \infty, \alpha_0 \neq 0$.

Since the zero solution of Eq. (1.3) is asymptotically stable on $\{X_1^{\mathrm{T}}, X_1, [0, +\infty)\}$, $\lim\limits_{t\to+\infty} \| X_1(t) \| = 0$. Thus

$$\begin{aligned}\lim_{n\to+\infty} X_2(t_n) &= \lim_{n\to+\infty} -[A_{21}X_1(t_n) + B_{21}X_1(t_n - r) + B_{22}X_2(t_n - r)] \\ &= \lim_{n\to+\infty} -B_{22}X_2(t_n - r).\end{aligned}$$

Since the sequence $\{X_2(t_n - r)\}$ are bounded, so it has a convergent subsequence denoted by itself. That is, there is an α_1, such that $\lim\limits_{n\to+\infty} X_2(t_n - r) = \alpha_1$.

Thus $\alpha_0 = B_{22}\alpha_1$. Repeat this process, we suppose that for any k, there exists an α_k, such that $\lim\limits_{n\to+\infty} X_2(t_n - kr) = \alpha_k, k = 1,2,3,\cdots$. Thus $\alpha_0 = B_{22}^k\alpha_k$. Since $\| B_{22} \| < 1$, $\{ \| \alpha_k \| \}$ are bounded, hence $\alpha_0 = 0$, it is in contradiction with that $\alpha_0 \neq 0$. Thus $\lim\limits_{t\to+\infty} \| X_2(t) \| = 0$, therefore the zero solution of Eq. (1. 3) is asymptotically stable on $\{X, [0, +\infty)\}$. Theorem 3. 1 is proved.

On the boundedness of $\| B_{ij} \|$ in Theorem 3. 1, we give a result.

At first, take matrices

$$D = \mathrm{diag}[-d \quad \cdots \quad -d], \quad G = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}, \tag{3. 10}$$

where the constant $d > 0$, D is an $n_1 \times n_1$ matrix; $G_{11} = \mathrm{diag}[g_1, \cdots, g_1]$ is an $n_1 \times n_1$ matrix, $G_{22} = \mathrm{diag}[g_2, \cdots, g_2]$ is an $n_2 \times n_2$ matrix, and constants $g_1, g_2 > 0$ which are to be determined.

For the given D, we find out the matrix E by (3. 4) firstly. From (3. 8) and (3. 10), we have

$$A_1 + B_1 + C_1 = \begin{bmatrix} (-d + g_1)I_1 + g_2A_{21}^{\mathrm{T}}A_{21} & E[B_{11} - A_{12}B_{21}] + g_2A_{21}^{\mathrm{T}}B_{21} & E[B_{11} - A_{12}B_{22}] + g_2A_{21}^{\mathrm{T}}B_{22} \\ [B_{11} - A_{12}B_{21}]^{\mathrm{T}}E + g_2B_{21}^{\mathrm{T}}A_{21} & g_2B_{21}^{\mathrm{T}}B_{21} - g_1I_1 & g_2B_{21}^{\mathrm{T}}B_{22} \\ [B_{11} - A_{12}B_{22}]^{\mathrm{T}}E + g_2B_{22}^{\mathrm{T}}A_{21} & g_2B_{22}^{\mathrm{T}}B_{21} & g_2[B_{22}^{\mathrm{T}}B_{22} - I_2] \end{bmatrix}. \tag{3. 11}$$

Take $g_1 = \frac{d}{4}$. If $A_{21} = 0$, then take $g_2 = \frac{d}{4}$; If $A_{21} \neq 0$, then take $g_2 = \frac{d}{4\| A_{21} \|^2}$. In the following, we always assume that $A_{21} \neq 0$.

Set $e = \max\{ \| B_{ij} \|, i,j = 1,2\}$. For that $\| B_{22} \| < 1$, thus there is an $\alpha \in (0,1)$, such that $\| B_{22} \| \leqslant \sqrt{1-\alpha}$.

Consider the matrix

$$\begin{aligned} &[(-d + g_1)I_1 + g_2A_{21}^{\mathrm{T}}A_{21}, E[B_{11} - A_{12}B_{21}] + g_2A_{21}^{\mathrm{T}}B_{21}, E[B_{11} - A_{12}B_{22}] \\ &+ g_2A_{21}^{\mathrm{T}}B_{22}], |-d + g_1| - \| g_2A_{21}^{T}A_{21} \| - \| E[B_{11} - A_{12}B_{21}] + g_2A_{21}^{\mathrm{T}}B_{21} \| \\ &- \| E[B_{11} - A_{12}B_{22}] + g_2A_{21}^{\mathrm{T}}B_{22} \| \\ &\geqslant \frac{3d}{4} - \frac{d}{4} - e[2\| E \| + 2\| EA_{12} \| + 2g_2\| A_{21} \|] = \frac{d}{2} - er_1, \end{aligned}$$

where $r_1 = 2[\| E \| + \| EA_{12} \| + g_2\| A_{21} \|]$.

Let $\frac{d}{2} - er_1 > \frac{d}{4}$, then we get that $e < \frac{d}{4r_1}$.

Let $\| B_{21} \| \leqslant 1$, noting that $\| B_{22} \| < 1$, consider the matrix

$$\begin{aligned} &[B_{11} - A_{12}B_{21}]^{\mathrm{T}}E + g_2B_{21}^{\mathrm{T}}A_{21}, g_2B_{21}^{\mathrm{T}}B_{21} - g_1I_1, g_2B_{21}^{\mathrm{T}}B_{22}], \frac{d}{4} - \| [B_{11} - A_{12}B_{21}]^{\mathrm{T}}E \\ &+ g_2B_{21}^{\mathrm{T}}A_{21} \| - g_2\| B_{21}^{\mathrm{T}}B_{21} \| - g_2\| B_{21}^{\mathrm{T}}B_{22} \| \\ &\geqslant \frac{d}{4} - e[\| E \| + \| A_{12}^{\mathrm{T}}E \| + g_2\| A_{21} \| - 2g_2] = \frac{d}{4} - er_2, \end{aligned}$$

where, $r_2 = [\| E \| + \| A_{12}^{\mathrm{T}}E \| + g_2\| A_{21} \| + 2g_2]$.

Let $\frac{d}{4} - er_2 > \frac{d}{8}$, then $e < \frac{d}{8r_2}$.

Consider the matrix

$$[[B_{11} - A_{12}B_{22}]^{\mathrm{T}}E + g_2B_{22}^{\mathrm{T}}A_{21}, g_2B_{22}^{\mathrm{T}}B_{21}, g_2[B_{22}^{\mathrm{T}}B_{22} - I_2]], g_2 - \| [B_{11} - A_{12}B_{22}]^{\mathrm{T}}E \|$$
$$- \| g_2B_{22}^{\mathrm{T}}A_{21} \| - \| g_2B_{22}^{\mathrm{T}}B_{21} \| - \| g_2[B_{22}^{\mathrm{T}}B_{22} \|$$
$$\geqslant \frac{d}{4 \| A_{21} \|^2} - e[\| E \| + \| A_{12} \| \| E \| + g_2 \| A_{21} \| + 2g_2 \|]$$
$$= \frac{d}{4 \| A_{21} \|^2} - er_3,$$

where, $r_3 = [\| E \| + \| A_{12} \| \| E \| + g_2 \| A_{21} \| + 2g_2]$.

Let $\frac{d}{4 \| A_{21} \|^2} - er_3 > \frac{d}{8 \| A_{21} \|^2}^2$, then $e < \frac{d}{8 \| A_{21} \|^2 r_3}$.

Let $\beta_0 = \min\{\frac{d}{4r_1}, \frac{d}{8r_2}, \frac{d}{8 \| A_{21} \|^2 r_3}\}$. Thus, if $e < \beta_0$, $\| B_{21} \| \leqslant 1$, and $\| B_{22} \| < 1$, then from (3.11) and the discussion above, $A_1 + B_1 + C_1$ is strongly diagonal dominant. By Gerschgorin Disc Theorem ([5]), $A_1 + B_1 + C_1$ is negative difinite . Thus we have obtained a boundedness of $\| B_{ij} \|$, $i, j = 1,2$. Obviously, the boundedness is related to the selected matrices D and G, and it is not unique. Thus the conditions of the following Theorem 3.2 is only sufficient and not necessary.

Theorem 3.2 For Eq. (1.3), if all eigenvalues of $A_{11} - A_{12}A_{21}$ have negative real parts, $e = \max\{ \| B_{ij} \| ; i, j = 1,2\} < \beta_0$, and $\| B_{21} \| \leqslant 1$, $\| B_{22} \| < 1$, then the zero solution of Eq. (1.3) is uniformly stable and asymptotically stable on $\{X_1^{\mathrm{T}}X_1, [0, +\infty)\}$. Furthermore, the zero solution of Eq. (1.3) is uniformly stable and asymptotically stable on $\{X, [0, +\infty)\}$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1. Noting the discussion above.

Lemma 3.1 If $n \times n$ matrix A_0 has a eigenvalue with positive real part, and $\lambda_i + \lambda_j \neq 0$, $i, j = 1,2,\cdots,n$, where $\lambda_i, i = 1,2,\cdots,n$ are eigenvalues of A_0, then for any given positive definite matrix C_0, there exists a matrix B_0 which is not semi-negative definite, such that $A_0^{\mathrm{T}}B_0 + B_0A_0 = C_0$. ([6])

Theorem 3.3 For Eq. (1.3), suppose that $A_{11} - A_{12}A_{21}$ has an eigenvalue with positive real part, $\lambda_i + \lambda_j \neq 0, i, j = 1,2,\cdots,n_1$, where $\lambda_i, i = 1,2,\cdots,n_1$ are all eigenvalues of $A_{11} - A_{12}A_{21}$, and that $\| B_{ij} \|$, $(i, j = 1,2)$ are small enough. Then the zero solution of Eq. (1.3) is unstable on $\{X, [0, +\infty)\}$.

Proof We only give the thoughts of the proof. By lemma 3.1, for a given positive definite matrix D_1, there is a matrix E_1 which is not semi-negative definite, such that

$$[A_{11} - A_{12}A_{21}]^{\mathrm{T}}E_1 + E_1[A_{11} - A_{12}A_{21}] = D_1.$$

$\forall\, \phi \in C([-r, 0], \mathbb{R}^n)$, take a V functional

$$V(\phi) = \phi_1^{\mathrm{T}}(0)E_1\phi_1(0) + \int_{-r}^{0} [\phi_1^{\mathrm{T}}(s), \phi_2^{\mathrm{T}}(s)]G_1\begin{bmatrix}\phi_1(s)\\ \phi_2(s)\end{bmatrix}ds,$$

where G_1 is an $n \times n$ negative definite matrix whose norm is small enough, which is to be determined. We can choose G_1 by the same manners in the proof of Theorem 3. 1.

Set $U = \{\phi \in C([-r,0],\mathbb{R}^n); V(\phi) > 0\}, \alpha > 0, q(t,X(t)) = X_1^T(t)X_1(t)$.

Then using Theorem 2. 2, we can prove Theorem 3. 3.

Until now, we have finished the discussion of stability and instability of zero solution to Eq. (1. 3).

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滞后广义系统的稳定性

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摘要：本文首先给出滞后广义系统解的稳定性的新概念，并建立了关于稳定与不稳定的判定定理；然后，讨论了一类 n 阶线性常系数滞后广义系统，得到了该类系统有关稳定与不稳定的一些结果.

关键词：广义系统；滞后；相容性条件；稳定

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