Stability Analysis of Random Hopfield Neural Networks*

Feng Zhaoshu, Wang Jian, Liu Hongwei and Liu Yongqing
(Department of Automatic Control Engineering, South China University of Technology · Guangzhou, 510640, P.R.China)

Abstract: In this paper, by applying the decomposition method of dynamic large-scale interconnected systems, a stability analysis is given for the random Hopfield neural networks described by Ito stochastic differential equations. Such neural networks are viewed as an interconnection of many single neurons. Stability results given in this paper are phrased in terms of the qualitative properties of the individual neurons and in terms of the properties of the interconnecting structure of the neural networks.

Key words: neural networks; Hopfield model; stability; stochastic differential equations

随机 Hopfield 神经网络的稳定性分析

冯昭枢　王　建　刘洪伟　刘永清
(华南理工大学自动控制工程系·广州, 510640)

摘要: 通过利用动态大规模互连系统的分解方法, 对由 Ito 随机微分方程描述的随机 Hopfield 神经网络给出了稳定性分析. 这样的神经网络被认为是许多神经元的一个互连. 在本文中给出的稳定性结论是以单个神经元和互连结构的性质来表示的.

关键词: 神经网络; Hopfield 模型; 稳定性; 随机微分方程

1 Introduction

Neural networks take advantage of distributed information processing and inherent potential for parallel computation. So neural networks have been used as novel computational systems with extremely robustness with respect to malfunctions of individual devices (see [1],[2],[3]). And since neural networks possess behavior of global stability in many cases, they can be used as an information processing system in which the evolution of an imperfect pattern towards the correct (stored) pattern is realized, which is analogous to the storage of information in an associative memory (refer to [4] and [5]). This shows that the qualitative analysis of neural networks is necessary and is very important.

In recent years, Hopfield-type dynamic neural networks have been studied by a lot of literature, such as [6] and [7]. Most of the authors discussed asymptotic behavior of the deterministic Hopfield-type neural networks model. Only a few authors obtained results for asymptotic behavior of the corresponding model with random perturbation.

In this paper, a stability analysis is given for random Hopfield neural networks described by Ito stochastic differential equations by applying the decomposition method of large-scale interconnected systems. First, stability analysis is given for neurons which are viewed as the isolated or free subsystems. Second, the entire neural networks are considered as the large-scale systems which consist of many neurons (isolated subsystems) and interconnecting structure.

2 System description and definitions

In the present paper we will discuss the random Hopfield neural networks described by the following Ito stochastic differential equations

$$C_i \mathrm{d}u_i = \sum_{j=1}^{N} T_{ij} v_j \mathrm{d}t - \frac{1}{\tau_i} u_i \mathrm{d}t + \sum_{j=1}^{N} E_{ij} u_j \mathrm{d}z_j + I_i(t)\mathrm{d}t, i = 1, \cdots, N \quad (1)$$

where $C_i > 0$, $T_{ij} = \frac{1}{R_{ij}}$, $R_{ij} \in \mathbb{R} = (-\infty, \infty)$, $\frac{1}{\tau_i} = \frac{1}{R_i} + \sum_{j=1}^{N} |T_{ij}|$, $R_i > 0$, $I_i: \mathbb{R}^+ = [0, \infty) \to \mathbb{R}$, I_i is continuous, $v_i = g_i(u_i)$, $g_i: \mathbb{R} \to (-1, 1)$, g_i is contin-

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uously differentiable and strictly monotonically increasing (i.e., $g_i(u'_i) > g_i(u''_i)$ if and only if $u'_i > u''_i$), $u_i g_i(u_i) > 0$ for all $u_i \neq 0$, $g_i(0) = 0$, $E_{ij} \in \mathbb{R}$, and $z_j = \{z_j(t), t \geqslant 0\}$ the scalar normalized Wiener process with independent components. In (1), C_j denotes capacitance, R_{ij} denotes resistance (possibly including a sign inversion due to an amplifier), $g_i(\cdot)$ denotes an amplifier nonlinearity, $I_i(\cdot)$ denotes an external input, and $E_{ij}u_j$ characterizes the strength of the random perturbation.

By using the appropriate transformations, (1) can be rewritten by the following equivalent form

$$dx_i = -b_i x_i dt + \sum_{j=1}^{N} A_{ij} G_j(x_j) dt + \sum_{j=1}^{N} e_{ij} x_j dz_j + U_i(t) dt, \quad i = 1, \cdots, N \tag{2}$$

where $G_i(0) = 0$, the origin $x = (x_1, \cdots, x_N)^{\mathrm{T}} = (0, \cdots, 0)^{\mathrm{T}} = 0$ is an isolated equilibrium of (2). And $b_i > 0$, $A_{ij} \in \mathbb{R}$, $e_{ij} \in \mathbb{R}$, $|e_{ij}| \leqslant e_j$, $e_j \in \mathbb{R}^+$. $G_j : \mathbb{R} \to (c_1, c_2) \subset (-2, 2)$ (where $c_1 < 0 < c_2$), G_j is continuous differentiable, G_j is strictly monotonically increasing in x_j, and $x_j G(x_j) > 0$ for all $x_j \neq 0$. Also, $U_i : \mathbb{R} \to \mathbb{R}$ is a continuous function.

By the existence and unique theorem for solution process of stochastic differential equation, under some appropriate assumptions (refer to [8] and [9]), for any given $x_0 = (x_{10}, \cdots, x_{N0})^{\mathrm{T}} \in \mathbb{R}^N$ and for a specific allowable external input $U(t) = (U_1(t), \cdots, U_N(t))^{\mathrm{T}}$, the system (2) possesses a unique solution process $\phi(t, t_0, x_0) = (\phi_1(t, t_0, x_0), \cdots, \phi_N(t, t_0, x_0))^{\mathrm{T}}$, with $\phi(t, t_0, x_0) = x_0$, which exists for all $t \geqslant t_0 \geqslant 0$ with probability one. On those occasions where the context is clear, we will frequently write $\phi(t) = (\phi_1(t), \cdots, \phi_N(t))^{\mathrm{T}}$ in place of $\phi(t, t_0, x_0)$.

It will be convenient to view system (2) as an interconnection of N free subsystems (or isolated subsystems) described by equations of the form

$$dp_i = -b_i p_i dt + A_{ii} G_i(p_i) dt + e_{ii} p_i dz_i + U_i(t) dt. \tag{3}$$

Under this viewpoint, the following terms

$$\sum_{j \neq i, j=1}^{N} [A_{ij} G_j(x_j) dt + e_{ij} x_j dz_j], \quad i = 1, \cdots, N \tag{4}$$

are considered as the interconnecting structure of the system (2).

Let $x = (x_1, \cdots, x_N)^{\mathrm{T}} \in \mathbb{R}^N$, $U(t) = (U_1(t), \cdots, U_N(t))^{\mathrm{T}} \in \mathbb{R}^N$, $A = (A_{ij})_{N \times N}$, $G(x) = (G_1(x_1), \cdots, G_N(x_N))^{\mathrm{T}}$, $E = (e_{ij})_{N \times N}$, $z = (z_1, \cdots, z_N)^{\mathrm{T}} \in \mathbb{R}^N$ and $B = \mathrm{diag}(b_1, \cdots, b_N)^{\mathrm{T}}$. By using the notations, we can rewrite system (2) in the following equivalent form

$$dx = -Bx dt + AG(x) dt + ExI dz + U(t) dt \tag{5}$$

where I denotes the identity matrix.

The entire neural networks (5) can be considered as an interconnected system or a composite system with decomposition (2).

Stability results for (5) involve the existence of Lyapunov function $V : \mathbb{R}^N \to \mathbb{R}$ which has second order continuous partial derivatives, and stability results for (5) require the differential operator $\mathscr{L}$ defined by

$$\mathscr{L}V_{(5)}(x) = \frac{\partial V}{\partial x}[-Bx + AG(x)] + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 V}{\partial x_i \partial x_j} e_{ij}^2 x_j^2. \tag{6}$$

3 Stability conditions for single neurons

The stability results for the entire neural network (5) (given in the next section) are closely related to the stability results for the individual free neurons (3). For this reason, we first give a stability analysis for these subsystems. It will be convenient to give the following hypotheses.

H1) For (3), $U_i(t) = 0$ for all $t \geqslant t_0 \geqslant 0$.

Let $f_i(p_i) = A_{ii} G_i(p_i) - b_i p_i + \frac{1}{2} e_{ii}^2 p_i^2$, $B(r_i) = \{p_i \in \mathbb{R} : -r_i \geqslant p_i \geqslant r_i\}$ for some $r_i > 0$.

H2) For (3), $f_i(p_i) < 0$ when $p_i > 0$, $f_i(p_i) > 0$ when $p_i < 0$, and $f_i(p_i) = 0$ when $p_i = 0$ for $p_i \in B(r_i)$. For the definitions of almost sure asymptotic stability and almost sure exponential stability of the equilibrium of (3), refer to [9], [10], [11] and [12].

By the assumption imposed on the functions $G_i(p_i)$ in Section 1, for $r_i > 0$ there exist constants σ_{i1} and σ_{i2} such that

$$\sigma_{i1} p_i^2 < p_i G_i(p_i) < \sigma_{i2} p_i^2, \quad -r_i < p_i < r_i \tag{7}$$

or equivalently, when $p_i \neq 0$,

$$\sigma_{i1} < \frac{G_i(p_i)}{p_i} < \sigma_{i2}, \quad -r_i < p_i < r_i. \tag{8}$$

We will also use the next hypothesis.

H3) For (3), $(-b_i + A_{ii}\delta_i + \frac{1}{2}e_{ii}^2) < 0$, where, $\delta_i = \sigma_{i1}$ when $A_{ii} < 0$ and $\delta_i = \sigma_{i2}$ when $A_{ii} > 0$.

We now give the following result for single neurons.

Proposition 1 Assume that for (3) hypothesis H1) is true.

i) If H2) holds, then equilibrium $p_i = 0$ of (3) is almost surely asymptotically stable.

ii) If H2) and H3) hold, then $p_i = 0$ of (3) is almost surely exponentially stable.

Proof i) For (3), we choose the following Lyapunov function

$$v_i(p_i) = \frac{1}{2}p_i^2. \tag{9}$$

By the assumption that H1) holds, we have

$$\mathscr{L}v_{i(3)}(p_i) = p_i(-b_ip_i + A_{ii}G_i(p_i) + \frac{1}{2}e_{ii}^2p_i). \tag{10}$$

And H2) guarantees that $v_i(p_i)$ is positive definite and $\mathscr{L}v_{i(3)}(p_i)$ is negative definite. Hence, the equilibrium $p_i = 0$ of (3) (with $U_i(t) \equiv 0$) is almost surely asymptotically stable.

ii) Choose (9) as a Lyapunov function for (3). By the assumption that H1), H2) and H3) hold, we get that

$$\mathscr{L}v_{i(3)}(p_i) \leqslant (-b_i + A_{ii}\delta_i + e_{ii}^2)p_i^2 \tag{11}$$

for $|p_i| < r_i$. Since $(-b_i + A_{ii}\delta_i + e_{ii}^2)p_i^2 < 0$, it follows that the equilibrium $p_i = 0$ of (3) is almost surely exponentially stable.

4 Stability conditions for neural networks

It will be convenient to give the following hypotheses.

A1) For system (5), the external inputs are all zero, i.e.,

$$U_i(t) \equiv 0, \quad i = 1,\cdots,N. \tag{12}$$

A2) For system (5), the interconnections satisfy the estimate

$$x_iA_{ij}G_i(x_j) \leqslant x_ia_{ij}x_j \tag{13}$$

for all $|x_i| < r_i$, $|x_j| < r_j$, $i,j = 1,\cdots,N$, where a_{ij} are real constants.

A3) There exists an N-vector $\alpha > 0$ (i.e., $\alpha^{\mathrm{T}} = (\alpha_1,\cdots,\alpha_N)$ and $\alpha_i > 0$, $i = 1,\cdots,N$) such that the test matrix $S = (s_{ij})$ which is given by

$$s_{ij} = \begin{cases} \alpha_i(-b_i + a_{ii} + \frac{1}{2}Ne_i^2), & i = j, \\ \frac{1}{2}(\alpha_ia_{ij} + \alpha_ja_{ji}), & i \neq j \end{cases} \tag{14}$$

is negative definite, where b_i is defined in (5) and a_{ij} is given in A2).

Now we establish the stability conditions for neural networks.

Theorem 1 The equilibrium $x = 0$ of the neural network (5) with decomposition (3) is almost surely exponentially stable if A1), A2) and A3) hold.

Proof For the entire system (5), we choose the following Lyapunov function

$$V(x) = \sum_{i=1}^{N}\frac{1}{2}\alpha_ix_i^2 \tag{15}$$

where the α_i is given in A3). The function given in (15) is obviously positive definite. The differential operator $\mathscr{L}$ acting on V (refer to Section 2), i.e., $\mathscr{L}V$, along the solution of (5) is given by

$$\mathscr{L}V_{(5)}(x) = \sum_{i=1}^{N}\frac{1}{2}\alpha_i(2x_i)[-b_ix_i + \sum_{j=1}^{N}A_{ij}G_j(x_i)] + \sum_{i,j=1}^{N}\frac{1}{2}\alpha_ie_{ij}^2x_j^2 \tag{16}$$

where A1) has been applied. By (A2) and $|e_{ij}| < e_j$, we have

$$\mathscr{L}V_{(5)} \leqslant \sum_{i=1}^{N}\alpha_i[(-b_i + \frac{1}{2}Ne_i^2)x_i^2 + x_i\sum_{j=1}^{N}a_{ij}x_j] = \omega^{\mathrm{T}}R\omega \tag{17}$$

for all $|x|_2 < r$, where $\omega^{\mathrm{T}} = (|x_1|,\cdots,|x_N|)$, $r = \min_i(r_i)$, $|x|_2 = (\sum_{i=1}^{N}x_i^2)^{\frac{1}{2}}$ and $R = (r_{ij})_{N\times N}$ is given by

$$r_{ij} = \begin{cases} \alpha_i(-b_i + a_{ii} + \frac{1}{2}Ne_i^2), & i = j, \\ \alpha_ia_{ij}, & i \neq j. \end{cases} \tag{18}$$

It is noted that

$$\omega^{\mathrm{T}}R\omega = \omega^{\mathrm{T}}(\frac{R + R^{\mathrm{T}}}{2})\omega = \omega^{\mathrm{T}}S\omega \leqslant \lambda_M(S)|x|_2^2 \tag{19}$$

where S is the matrix given in A3) and $\lambda_M(S)$ denotes the largest eigenvalue of the real symmetric matrix S. By the assumption, S is negative definite. So we get that $\lambda_M(S) < 0$. By (15) and (19), in some neighborhood of the origin $x = 0$, the following inequalities hold

$$c_1|x|_2^2 \leqslant v(x) \leqslant c_2|x|_2^2, \tag{20}$$

$$\mathscr{L}V_{(5)}(x) \leqslant -c_3|x|_2^2 \tag{21}$$

where $c_1 = \frac{1}{2}\min(\alpha_i) > 0$, $c_2 = \frac{1}{2}\max(\alpha_i) > 0$, and $c_3 = -\lambda_M(S) > 0$. Hence, the equilibrium $x = 0$ of the neural network (5) is almost surely exponentially stable.

The above discussion shows that the conditions of Theorem 1 provide a means of analyzing a complex neural network in terms of the qualitative properties of the free subsystems (3) and in terms of the interconnecting structure of the networks (5). If in A2), we take absolute values of both sides of the inequality for $i \neq j$ and if we apply (7), we obtain

$$|x_iA_{ij}G_j(x_j)| \leqslant |x_i||A_{ij}|\sigma_{j2}|x_j|. \tag{22}$$

In this case we may rephrase hypotheses A2) and A3) in the following manner.

A4) For system (2), the interconnections satisfy the estimates

$$x_iA_{ii}G_i(x_i) \leqslant \delta A_{ii}x_i^2 \tag{23}$$

and

$$|x_iA_{ij}G_j(x_j)| \leqslant |x_i||A_{ij}|\sigma_{j2}|x_j|, i \neq j \tag{24}$$

where $\delta_i = \sigma_{i1}$ when $A_{ii} < 0$ and $\delta_i = \sigma_{i2}$ when $A_{ii} > 0$ for all $|x_i| < r_i$, $|x_j| < r_j$, $i, j = 1, \cdots, N$.

A5) There exists an N-vector $\alpha > 0$ such that the text matrix $S = (S_{ij})_{N\times N}$ specified by

$$s_{ij} = \begin{cases} \alpha_i(-b_i + \delta_iA_{ii} + \frac{1}{2}Ne_i^2), & i = j, \\ \frac{1}{2}(\alpha_i|A_{ij}|\sigma_{j2} + \alpha_j|A_{ji}|\sigma_{i2}), & i \neq j \end{cases} \tag{25}$$

is negative definite.

Similarly as in Theorem 1, we now obtain the following result.

Theorem 2 The equilibrium $x = 0$ of the neural system (5) with decomposition (2) is almost surely exponentially stable if hypotheses A1), A4) and A5) are satisfied.

5 Conclusion

In this paper, a stability analysis is given for the random Hopfield neural networks described by Itô stochastic differential equations by applying the decomposition method of large-scale interconnected systems. The dynamic neural networks are considered as an interconnection of many free (or isolated) neurons. Sufficient conditions are first given for the neurons (free subsystems) and second for the neural networks (interconnected systems, or large-scale systems, or composite systems).

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本文作者简介

冯昭枢 1962年生.1990年在华南理工大学获博士学位,1994年被晋升为教授.发表论文100多篇.目前的研究兴趣是滞后系统,随机系统,神经网络,动态系统的分支与混沌等.

王 建 1968年生.1992年毕业于华中理工大学自动控制系,1998年毕业于华南理工大学自动控制工程系,获硕士学位,现为华南理工大学博士研究生.目前的研究领域是电力系统运行与控制.

刘洪伟 1962年生.1993年在华南理工大学自动化系获硕士学位,1995年考入华南理工大学自动化系攻读博士学位至今.曾作为主要研究者获国家教委科技进步奖二等奖等奖励,已在国内外发表20多篇论文.目前的研究领域是大系统的智能分析与综合.

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